The Core of an Economy with a Common Pool Resource:
A Partition Function Form Approach[#]

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1. Introduction

The “tragedy of the commons,” in which common-pool resources are overused is a crucial problem in modern societies (see Hardin ,1968) . Notable examples of this phenomenon include recent decreases of fish stocks and the rapid deforestation taking place in tropical countries. Consider a society of fishermen who fish on a commonly owned lake. If they behave non-cooperatively, each will choose his labor input to maximize his own income given the labor inputs of the others. When there are decreasing returns to labor the total amount of labor inputs in any Nash equilibrium is larger than the Pareto efficient level. In other words, the lake is overfished and the tragedy of the commons occurs.

However, do fishermen always behave non-cooperatively? If sufficient communication is feasible, it may be possible for cooperation to arise. In this paper, we examine whether it is possible to achieve Pareto efficiency and avoid the tragedy of the commons through cooperation among fishermen when free negotiations are possible.

Suppose that after sufficient discussion, fishermen in group S agree to cooperate. They will then coordinate their labor inputs to maximize the sum of their incomes and share the quantity of fish they catch. As there is a negative externality present, both the labor input decision of coalition S as well as the income that S can obtain depend on the behavior of the fishermen who do not belong to S. Hence it is important to specify the coalitions formed among fishermen as a whole. There are many possible cases. For example, fishermen outside S may act non-cooperatively.

Alternatively, they may cooperate and form a single coalition. Another possibility is that several coalitions may form and coexist.[1] We use the concept of a *coalition structure,* that is, a partition of the set N of all fishermen, in order to describe which coalitions are formed.

If a coalition structure $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$ is formed, then in equilibrium each coalition S in $\mathcal{P}$ will choose the total amount of labor input to maximize the sum of members' incomes, given the total labor inputs for the other coalitions in $\mathcal{P}$. The total income that the coalition S obtains in equilibrium under the coalition structure $\mathcal{P}$ is defined as the *worth* that S can obtain by cooperating. Notice that the worth of coalition S under one coalition structure may be different from that under another coalition structure. Because of this feature, we cannot employ the usual definition of a *game in characteristic function form* to describe the model. Here we apply the concept of a *game in partition function form* due to Thrall (1962) and Thrall and Lucas (1963): a partition function assigns a non-negative value to each pair of a coalition and coalition structure which includes that coalition.

In this paper, we investigate whether it is possible for all fishermen to agree to form a "grand coalition" to coordinate their labor inputs. In order to reach such an agreement, there should exist an income distribution such that any coalition S cannot be better off by redistributing the income that S can get by itself among its members. The set of such income distributions is called the *core*. However, as mentioned above, the income that coalition S can obtain by itself depends on the coalitions which the players outside S form. Therefore, we cannot use the ordinal notion of the core of a game in characteristic function form. Instead we introduce several different core concepts for a game in partition function form. These will depend on the expectation

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[1] When examining the core of an economy with environmental externalities, Chander and Tulkens (1997) assume that if the members in a group $S \subseteq N$ form a coalition, then those do not belonging to S will act non-cooperatively and play only individual best reply strategies. They do not consider the possibility that players outside S may form coalitions and take coalitional actions.

of each coalition S regarding the coalition formation of outsiders should coalition S break the agreement and operate on its own.

Imagine that the members of coalition S pessimistically expect that the coalitions of outsiders form in the worst possible way for S. In the present model, this corresponds to the case in which every player outside S acts independently and non-cooperatively. We prove that the core exists if every coalition has these pessimistic expectations. In particular, equal division of total income belongs to the core. Therefore the grand coalition is formed and Pareto efficiency is achieved; the tragedy of the commons can be avoided. On the other hand, the core is empty if each coalition S optimistically expects that all players outside S will form the largest possible coalition, which is best for S in our model. Whether or not the core exists depends crucially on the expectation of each coalition regarding the coalition formation of the outsiders.

The paper is organized as follows: Section 2 describes our basic model of an economy with a common pool resource. In Section 3, we define an equilibrium concept under a given coalition structure and examine the properties of equilibrium labor inputs and incomes. The following section studies the core when each coalition has a pessimistic view regarding outsiders' coalition formation. Section 5 investigates the core when some coalitions have optimistic expectations regarding outsiders. In the final section, we offer concluding remarks and some speculations on possible future research.

2. The Model

We consider the following model of an economy with a common pool resource as examined by Weitzman (1974) and Roemer (1989). There are $n \geq 2$ fishermen employed on a commonly owned lake. Let $N = \{1,2,..,n\}$ be the set of fishermen, with generic element j. Initially there are no fish. Let $x_j \geq 0$ denote the amount of labor fisherman i expends to catch fish.[2] The total amount of labor is given by $x_N \equiv \sum_{j \in N} x_j$. We represent technology by a production function f which specifies the amount of fish caught for each value of the total amount of labor x_N. We assume that $f(0)=0$, $f'(x_N)>0$, $f''(x_N)<0$, and $\lim_{x_N \to \infty} f'(x_N)=0$; that is, there are decreasing returns to labor. The distribution of fish is proportional to the quantity of labor expended among fishermen, since the input is homogeneous and all fishermen are equally likely to catch a fish per unit of time. In other words, the amount of fish for fisherman j is given by $(x_j / x_N) f(x_N)$. Notice that the distribution of fish is not a result of negotiations among fishermen; it is simply a reflection of our technological assumptions. We normalize the price of fish as one and denote the personal cost of labor by q. Further, suppose:

(1) $$0 < q < f'(0).$$

As we will see below, assumption (1) guarantees that an interior solution is obtained. The income of fisherman j is given by

$$m_j(x_1, x_2, \ldots, x_n) = \frac{x_j}{x_N} f(x_N) - q x_j \qquad (j = 1, 2, \ldots, n),$$

where $m_j(0,0,\ldots,0)=0$.

