Testable implications of no envy allocations

Yasushi Agatsuma*

Abstract

We investigate testable implications of no envy allocations via revealed preference approach. For an observed data which consists of finitely many allocations for a given group of agents, we seek what conditions must be satisfied by the data in order to rationalize that the data is consistent with envy free allocations. To achieve our goal, we follow the method used in Brown and Matzkin [3] and Snyder [7, 8] which consists of two steps. First, we show that a system of inequalities whose solvability is equivalent to rationalization. Then, we apply Tarski-Seidenburg theorem to the system of inequalities to obtain quantifier-free conditions for rationalization for the case when allocation data consists only two allocations. We also deal with testable implications of no envy and efficient allocations, and we show that efficiency puts no further observable restriction to no envyness when the number of agents and observations are two. Trough numerical example, we demonstrate these conditions are not vacuous.

KEYWORDS: Testable implications; revealed preference; fair division; no envyness; Pareto efficient

1 Introduction

We investigate testable implications of no envy allocations via revealed preference approach. Specifically, we answer the following question; given observed allocation data, when can we justify the allocations as a result of agents choosing envy free allocations? Our aim through answering this question is to understand the empirical contents of no envy allocations, that is, to understand what restrictions are imposed on the real world by no envyness of allocations. Put differently, we clarify what pattern of observed behaviour is ruled out when we construct a model which employ no envyness concept.

Since its first introduction by Foley [6], no envyness has been considered to be a prominent concept of fairness in economics. Although it has been a subject of many studies, no attempt has been made to understand testable implications of no envy allocations. The issue which makes our task challenging is that the model does not involve any maximization problem. Consider, for instance, the canonical example of revealed preference excercise on consumer theory. An observation of a consumer choosing a commodity bundle under a market price reveales that the consumer thinks the commodity is at least as good as any other commodity bundles which are affordable to him/her under the market price. On the other hand, under the hypothesis of agents choosing no envy allocations, an observation only reveales the agents' preference such that they thinks their own commodity bundle is at

*Faculty of Political Science and Economics, Waseda University. email: yasushi.agatsuma@gmail.com

least as good as commodity bundle of others'. Therefore whether no envy allocations have non-vacuous testable impolications is not trivial question.

In order to obtain testable implications of no envy allocations, we follow the method used in Brown and Matzkin [3], Snyder [7, 8], which explore testable implications of competitive equilibrium, public good production, and household consumption behaviour, respectively. The method consists of two steps.

First, we characterize the testable implications of no envy allocations through a system of inequalities. Such characterization frequently appeared in the literature, and the system is known as the Afriat type inequality system. Afriat [1] shows that the utility maximizing hypothesis can be tested from given collection of price-consumption data by checking the solvability of a system of inequalities (Afriat inequalities). Our first characterization has similar structure; we prove that the observed allocation data is consistent with no envyness concept if and only if a system of inequalities has a solution.

Second, we apply the technique of quantifier elimination to the system of inequalities provided in the first step. Tarski-Seidenburg theorem guarantees that for any finite polynomial inequalities, there exists an equivalent system which does not involve any quantifiers. The characterization obtained in the first step allows us to apply the technique, which leads us a quantifier-free equivalent form of testable implications when the number of observations are two.

We also deal with testable implications of no envy and efficient allocations. Compatibility of fairness and efficiency has been a classical problem in economics (see for example, Varian [10] and Svensson [9]), it is natural to extend our investigation on testable implications of no envy allocations to no envy and efficient allocations. We will show that, in fact, efficiency puts no further observable restriction to no envyness. In other words, allocations which are merely no envy and allocations which are no envy and efficient are indistinguishable by observing allocations.

This paper is organized as follows. We begin with introducing notations and preliminary definitions next section. Section 3 concerns testable implications of no envy allocations, and section 4 deals with no envy and efficient allocations. Appendix contains proofs of main results.