First of all, we consider the case in which all fishermen behave non-cooperatively. A list of labor inputs $(x_1^*, x_2^*, \ldots, x_n^*)$ is said to be a *Nash equilibrium* if for all $j \in N$ and all $x_j \geq 0$,

$$m_j(x_j^*, x_{-j}^*) \geq m_j(x_j, x_{-j}^*),$$

where $x_{-j}^* = (x_1^*, .., x_{j-1}^*, x_{j+1}^*, .., x_n^*)$ and $x_j^* \geq 0$ for all $j \in N$. In other words, each fisherman chooses his labor input to maximize his income, given the labor inputs of the other fishermen. Under the present assumptions, a Nash equilibrium exists (see

[2] Our main results hold independent of whether fishermen are initially endowed with labor.

Theorem 1). Furthermore, in any Nash equilibrium the total amount of labor input is larger than the Pareto efficient level (for example see Roemer, 1989). Each fisherman exerts a negative externality on the others which he does not take into account in his own utility maximization, and he therefore fishes too much; the lake is overfished and the tragedy of the commons occurs.[3]

3. Cooperation and Coalition Structures

It is commonly assumed in the literature that each agent behaves independently and non-cooperatively. But with sufficient communication is it possible that some kind of cooperation may arise and the tragedy of the commons somehow be avoided? In an attempt to answer this question, we consider the case in which cooperation among fishermen is possible.

Suppose that after sufficient discussion, the fishermen in some group $S \subseteq N$ agree to cooperate. It is natural to assume that they would choose their total labor input, $x_S \equiv \sum_{j \in S} x_j$, to maximize the sum of their incomes

$$m_S \equiv \sum_{j \in S} m_j = \frac{x_S}{x_N} f(x_N) - q x_S ,$$

given the labor input of the fishermen outside S. We first investigate how group S chooses its total labor input x_S. In the next section, we discuss the question of how to distribute the income m_S among the fishermen in S.

The decision on the total amount of labor chosen by the group S depends crucially on the behavior of fishermen outside S. There are many possible cases. Fishermen outside S may act non-cooperatively, or they may cooperate and form one coalition. Alternatively several coalitions may formed and coexist. We use the

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[3] Roemer (1989) shows that when the technology exhibits either decreasing or increasing returns, a Nash equilibrium is not Pareto efficient. Only under constant returns to labor is a Nash equilibrium Pareto efficient.

concept of a coalition structure to describe the coalitions formed among fishermen. A *coalition structure* is a partition of the set N of fishermen, $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$, where $1 \le k \le n$, $S_i \ne \varnothing$ for $i = 1,..,k$, $S_i \cap S_j = \varnothing$ for $i, j = 1,..,k$, $i \ne j$, and $S_1 \cup \ldots \cup S_k = N$. An element of a coalition structure, $S_i \in \mathcal{P}$, is called an *admissible coalition in* $\mathcal{P}$.

Suppose that a coalition structure $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$ is formed. Total labor input for an admissible coalition S_i in $\mathcal{P}$ is denoted by $x_{S_i} \equiv \sum_{j \in S_i} x_j$ $(i = 1, \ldots, k)$. A vector $(x^*_{S_1}, x^*_{S_2}, \ldots, x^*_{S_k})$ is an *equilibrium under the coalition structure* $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$ if for all $i = 1,..,k$ and all $x_{S_i} \ge 0$,

$$m_{S_i}(x^*_{S_i}, x^*_{-S_i}) \ge m_{S_i}(x_{S_i}, x^*_{-S_i}),$$

where $x^*_{-S_i} = (x^*_{S_1}, .., x^*_{S_{i-1}}, x^*_{S_{i+1}}, .., x^*_{S_k})$ and $x^*_{S_i} \ge 0$ for all $i = 1,..,k$. In other words, each admissible coalition chooses its labor input to maximize the sum of its members' incomes, given the total amounts of labor inputs for the other admissible coalitions. If $k = n$, then this reduces to the definition of a Nash equilibrium. Moreover, if $k = 1$, then it reduces to the definition of Pareto efficiency.

We now characterize equilibria under a given coalition structure.

Theorem 1. For any coalition structure $\mathcal{P} = \{S_1, \ldots, S_k\}$, *there exists a unique equilibrium under* $\mathcal{P}$, $(x^*_{S_1}, \ldots, x^*_{S_k})$, *which satisfies*

(2) $$f'(x^*_N) + (k-1) f(x^*_N) / x^*_N = kq,$$

(3) $$x^*_{S_i} = x^*_N / k \text{ for all } i = 1, \ldots, k, \text{ and}$$

(4) $$x^*_{S_i} > 0 \text{ for all } i = 1, \ldots, k,$$

where $x^*_N \equiv \sum_{i=1}^{k} x^*_{S_i}$.