2 Preliminary

We adopt standard fair division model in which commodities are assumed to be perfectly divisible. Precisely, a *fair division model* is a tuple $(N, \{u_i\}_{i \in N}, \omega)$, where each of the components is interpreted as:

- $N = \{1, \dots, n\}$ is a finite set of agents,
- $u_i : \mathbb{R}^L_+ \to \mathbb{R}$ is a utility function of the agent i,
- $\omega \in \mathbb{R}^L_{++}$ is an initial resource to be divided among agents.

Notice that absence of subscript for ω reflects that the initial resource is not owned privately. An *allocation* is a vector $x \in (\mathbb{R}^L_+)^N$ with $\sum_{i \in N} x_i = \omega$. We say x is a *no envy allocation* if there are no i and j with $u_i(x_j) > u_i(x_i)$. In words, an allocation is no envy if no agent feels that another agent has more desirable commodity bundle than his/her own. An allocation x is *Pareto efficient* (or simply *efficient*) if there is no other allocation y with $u_i(y_i) \geq u_i(x_i)$ for all $i \in N$ and $u_i(y_i) > u_i(x_i)$ for some i.

For our purpose of investigating testable implications of no envy allocations, our given is a finite set of allocations for agents. We say that $\{x^t\}_{t=1}^T$ with $x^t = (x_i^t)_{i\in N} \in (\mathbb{R}_+^L)^N$ as a *finite allocation data*. For a given finite allocation data, we investigate when the allocations in the data can be regarded as no envy allocations of some fair division model. The formal definition is given as follows.

Definition 2.1. A finite allocation data $\{x^t\}_{t=1}^T$ is no-envy-rationalizable (NE-rationalizable) if there exists a set of continuous, monotone, and concave utility functions $\{u_i(\cdot)\}$ such that x^t is an envy free allocation of $(N, \{u_i\}_{i\in N}, \omega^t)$ for each $t = 1, \ldots, T$, where $\omega^t = \sum_{i\in N} x_i^t$.

3 Testable implications of no envy allocations

In order to obtain testable implications of no envy allocations, we employ the method used in Brown and Matzkin [3], and Snyder [7, 8], which explore testable implications of competitive equilibrium, public good production, and household consumption behaviour, respectively. The method consists of two steps. First, they provide a system of inequalities whose solvability is equivalent to rationalizability of data. Second, apply the technique of quantifier elimination to the system of inequalities provided in the first step, they obtain another quantifier-free equivalent condition of rationalization.

Our first result provides a system inequalities whose solvability is equivalent to NE-rationalizability of an allocation data, which is formally stated as follows.

Theorem 3.1. *A finite allocation data $\{x^t\}_{t=1}^T$ is NE-rationalizable if and only if there exist numbers and vectors*

- $u_{ij}^t \in \mathbb{R}$ *for* $t = 1, \ldots, T; i, j \in N$,
- $\lambda_{ij}^t \in \mathbb{R}_+^L \setminus \{0\}$ *for* $t = 1, \ldots, T; i, j \in N$

such that

$$u_{ii}^s - u_{ii}^t \le \langle \lambda_{ii}^t, x_i^s - x_i^t \rangle \text{ for } s, t = 1, \ldots, T; i \in N, \tag{3.1}$$

$$u_{ii}^s - u_{ij}^t \le \langle \lambda_{ij}^t, x_i^s - x_j^t \rangle \text{ for } s, t = 1, \ldots, T; i, j \in N, \tag{3.2}$$

$$u_{ii}^t \ge u_{ij}^t \text{ for } t = 1, \ldots, T; i, j \in N. \tag{3.3}$$

Proof. Suppose that a finite allocation data $\{x^t\}_{t=1}^T$ is NE-rationalizable. Note that concavity of $u_i(\cdot)$ guarantees that its superdifferential $\partial u_i(x)$ is non empty, and monotonicity implies it is a subset of $\mathbb{R}_+^L \setminus \{0\}$. Set numbers u_{ij}^t and vectors λ_{ij}^t as follows

$$u_{ij}^t = u_i(x_j^t),\ \lambda_{ij}^t \in \partial u_i(x_j^t) \tag{3.4}$$

for $i, j \in N$, and $t = 1, \ldots, T$. Then it is obvious that inequalities (3.1) and (3.2) are hold. Inequalities (3.3) are implied by no envyness.