Proof. We first prove (2) and (3). The first-order conditions for all coalitions S_i are

(5) $$\frac{\partial m_{S_i}(x^*_{S_1},\dots,x^*_{S_k})}{\partial x_{S_i}} = \frac{(x^*_N - x^*_{S_i})f(x^*_N) + x^*_{S_i}x^*_N f'(x^*_N)}{(x^*_N)^2} - q = 0 \quad (i = 1,..,k).$$

By summing up these equations, we obtain

$$\textstyle\sum_{i=1}^{k} \frac{\partial m_{S_i}(x^*_{S_1},\dots,x^*_{S_k})}{\partial x_{S_i}} = \frac{(k-1)x^*_N f(x^*_N) + (x^*_N)^2 f'(x^*_N)}{(x^*_N)^2} - kq = 0,$$

which implies (2). Using (2) and (5), we obtain (3):

$$x^*_{S_i} = \frac{x^*_N(qx^*_N - f(x^*_N))}{x^*_N f'(x^*_N) - f(x^*_N)} = \frac{x^*_N(qx^*_N - f(x^*_N))}{kqx^*_N - (k-1)f(x^*_N) - f(x^*_N)} = \frac{x^*_N}{k} \quad (i = 1,..,k).$$

Next we prove that there exists a unique value x^*_N satisfying (2) and $x^*_N > 0$. Let

$$g(x) \equiv f'(x) + (k-1)f(x)/x.$$

We will show that $\lim_{x\to 0} g(x) > kq$, $\lim_{x\to\infty} g(x) = 0$, and that g is strictly decreasing in x if $x > 0$, which together imply the existence of a unique value x^*_N satisfying (2) and $x^*_N > 0$. First, we prove that $\lim_{x\to 0} g(x) > kq$. By L'Hôpital's rule, $\lim_{x\to 0} g(x)$ = $f'(0) + (k-1)f'(0)$ = $kf'(0)$. Since $kf'(0) > kq$ by assumption (1), $\lim_{x\to 0} g(x) > kq$. Now it is true that

(6) $$g'(x) = f''(x) + (k-1)[xf'(x) - f(x)]/x^2 < 0 \text{ if } x > 0.$$

By our assumptions, $f''(x) < 0$ and $k \geq 1$. Moreover,

(7) $$xf'(x) - f(x) < 0 \text{ if } x > 0,$$

since $xf'(x) - f(x) = 0$ if $x = 0$ and $\frac{d}{dx}(xf'(x) - f(x)) = xf''(x) < 0$ if $x > 0$. Hence, $g'(x) < 0$ if $x > 0$. Finally, we prove that $\lim_{x \to \infty} g(x) = 0$. By L'Hôpital's rule and our assumption on the production function,

(8) $$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} (f'(x) + (k-1) f'(x)) = \lim_{x \to \infty} kf'(x) = 0.$$

Turn to the proof of (4). Since $x_N^* > 0$, it follows from (3) that $x_{S_i}^* = x_N^* / k > 0$ for all i = 1,…,k.

Finally, we check the second-order conditions. By twice differentiating m_{S_i} with respect to x_{S_i} and using (3), we have

$$\frac{\partial^2 m_{S_i}(x_{S_1}^*, \ldots, x_{S_k}^*)}{\partial x_{S_i}{}^2} = \frac{f''(x_N^*)}{k} + \frac{2(k-1)[x_N^* f'(x_N^*) - f(x_N^*)]}{k(x_N^*)^2} \quad (i = 1,..,k).$$

Since $x_N^* > 0$, $f''(x_N^*) < 0$, $k \geq 1$, and (7) holds, the second order conditions are satisfied with strict inequalities: $\frac{\partial^2 m_{S_i}(x_{S_1}^*, \ldots, x_{S_k}^*)}{\partial x_{S_i}{}^2} < 0$ for $i = 1,..,k$. Q.E.D.

Equation (2) in Theorem 1 indicates that the total amount of labor input in this economy is determined by the cost of labor q and the number of coalitions k. From (3), the total labor input for each coalition is the same independent of coalition size. Moreover, expression (4) shows that interior solutions are obtained: each coalition expends a positive amount of labor to catch fish.

Next we examine how the equilibrium labor input, the average income per head, and the equilibrium income of a coalition depend on the number of admissible coalitions in a coalition structure. Given a coalition structure $\mathcal{P} = \{S_1, \ldots, S_k\}$, let $(x_{S_1}^*(\mathcal{P}), \ldots, x_{S_k}^*(\mathcal{P}))$ be a unique equilibrium under $\mathcal{P}$ and let $x_N^*(\mathcal{P}) = \sum_{i=1}^k x_{S_i}^*(\mathcal{P})$. Moreover, let $m_{S_i}^*(\mathcal{P})$ = $m_{S_i}(x_{S_1}^*(\mathcal{P}), \ldots, x_{S_k}^*(\mathcal{P}))$ be the equilibrium income of coalition S_i for $i = 1,..,k$ and therefore $m_N^*(\mathcal{P})$ = $\sum_{i=1}^k m_{S_i}(x_{S_1}^*(\mathcal{P}), \ldots, x_{S_k}^*(\mathcal{P}))$. Then the following result holds:

Theorem 2. Consider any two coalition structures $\mathcal{P}_k = \{S_1, \ldots, S_k\}$ and $\mathcal{P}'_{k'} = \{S'_1, \ldots, S'_{k'}\}$ *such that* $k < k'$. *Then*

(9) $\quad x_N^*(\mathcal{P}_k) < x_N^*(\mathcal{P}'_{k'})$;

(10) $\quad m_N^*(\mathcal{P}_k) / n > m_N^*(\mathcal{P}'_{k'}) / n$; and

(11) $\quad$ if $S \in \mathcal{P}_k$ and $S \in \mathcal{P}'_{k'}$, then $m_S^*(\mathcal{P}_k) > m_S^*(\mathcal{P}'_{k'})$.

Proof. First, we prove (9). Since x_N^* depends only on the number of coalitions in a coalition structure as we saw in Theorem 1, it is sufficient to show that $\frac{dx_N^*}{dk} > 0$. Let $h(x,k) \equiv f'(x) + (k-1) f(x) / x - kq$. By (2), $h(x_N^*, k) = 0$. Therefore,

(12) $$\frac{dx_N^*}{dk} = -\frac{\partial h(x_N^*, k) / \partial k}{\partial h(x_N^*, k) / \partial x_N^*}.$$

It is easy to see that

(13) $$\frac{\partial h(x_N^*, k)}{\partial x_N^*} = f''(x_N^*) + (k-1)\frac{x_N^* f'(x_N^*) - f(x_N^*)}{(x_N^*)^2} < 0$$

since $f''(x_N^*) < 0$, $k \geq 1$, $x_N^* > 0$, and from (7), $x_N^* f'(x_N^*) - f(x_N^*) < 0$. Further, it is true that

$$\frac{\partial h(x_N^*, k)}{\partial k} = \frac{f(x_N^*)}{x_N^*} - q.$$

By (2) and (7),

(14) $$\frac{f(x_N^*)}{x_N^*} - q = \frac{1}{k}\left[\frac{f(x_N^*)}{x_N^*} - f'(x_N^*)\right] > 0.$$

Thus, $\frac{\partial h(x_N^*, k)}{\partial k} > 0$. This inequality, (12), and (13) together imply $\frac{dx_N^*}{dk} > 0$.

Looking now at (10), since $m_N^* / n = [f(x_N^*) - qx_N^*] / n$ depends only on the number of coalitions in a coalition structure, it is sufficient to show that $\frac{d(m_N^* / n)}{dk} < 0$. Differentiating the function m_N^* / n with respect to *k*, we can show that

(15) $$\frac{d(m_N^* / n)}{dk} = \frac{dx_N^*}{dk}(f'(x_N^*) - q) / n .$$

By (2) and (14),

(16) $$f'(x_N^*) - q = (k-1)(q - \frac{f(x_N^*)}{x_N^*}) < 0 .$$

Since $\frac{dx_N^*}{dk} > 0$, it follows from (15) and (16) that $\frac{d(m_N^* / n)}{dk} < 0$.

Finally, to show (11), since m_S^* depends only on the number of coalitions in a coalition structure, it is sufficient to show that $\frac{dm_S^*}{dk} < 0$. By (3), m_S^* $= (f(x_N^*) - qx_N^*) / k$. Thus,

$$\frac{dm_S^*}{dk} = \frac{\partial m_S^*}{\partial x_N^*}\frac{dx_N^*}{dk} + \frac{\partial m_S^*}{\partial k} .$$

By the above argument, $\frac{dx_N^*}{dk} > 0$. In order to prove that $\frac{dm_S^*}{dk} < 0$, it remains to show that $\frac{\partial m_S^*}{\partial x_N^*} < 0$ and $\frac{\partial m_S^*}{\partial k} < 0$. Partially differentiating m_S^* with respect to x_N^* and using (16), we have

$$\frac{\partial m_S^*}{\partial x_N^*} = \frac{1}{k}(f'(x_N^*) - q) < 0 .$$

Moreover, partially differentiating m_S^* with respect to *k* and using (14), we obtain

$$\frac{\partial m_S^*}{\partial k} = -\frac{x_N^*}{k^2}(\frac{f(x_N^*)}{x_N^*} - q) < 0 .$$ Q.E.D.

Theorem 2 shows that as the number of coalitions decreases, the total amount of labor input decreases, while average income increases. Further, if the number of admissible coalitions in one coalition structure is smaller than that in another coalition structure, then the total income under the former structure is larger than that under the latter.