For the converse direction, suppose that we have found numbers $u_{ij}^t \in \mathbb{R}$ and vectors $\lambda_{ij}^t \in \mathbb{R}_+^L \setminus \{0\}$ for $i, j \in N$ and $t = 1, \ldots, T$ by which the system of inequalities (3.1) - (3.3) is satisfied. Define $u_i : \mathbb{R}_+^L \to \mathbb{R}$ as

$$u_i(x) = \min\{u_{ij}^t + \langle \lambda_{ij}^t, x - x_j^t \rangle \mid j \in N,\ t = 1, \ldots, T.\}. \tag{3.5}$$

It is obvious that $u_i(\cdot)$ is continuous, concave and monotonic by its definition. From inequalities (3.1) and (3.2), we have that $u_i(x_i^t) = u_{ii}^t$ and $u_i(x_j^t) \le u_{ij}^t$. Then by (3.3) we have $u_i(x_i^t) \ge u_i(x_j^t)$ which shows that x^t is a no envy allocation. $\square$

Theorem 3.1 says that an allocation data is NE-rationalizable if and only if the system of inequalities (3.1)-(3.3) has a solution. There are two advantages to obtain such a characterization.

Firstly, it reduces the space we need to look for. Definition of NE-rationalizability basically requires us to seek all possible utility functions to check whether there is some function which rationalize the allocation data. This is obviously impossible task. According to Theorem 3.1, however, we only need to find finitely many numbers which solves the inequality system (3.1)-(3.3). For more extensive discussion on this view point, see Chambers and Echenique [5].

Secondly, it allows us to apply semialgebraic method of quantifier elimination. Tarski-Seidenburg theorem guarantees that for any finite polynomial inequalities, there exists an equivalent system which does not involve any quantifiers. In our context, this means that we can obtain an equivalent condition of NE-rationalization which only consists of observed allocations. See Carvajal et al. [4] for the detailed exposition of Tarski-Seidenburg theorem.

Before proceeding to the next result, we shall comment on the refutability of NE-rationalization. At this moment, there are three possible scenarios. The first is that the system (3.1)-(3.3) does not restrict anything, that is, for any allocation data, we can always find a solution to the system. In other words, NE-rationalizability is non-refutable. The second is the opposite polar of the first; for any allocation data, we never find a solution to the system. The third is that intermediate of the two; there are both allocation data which we can find a solution to the system and which we cannot. In the third case, we say that the testable implications of no envyness are non-vacuous.

Among the above three scenarios, it is easy to see the second is not true for NE-rationalization. Indeed, under monotone, continuous, and concave utility functions, any fair division model possesses no envy allocations. Picking such allocations finitely many times, we will end up with an allocation data which is NE-rationalizable. The following numerical example shows that there exists an allocation data which is not NE-rationalizable, indicating that no envyness has non-vacuous testable implications.

Example 3.1. Suppose that $N = \{1, 2\}$, $L = \{1, 2\}$. Consider the following finite allocation data:

$$x_1^1 = (1, 3), x_2^1 = (4, 2), x_1^2 = (3, 1), x_2^2 = (2, 4). \tag{3.6}$$

This data is not NE-rationalizable. To see why, suppose on the contrary, that this data is NE-rationalizable. Then by Theorem 3.1, there exists numbers and vectors by which (3.1)-(3.3) are satisfied. For $i = 1, j = 2$, the inequality (3.2) implies

$$u_{11}^1 - u_{12}^2 \leq \langle \lambda_{12}^2, x_1^1 - x_2^2 \rangle < 0, \text{ and } u_{11}^2 - u_{12}^1 \leq \langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle < 0. \tag{3.7}$$

Therefore we have $u_{11}^1 < u_{12}^2$ and $u_{11}^2 < u_{12}^1$. However this is incompatible with the inequality (3.3).

Remark 3.1. Example 3.1 shows that testable implications of no envy allocations are non-vacuous. This should be contrasted to Agatsuma [2], which shows that the core of market game non-refutable unless we can observe initial endowments or we impose more stringent assumption on utility functions such as smoothness.

Now we proceed to the second result. As we mentioned above, we shall obtain quantifier-free condition for NE-rationalizability. However, quantifier elimination is typically difficult to execute. We therefore follow Brown

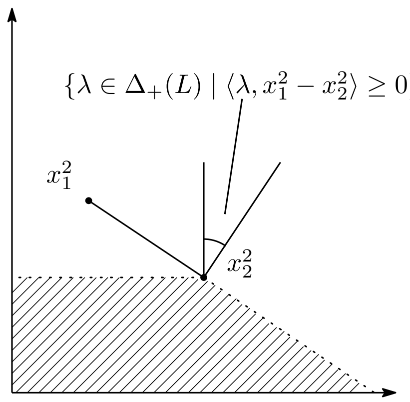

Figure 1: The area in which the maximized value becomes negative.

and Matzkin [3], Snyder [7, 8]; we restrict our attention to the simple case where allocation data consists only two observations.[1] The formal statement goes as follows.

Theorem 3.2. *Suppose that we have a finite allocation data $\{x^1, x^2\}$ which consists of two allocations. Then the data is NE-rationalizable if and only if the following conditions are satisfied:*

1. *For any $t = 1, 2$ and any pair of $i, j \in N$, we have $x_i^t \not\gg x_j^t$,*
2. *For any pair of $i, j \in N$,* $\max\{\langle \lambda, x_i^1 - x_j^2 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x_i^2 - x_j^2 \rangle \geq 0\} \leq (<) 0$ *implies* $\max\{\langle \lambda, x_i^2 - x_j^1 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x_i^1 - x_j^1 \rangle \geq 0\} \geq (>) 0$.

Proof. See Appendix A. □

Let us call the conditions stated in Theorem 3.2 as *weak axiom of revealed no envy (WARNE)*. It is easy to understand why the first condition in WARNE is involved for NE-rationalization: indeed if there are some t, i and j with $x_i^t \gg x_j^t$, monotonicity implies that agent j envies i.

To understand the role of the second condition of WARNE, consider the case when the presumption of the condition for $i = 1$ and $j = 2$ becomes negative, that is, when $\max\{\langle \lambda, x_1^1 - x_2^2 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x_1^2 - x_2^2 \rangle \geq 0\} < 0$. Suppose that an allocation x^2 is obtained as in the Figure 1. If x_1^1 is in the shaded area in the figure, the maximized value becomes negative. Indeed, when x_1^1 is in the shaded area, any vector λ which makes an acute angle with $x_1^2 - x_2^2$, makes an obtuse angle with $x_1^1 - x_2^2$.

In order to understand the consequence of the negativity of the maximized value, suppose that x_1^1 is obtained as in the Figure 2. It is clear from Figure 2, there exists a commodity bundle on the line which connects x_1^2 and x_2^2, say y, which dominates x_1^1. Since three points x_1^2, x_2^2 and y are on the same line there is some $\alpha \in [0, 1]$ such

[1] Brown and Matzkin [3] and Snyder [7, 8] actually restrict their cases to the number of agents are two as well as the number of observations are two. The difference comes from the relationship of inequalities involved in the inequality system for rationalization. In our system given in Theorem 3.1, we can separate the system (3.1)-(3.3) for each agent. In other words, the system consists of n independent subsystems. This enables us to execute quantifier-elimination technique with only reducing the number of observations. This changes when we require efficiency as well as no envyness to rationalize observed allocations. In the next section, we will see that the inequality system become interrelated across the index of agents as a result of efficiency. Consequently we will need to restrict the number of agents as well.