4. The Core of a Game in Partition Function Form

This section considers the final income distribution which fishermen can agree upon. For this purpose we introduce the notion of a *TU(transferable utility) game.* An ordinal TU game is represented by the pair (N, v), where N is a *player set* and v is a *characteristic function* which assigns a real number $v(S)$ for each S in N. The real number $v(S)$ is defined as the worth, which members of S can obtain by cooperating. However, we cannot use this type of TU game to analyze our model because there exists a negative externality: S's payoff depends not only on the labor inputs of members of S but also on the labor inputs of outsiders. Instead, we employ a new approach based on games in partition function form, introduced by Thrall (1962) and Thrall and Lucas (1963).

An *n-person cooperative game in partition function form* is defined by a triple (N, P, $\{v_\mathcal{P}\}_{\mathcal{P}\in\mathsf{P}}$). Here N is a player set, P is the set of all coalition structures $\mathcal{P}$ of N, and $_\mathcal{P}$ is a partition function that associates with each admissible coalition in $\mathcal{P}$ a real number $S)$. The worth $v_\mathcal{P}(S)$ depends on how players outside S form coalitions; that is, $v_\mathcal{P}(\ )$ $v_\mathcal{P}'(S)$ may be different if $\mathcal{P}\neq\mathcal{P}'$. In our model, the value $v_\mathcal{P}(S_i)$ under coalition structure $\mathcal{P}$ is given by

$$v_\mathcal{P}(S_i)\equiv\sum_{j\in S_i} m_j(x^*_{S_1}(\mathcal{P}),x^*_{S_2}(\mathcal{P}),\dots,x^*_{S_k}(\mathcal{P})) \qquad (i=1,..,k),$$

where $(x^*_{S_1}(\mathcal{P}),x^*_{S_2}(\mathcal{P}),\dots,x^*_{S_k}(\mathcal{P}))$ is an equilibrium vector of labor inputs under the coalition structure $\mathcal{P}=\{S_1,S_2,\dots,S_k\}$.

Given a coalition structure $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$, a payoff vector $z \in R^N$ is said to be *feasible under* $\mathcal{P}$ if it satisfies

$$\sum_{j \in S_i} z_j \le v_{\mathcal{P}}(S_i) \qquad (i = 1, \ldots, k).$$

Let $\mathcal{I}^{\mathcal{P}}$ be the set of all feasible payoff vectors under $\mathcal{P}$ and $\mathcal{I} \equiv \cup_{\mathcal{P} \in \mathrm{P}} \mathcal{I}^{\mathcal{P}}$. We now introduce a domination relation for two payoff vectors in $\mathcal{I}$. Consider two payoff vectors z, z' in $\mathcal{I}$ and a coalition S in N. We say z *dominates* z' *via* S and denote $z \ dom_S \ z'$ if the following two conditions hold:

(i) $\sum_{j \in S} z_j \le v_{\mathcal{P}}(S)$ for all $\mathcal{P} \ni S$; and

(ii) $z_j > z'_j$ for all j with $j \in S$.

Each member of S can get a larger payoff under the feasible distribution z than under the present distribution z' independent of coalition formation among outsiders. In other words, the members of S have a pessimistic view of the coalition formation of outsiders. In addition, we simply say z *dominates* z' if there exists $S \subseteq N$ such that $z \ dom_S \ z'$, and denote $z \ dom \ z'$.

In order to find a reasonable final agreement vector, we consider payoff vectors that are not dominated by any other vectors in $\mathcal{I}$. The set of feasible payoff vectors that satisfies this condition is called the *core* and is denoted by C. Formally the core C is given by

$$C = \{ z \in \mathcal{I} \mid \nexists \, z' \in \mathcal{I} \text{ s.t. } z' \ dom \ z \}.$$

The core is typically defined in the context of a TU game. We transform our partition function form game to an ordinal TU game, and compare the core C and the core of the TU game. To accomplish this we first consider a payoff vector that is not dominated by any other vectors in I *via* N. Of course, the payoff vector in the core C

satisfies this condition. In our model, any feasible payoff vector z under $\mathcal{P}$ = $\{S_1, S_2, \ldots, S_k\}$ other than that under $\mathcal{P}^N \equiv \{N\}$ satisfies

$$v_{\mathcal{P}^N}(N) > \sum_{i=1}^{k} v_{\mathcal{P}}(S_i) \geq \sum_{i=1}^{k} \sum_{j \in S_i} z_j \,,$$

where the first inequality is implied by (10). Hence z is dominated by some feasible payoff vector under $\mathcal{P}^N$ via N. Further, any feasible payoff vector z' under $\mathcal{P}^N$ satisfying $\sum_{j \in N} z'_j < v_{\mathcal{P}^N}(N)$ is also dominated by some feasible payoff vector z under $\mathcal{P}^N$ satisfying $\sum_{j \in N} z_j = v_{\mathcal{P}^N}(N)$ via N. The last equality corresponds to the Pareto efficiency condition. Thus the core C is included in the following set $\mathcal{E}$:

$$\mathcal{E} \equiv \{z \in \mathcal{I} | \sum_{j \in N} z_j = v_{\mathcal{P}^N}(N)\}.$$

The core of an ordinal TU game is defined as the set of feasible payoff vectors such that any coalition S receives a payoff not less than the corresponding worth $v(S)$. By the above argument, given a game $(N, \mathrm{P}, \{v_{\mathcal{P}}\}_{\mathcal{P} \in \mathrm{P}})$ in partition function form, if a TU game (N, v) satisfies $v(N) = v_{\mathcal{P}^N}(N)$, the *core of the TU game* $C(v)$ should be given by

(17) $$C(v) \equiv \{z \in \mathcal{E} | \sum_{j \in S} z_j \geq v(S), \forall S \subset N, S \neq N\}.$$

We now give an equivalence theorem about the core C of a partition function form game and the core $C(v)$ of a TU game.