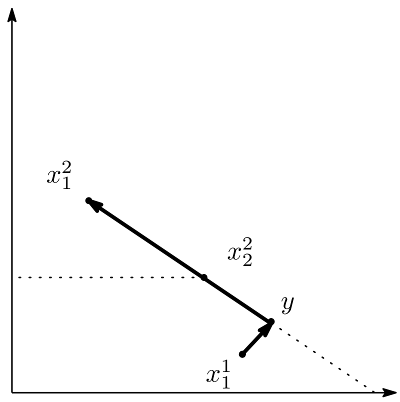

Figure 2: Consequence of negativity of the maximized value.

that $x_2^2 = \alpha x_1^2 + (1-\alpha)y$. By concavity of $u_1(\cdot)$ we have $u_1(x_2^2) \geq \alpha u_1(x_1^2) + (1-\alpha)u_1(y)$. If the allocation data is NE-rationalizable, we have $u_1(x_1^2) \geq u_1(x_2^2)$, and consequently, $u_1(x_1^2) \geq u_1(y)$. Since y is taken to dominate x_1^1, by monotonicity, we have $u_1(y) > u_1(x_1^2)$. Therefore, we must have $u_1(x_1^2) > u_1(x_1^1)$. In Figure 2, these preference relation is indicated by thick arrows.

We can, therefore, summarize the role of the second condition of WARNE as follows; under the assumption of NE-rationalizability, $\max\{\langle \lambda, x_1^1 - x_2^2 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x_1^2 - x_2^2 \rangle \geq 0\} < 0$ implies that agent 1 strictry prefers x_1^2 to x_1^1. Hence both $\max\{\langle \lambda, x_1^1 - x_2^2 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x_1^2 - x_2^2 \rangle \geq 0\} < 0$ and $\max\{\langle \lambda, x_1^2 - x_2^1 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x_1^1 - x_2^1 \rangle \geq 0\} < 0$ are incompatible with NE-rationalizability, since otherwise we have preference reversal ($u_1(x_1^2) > u_1(x_1^1)$ and $u_1(x_1^1) > u_1(x_1^2)$). The second condition of WARNE prevents such preference reversal happens.

4 Testable implications of no envy and efficient allocations

Let us now turn our investigation on testable implications of no envy and efficient allocations. Since compatibility of no envyness and efficiency has been a subject of many economic studies, it is natural to be curious about what observable restrictions are added when efficiency is required for rationalizations as well as no envyness.

Definition 4.1. A finite allocation data $\{x^t\}_{t=1}^T$ is no-envy-Pareto-rationalizable (NEP-rationalizable) if there exist a set of continuous, monotone, and concave utility functions $\{u_i(\cdot)\}$ such that x^t is a no envy and Pareto efficient allocation of $(N, \{u_i\}_{i\in N}, \omega^t)$ for each $t = 1, \ldots, T$, where $\omega^t = \sum_{i\in N} x_i^t$.

As we have done in the previous section, we first show an inequality type characterization of NEP-rationalizability. After that, we again employ the technique of quantifier-elimination to obtain a quantifier-free equivalent condition of NEP-rationalizability.

Theorem 4.1. *A finite allocation data $\{x^t\}_{t=1}^T$ is NEP-rationalizable by a set of monotone, continuous and concave utility functions $\{u_i(\cdot)\}$ if and only if there exist numbers*

- $u_{ij}^t \in \mathbb{R}$ *for* $t = 1, \ldots, T; i, j \in N$,
- $\lambda_{ij}^t \in \mathbb{R}_+^L \setminus \{0\}$ *for* $t = 1, \ldots, T; i, j \in N$ *with* $i \neq j$,

- $\lambda^t \in \mathbb{R}^L_+ \setminus \{0\}$ *for* $t = 1, \ldots, T$
- $\theta_i^t \in \mathbb{R}_{++}$ *for* $t = 1, \ldots, T; i, \in N$

such that

$$u_{ii}^s - u_{ii}^t \leq \frac{1}{\theta_i^t} \langle \lambda^t, x_i^s - x_i^t \rangle \text{ for } s, t = 1, \ldots, T; i \in N, \tag{4.1}$$

$$u_{ii}^s - u_{ij}^t \leq \langle \lambda_{ij}^t, x_i^s - x_j^t \rangle \text{ for } s, t = 1, \ldots, T; i, j \in N \text{ with } i \neq j, \tag{4.2}$$

$$u_{ii}^t \geq u_{ij}^t \text{ for } t = 1, \ldots, T; i, j \in N. \tag{4.3}$$

Proof. Suppose that a finite allocation data $\{x^t\}_{t=1}^T$ is NEP-rationalizable by a set of utility function $\{u_i(\cdot)\}$. As we have done in the proof of Theorem 3.1, set numbers $u_{ij}^t = u_i(x_j^t)$ for $i, j \in N$, and vectors $\lambda_{ij}^t \in \partial u_i(x_j^t)$ for $i, j \in N$ with $i \neq j$. We have shown that the definition and no envyness imply (4.2) and (4.3). Pareto efficiency of x^t implies that there exist some weights $\theta_i^t > 0$ for $i \in N$ such that

$$\sum_{i \in N} \theta_i^t u_i(x_i^t) = \max \sum_{i \in N} \theta_i^t u_i(x_i) \tag{4.4}$$

$$\text{s.t} \sum_{i \in N} x_i = \sum_{i \in N} x_i^t. \tag{4.5}$$

Then there exists λ^t such that $\frac{1}{\theta_i^t} \lambda^t \in \partial u_i(x_i^t)$. Consequently, we have

$$u_{ii}^s - u_{ii}^t \leq \frac{1}{\theta_i^t} \langle \lambda^t, x_i^s - x_i^t \rangle \text{ for } s, t = 1, \ldots, T; i \in N, \tag{4.6}$$

that is (4.1).

For the converse direction, suppose that we have found numbers $u_{ij}^t \in \mathbb{R}$, $\theta_i^t > 0$, and vectors $\lambda^t \in \mathbb{R}^L_+ \setminus \{0\}$, $\lambda_{ij}^t \in \mathbb{R}^L_+ \setminus \{0\}$ by which the system (4.1)-(4.3) is satisfied. Define $u_i : \mathbb{R}^L_+ \to \mathbb{R}$ as

$$u_i(x) = \min\{u_{ij}^t + \langle \lambda_{ij}^t, x - x_j^t \rangle \mid j \in N, \ t = 1, \ldots, T.\}. \tag{4.7}$$

where $\lambda_{ii}^t = \theta_i^t \lambda^t$. Theorem 3.1 shows that x^t is a no envy allocation of $(N, \{u_i(\cdot)\}_{i \in N}, \omega^t)$. To show Pareto efficiency of x^t, take $(x_i)_{i \in N}$ with $\sum_{i \in N} x_i \leq \sum_{i \in N} x_i^t$. Then, we have

$$\begin{aligned}
\sum_{i \in N} \theta_i^t u_i(x_i) &\leq \sum_{i \in N} \theta_i^t \left(u_i^t + \frac{1}{\theta_i^t} \langle \lambda^t, x_i - x_i^t \rangle \right) && (4.8)\\
&= \sum_{i \in N} \theta_i^t u_i^t + \sum_{i \in N} \langle \lambda^t, x_i - x_i^t \rangle && (4.9)\\
&\leq \sum_{i \in N} \theta_i^t u_i^t + \sum_{i \in N} \langle \lambda^t, x_i^t - x_i^t \rangle && (4.10)\\
&= \sum_{i \in N} \theta_i^t u_i^t = \sum_{i \in N} \theta_i^t u^t(x_i^t) && (4.11)
\end{aligned}$$

where the second inequality holds because $\sum_{i \in N} x_i \leq \sum_{i \in N} x_i^t$. The above inequality and the positivity of θ_i^t imply that it is impossible to have $u_i(x_i) > u_i(x_i^t)$ for all $i \in N$. □