Theorem 3. Suppose $v_{\mathcal{P}^N}(N) > \sum_{i=1}^{k} v_{\mathcal{P}}(S_i)$ *for any* $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$ *other than* $\mathcal{P}^N$. *If we consider the transformation of* $(N, \mathrm{P}, \{v_{\mathcal{P}}\}_{\mathcal{P} \in \mathrm{P}})$ *to* $(N, v_{\min})$ *given by*

(18) $$v_{\min}(S) = \min_{\mathcal{P} \ni S} v_{\mathcal{P}}(S) \qquad \forall S \subseteq N,$$

then $C = C(v_{\min})$.

Proof. Left to the readres.

To determine $v_{\min}(S)$ in (18), we provide the following lemma:

Lemma 1. Consider any coalition $S \subseteq N$. *Then* $v_{\mathcal{P}_k}(S) > v_{\mathcal{P}'_{k'}}(S)$ *for any two coalition structures* $\mathcal{P}_k = \{S_1, \ldots, S_k\}$ *and* $\mathcal{P}'_{k'} = \{S'_1, \ldots, S'_{k'}\}$ *such that* $k < k'$ *and* $S \in \mathcal{P}_k \cap \mathcal{P}'_{k'}$. *Moreover,*

$$v_{\min}(S) = \frac{1}{n - |S| + 1}\left(f(x_N^*(\overline{\mathcal{P}}^S)) - q x_N^*(\overline{\mathcal{P}}^S)\right)$$

where $|S|$ *is the cardinality of S and* $\overline{\mathcal{P}}^S = \arg\min_{\mathcal{P} \ni S} v_{\mathcal{P}}(S)$.

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Proof. The first inequality $v_{\mathcal{P}_k}(S) > v_{\mathcal{P}'_{k'}}(S)$ is implied by equation (11) of Theorem 2 because $v_{\mathcal{P}}(S) = m_S^*(\mathcal{P})$. Hence the number of elements in $\overline{\mathcal{P}}^S$ should be as large as possible. Thus we have $\overline{\mathcal{P}}^S = \{S_0, S_1, \ldots, S_{n-|S|}\}$, where $S_0 = S$ and $|S_j| = 1, j = 1, \ldots, n - |S|$, that is,

$$v_{\min}(S) = v_{\overline{\mathcal{P}}^S}(S) = \frac{1}{n - |S| + 1}\left(f(x_N^*(\overline{\mathcal{P}}^S)) - q x_N^*(\overline{\mathcal{P}}^S)\right).$$

Q.E.D.

According to Lemma 1, when insiders of S entertain pessimistic expectations, they act as though outsiders behave non-cooperatively. The following theorem shows that the core in the case is non-empty:

Theorem 4. For the TU game (N, $v_{\min}$*) defined by* (18),

$C(v_{\min}) \neq \emptyset$.

Proof. First we show that $\frac{v_{\min}(N)}{n} \geq \frac{v_{\min}(S)}{s}$ for all $S \subseteq N$,where $|S| = s$[4]. Consider any coalition $S \subset N$ with $S \neq N$ and set $\overline{\mathcal{P}}^S = \arg\min_{\mathcal{P} \ni S} v_{\mathcal{P}}(S)$. Then by Lemma 1, we have

$$\begin{aligned}\frac{v_{\min}(N)}{n} - \frac{v_{\min}(S)}{s} &= \frac{f(x_N^*(\mathcal{P}^N)) - qx_N^*(\mathcal{P}^N)}{n} - \frac{f(x_N^*(\overline{\mathcal{P}}^S)) - qx_N^*(\overline{\mathcal{P}}^S)}{s(n-s+1)} \\ &\geq \frac{f(x_N^*(\overline{\mathcal{P}}^S)) - qx_N^*(\overline{\mathcal{P}}^S)}{n} - \frac{f(x_N^*(\overline{\mathcal{P}}^S)) - qx_N^*(\overline{\mathcal{P}}^S)}{s(n-s+1)},\end{aligned}$$

where the last inequality follows since $x_N^*(\mathcal{P}^N) < x_N^*(\overline{\mathcal{P}}^S)$ by (9) and $f(x) - qx$ is decreasing for $x > 0$ by (16). Hence

$$\begin{aligned}\frac{v_{\min}(N)}{n} - \frac{v_{\min}(S)}{s} &\geq (f(x_N^*(\overline{\mathcal{P}}^S)) - qx_N^*(\overline{\mathcal{P}}^S))(\frac{1}{n} - \frac{1}{s(n-s+1)}) \\ &= (f(x_N^*(\overline{\mathcal{P}}^S)) - qx_N^*(\overline{\mathcal{P}}^S))\frac{(n-s)(s-1)}{ns(n-s+1)} \geq 0,\end{aligned}$$

because $f(x_N^*(\overline{\mathcal{P}}^S)) - qx_N^*(\overline{\mathcal{P}}^S) > 0$ by (14). Since it holds that $|S|\frac{v_{\min}(N)}{n} \geq v_{\min}(S)$ for any $S \subseteq N$, $(\frac{v_{\min}(N)}{n}, \frac{v_{\min}(N)}{n}, \ldots, \frac{v_{\min}(N)}{n}) \in C(v_{\min}) = C$.

Q.E.D.

The above proof demonstrates that the egalitarian distribution among the grand coalition N always belongs to the core. With pessimistic expectations regarding coalition formation among outsiders, we would therefore expect fishermen to make an agreement dividing the total income equally among all players. In this situation, the

[4] In fact, it is easy to prove that this condition is both necessary and sufficient for a symmetric TU game to have a non-empty core.

tragedy of the commons could then be avoided.

5. The Core under an Optimistic Expectations

In this section, we consider the opposite case, where fishermen's expectations about outsiders' coalition formation are optimistic. We modify the definition of the domination relation *dom* and introduce a new domination relation $\underline{dom}$ as follows: Given *S* in *N*, and $z,\ z' \in \mathcal{I}$,

$$z \ \underline{dom}_S \ z' \ \Leftrightarrow \ \exists \mathcal{P} \ni S \ \text{ s.t. (i) } \textstyle\sum_{j \in S} z_j \le v_{\mathcal{P}}(S) \text{; and}$$

$$\text{(ii) } z_j > z'_j \quad \text{for all } j \text{ with } j \in S.$$

The insiders of *S* now suppose that the most favorable coalition structure occurs. Thus we say coalition *S* has optimistic expectations. We also define; for $z,\ z' \in \mathcal{I}$,

$$z \ \underline{dom} \ z' \ \Leftrightarrow \ \exists S \subseteq N \ \text{ s.t. } \ z \ \underline{dom}_S \ z'.$$

When all coalitions have optimistic expectations, we consider the core $\underline{C}$ defined by this new domination relation. Formally we have:

$$\underline{C} = \{ z \in \mathcal{I} \mid \ \nexists\, z' \in \mathcal{I} \ \text{ s.t. } z' \ \underline{dom} \ z \ \}.$$

The following equivalence theorem holds for the core $\underline{C}$:

Theorem 5. Suppose $v_{\mathcal{P}^N}(N) > \sum_{i=1}^{k} v_{\mathcal{P}}(S_i)$ *for any* $\mathcal{P} = \{S_1, S_2, \ldots, S_k\}$ *other than* $\mathcal{P}^N$. *Under the transformation of* $(N,\ \mathsf{P}\ ,\ \{v_{\mathcal{P}}\}_{\mathcal{P} \in \mathsf{P}})$ *to* $(N, v_{\max})$ *given by*

$$v_{\max}(S) = \max_{\mathcal{P} \ni S} v_{\mathcal{P}}(S) \qquad \forall S \subset N, \tag{19}$$

then $\underline{C} = C(v_{\max})$.

Proof. The proof is similar to that of Theorem 3.

We will now prove the core given by $\underline{C} = C(v_{\max})$ is empty, which is opposite to the result obtained under pessimistic expectations.

Theorem 6. Let $n \geq 4$ *. Then for the TU-game* $(N, v_{\max})$ *defined by* (19),

$$C(v_{\max}) = \emptyset .$$

Proof. We will show that $\dfrac{v_{\max}(N)}{n} < v_{\max}(R)$ for all $R \subset N$ with $|R| = 1$, which implies emptiness of the core of symmetric TU games. Set $\underline{\mathcal{P}} = \arg\max\limits_{\mathcal{P} \ni R} v_{\mathcal{P}}(R)$, and note that $v_{\max}(N) = v_{\mathcal{P}^N}(N)$. Then, by equation (11) in Theorem 2, we have $\underline{\mathcal{P}} = \{R, N \setminus R\}$ and

$$v_{\max}(R) = v_{\underline{\mathcal{P}}}(R) = \frac{1}{2}(f(x_N^*(\underline{\mathcal{P}})) - qx_N^*(\underline{\mathcal{P}})).$$

Then equation (2) of Theorem 1 implies

$$f'(x_N^*(\mathcal{P}^N)) = q \quad \text{and} \quad \frac{1}{2}\left\{x_N^*(\underline{\mathcal{P}}) f'(x_N^*(\underline{\mathcal{P}})) + f(x_N^*(\underline{\mathcal{P}}))\right\} = qx_N^*(\underline{\mathcal{P}}).$$