By comparing Theorem 3.1 and Theorem 4.1, we see that (4.1) is changed as a result of requiring Pareto efficiency in addition to no envyness. Precisely, in order to rationalize as Pareto efficient allocation, we need to find λ_{ii}^t which orients the same direction for all $i \in N$, that is, it must be a scalar multiplication of common λ^t. In this way, the inequalities are interrelated across the index of agents. Consequently, we need to restrict the number of agents as well as the number of observations in order to apply quantifier-elimination.

Theorem 4.2. *Suppose that $N = \{1, 2\}$, and we have a finite allocation data $\{x^1, x^2\}$ which consists of two allocations. Then the data is NEP-rationalizable if and only if it satisfies WARNE.*

Proof. See Appendix B. □

Theorem 4.2 says that when the number of agents and the number of observations are two, NEP-rationalizability will end up with the same testable implications as NE-rationalizability, that is, WARNE. This means that Pareto efficiency puts no further observable restriction to no envyness. Put differently, no envy allocations and efficient and no envy allocations are indistinguishable from observed allocation data.

Corollary 4.1. *When the number of agents and the number of observations are two, no envyness and no envy efficiency are observationally equivalent.*

Appendix A: proof of Theorem 3.2

By Theorem 3.1, $\{x^1, x^2\}$ is NE-rationalizable if and only if there exist numbers $u^1_{ii}, u^1_{ij}, u^2_{ii}, u^2_{ij}$ and vectors $\lambda^1_{ii}, \lambda^1_{ij}, \lambda^2_{ii}, \lambda^2_{ij}$ for $i, j \in N$ such that

$$u^1_{ii} - u^2_{ii} \leq \langle \lambda^2_{ii}, x^1_i - x^2_i \rangle \tag{A1}$$
$$u^2_{ii} - u^1_{ii} \leq \langle \lambda^1_{ii}, x^2_i - x^1_i \rangle \tag{A2}$$
$$u^1_{ii} - u^2_{ij} \leq \langle \lambda^2_{ij}, x^1_i - x^2_j \rangle \tag{A3}$$
$$u^2_{ii} - u^1_{ij} \leq \langle \lambda^1_{ij}, x^2_i - x^1_j \rangle \tag{A4}$$
$$u^1_{ii} - u^1_{ij} \leq \langle \lambda^1_{ij}, x^1_i - x^1_j \rangle \tag{A5}$$
$$u^2_{ii} - u^2_{ij} \leq \langle \lambda^2_{ij}, x^2_i - x^2_j \rangle \tag{A6}$$
$$u^1_{ii} \geq u^1_{ij} \tag{A7}$$
$$u^2_{ii} \geq u^2_{ij} \tag{A8}$$

holds for $i, j \in N$ with $i \neq j$. This, in turn, is equivalent to the following: there exist numbers $u^1_{ii}, u^1_{ij}, u^2_{ii}, u^2_{ij}$ and vectors $\lambda^1_{ii}, \lambda^1_{ij}, \lambda^2_{ii}, \lambda^2_{ij}$ for $i, j \in N$ such that