Hence,

$$\frac{v_{\max}(N)}{n} - v_{\max}(R) = \frac{f(x_N^*(\mathcal{P}^N)) - qx_N^*(\mathcal{P}^N)}{n} - \frac{f(x_N^*(\underline{\mathcal{P}})) - qx_N^*(\underline{\mathcal{P}})}{2}$$

$$= \frac{1}{2n}\Big\{2(f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}^N))x_N^*(\mathcal{P}^N)) \\ -n(f(x_N^*(\underline{\mathcal{P}})) - 1/2(x_N^*(\underline{\mathcal{P}}) f'(x_N^*(\underline{\mathcal{P}})) + f(x_N^*(\underline{\mathcal{P}}))))\Big\}$$

$$= \frac{1}{4n}\Big\{4(f(x_N^*(\mathcal{P}^N)) - x_N^*(\mathcal{P}^N) f'(x_N^*(\mathcal{P}^N))) - n(f(x_N^*(\underline{\mathcal{P}})) - x_N^*(\underline{\mathcal{P}}) f'(x_N^*(\underline{\mathcal{P}})))\Big\}.$$

Here, $0 < f(x_N^*(\mathcal{P}^N)) - x_N^*(\mathcal{P}^N) f'(x_N^*(\mathcal{P}^N)) < f(x_N^*(\underline{\mathcal{P}})) - x_N^*(\underline{\mathcal{P}}) f'(x_N^*(\underline{\mathcal{P}}))$ holds because $f(x) - xf'(x)$ is increasing for $x > 0$ (see the proof of (7) in Theorem 1) , and $x_N^*(\mathcal{P}^N) < x_N^*(\underline{\mathcal{P}})$. This and $n \geq 4$ together imply $\dfrac{v_{\max}(N)}{n} - v_{\max}(R) < 0$. Q.E.D.

Theorem 1 states that the core of any game with 4 or more players is empty if every coalition has the optimistic expectations. On the other hand, the core exists in the case of a 2-person game. For 3-person games, it is possible to find examples both of the existence and non-existence of the core. Indeed, it is easy to see that the core of $(N, v_{\max})$ is empty if the production function is given by $f(x) = x^{0.5}$, while it is non-empty if $f(x) = x^{0.2}$.

The existence or non-existence of the core depends on insiders' expectations about the behavior of outsiders. Thus the possibility for a resolution of the tragedy of the commons also depends on these expectations[5].

6. Concluding Remarks

We interpret the difference between $v_{\min}(S)$ and $v_{\max}(S)$ as based on differences in the expectations of coalition S about the coalition formation of outsiders. If the expectations of the members of S are optimistic, then the core is empty and hence the tragedy of the commons cannot be avoided. In other words, if we could find a way to change these expectations from optimistic to pessimistic, the tragedy might still be avoided. However, our analysis in this paper has proceeded under the assumption of given expectations. In order to study how players form these specific expectations, it would be necessary to investigate a dynamic or repeated non-cooperative game based on our model. This important and interesting problem is left for future research.

Another interpretation of the difference between $v_{\min}(S)$ and $v_{\max}(S)$ exists. In non-zero-sum games in strategic form, von-Neumann and Morgenstern (1953) introduce a method for deriving $v(S)$ using maximin strategies between the two coalitions S and $N \setminus S$. We will now apply this method to determine $v(S)$. The strategy

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[5] We have focused on the two extreme cases of pessimistic and optimistic expectations of coalition formation: For intermediate cases, it is easy to construct examples of both existence and non-existence of the core. Therefore it is not *necessary* for the existence of the core that all coalitions have pessimistic expectations. Similarly, it is not necessary for the non-existence of the core that all coalitions have optimistic expectations.

of S (respectively $N \setminus S$) is to partition the coalition S ($N \setminus S$) itself. That is, the strategy set P^S ($\mathrm{P}^{N \setminus S}$) is given by the set of all partitions of S ($N \setminus S$). Like von-Neumann and Morgenstern (1953), we define $\hat{v}(S)$ as the maximin value:

$$\hat{v}(S) = \max_{\mathcal{P}^S \in \mathrm{P}^S} \min_{\mathcal{P}^{N \setminus S} \in \mathrm{P}^{N \setminus S}} v_{(\mathcal{P}^S, \mathcal{P}^{N \setminus S})}(S).$$

By (18) and Lemma 1, we have $\hat{v}(S) = v_{\min}(S)$ for all S in N. That is, this case is equivalent to the pessimistic expectations case.

It is possible to apply the notion of Nash equilibrium instead of maximin strategies. Under the same strategy sets as above, if we define $\tilde{v}(S)$ as a Nash equilibrium value, we have:

$$\tilde{v}(S) = v_{(\mathcal{P}^{*S}, \mathcal{P}^{*N \setminus S})}(S)$$

$$\text{such that} \quad v_{(\mathcal{P}^{*S}, \mathcal{P}^{*N \setminus S})}(S) \geq v_{(\mathcal{P}^{S}, \mathcal{P}^{*N \setminus S})}(S) \qquad \text{for all } \mathcal{P}^S \in \Pi^S$$

$$v_{(\mathcal{P}^{*S}, \mathcal{P}^{*N \setminus S})}(N \setminus S) \geq v_{(\mathcal{P}^{*S}, \mathcal{P}^{N \setminus S})}(N \setminus S) \quad \text{for all } \mathcal{P}^{N \setminus S} \in \Pi^{N \setminus S}.$$

In this case there is a unique Nash equilibrium which is also a dominant equilibrium. Using (11) and (19) we can show that $\tilde{v}(S) = v_{\max}(S)$. This case is equivalent to the optimistic expectations case.

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