$$u^1_{ii} - u^2_{ii} \leq \langle \lambda^2_{ii}, x^1_i - x^2_i \rangle \tag{A1}$$
$$u^2_{ii} - u^1_{ii} \leq \langle \lambda^1_{ii}, x^2_i - x^1_i \rangle \tag{A2}$$
$$u^1_{ii} - u^2_{ij} \leq \langle \lambda^2_{ij}, x^1_i - x^2_j \rangle \tag{A3}$$
$$u^2_{ii} - u^1_{ij} \leq \langle \lambda^1_{ij}, x^2_i - x^1_j \rangle \tag{A4}$$
$$0 \leq \langle \lambda^1_{ij}, x^1_i - x^1_j \rangle \tag{A5'}$$
$$0 \leq \langle \lambda^2_{ij}, x^2_i - x^2_j \rangle \tag{A6'}$$
$$u^1_{ii} \geq u^1_{ij} \tag{A7}$$
$$u^2_{ii} \geq u^2_{ij} \tag{A8}$$

holds for $i, j \in N$ with $i \neq j$. Indeed, it is obvious that the numbers and vectors which satisfy (A1)-(A8) also satisfies (A5') and (A6'). For the other direction, suppose numbers $u^1_{ii}, u^1_{ij}, u^2_{ii}, u^2_{ij}$ and vectors $\lambda^1_{ii}, \lambda^1_{ij}, \lambda^2_{ii}, \lambda^2_{ij}$

satisfy (A1)-(A4), (A5'), (A6'), (A7), and (A8). Assume that (A5) is violated, that is, $u^1_{ii} - u^1_{ij} > \langle \lambda^1_{ij}, x^1_i - x^1_j \rangle$. Take $A > 0$ such that $u^1_{ii} - u^1_{ij} - A = \langle \lambda^1_{ij}, x^1_i - x^1_j \rangle$. Set a number $\bar{u}^1_{ij} = u^1_{ij} + A$. Alteration of u^1_{ij} to $\bar{u}^1_{ij}$ only affects (A4), (A5), and (A7). By the definition, (A5) is satisfied. $u^2_{ii} - \bar{u}^1_{ij} < u^2_{ii} - u^1_{ij} \leq \langle \lambda^1_{ij}, x^2_i - x^1_j \rangle$ so (A4) is satisfied. Finally, $u^1_{ii} - \bar{u}^1_{ij} = \langle \lambda^1_{ij}, x^1_i - x^1_j \rangle \geq 0$ by (A5'). Therefore the numbers $u^1_{ii}, \bar{u}^1_{ij}, u^2_{ii}, u^2_{ij}$ and the vectors $\lambda^1_{ii}, \lambda^1_{ij}, \lambda^2_{ii}, \lambda^2_{ij}$ satisfy (A1)-(A5), (A7), and (A8). If these numbers and vectors violates (A6), similar alteration of u^2_{ij} leads us numbers and vectors which satisfy (A1)-(A8).

Next, we will show that the existence of numbers and vectors which satisfy (A1)-(A4), (A5'), (A6'), (A7), and (A8) is equivalent to the following: there exist numbers $u^1_{ii}, u^1_{ij}, u^2_{ii}, u^2_{ij}$ and vectors $\lambda^1_{ii}, \lambda^1_{ij}, \lambda^2_{ii}, \lambda^2_{ij}$ for $i, j \in N$ such that

$$\langle \lambda^2_{ii}, x^1_i - x^2_i \rangle \leq (<) 0 \implies u^1_{ii} - u^2_{ii} \leq (<) 0 \tag{A1'}$$

$$\langle \lambda^1_{ii}, x^2_i - x^1_i \rangle \leq (<) 0 \implies u^2_{ii} - u^1_{ii} \leq (<) 0 \tag{A2'}$$

$$\langle \lambda^2_{ij}, x^1_i - x^2_j \rangle \leq (<) 0 \implies u^1_{ii} - u^2_{ij} \leq (<) 0 \tag{A3'}$$

$$\langle \lambda^1_{ij}, x^2_i - x^1_j \rangle \leq (<) 0 \implies u^2_{ii} - u^1_{ij} \leq (<) 0 \tag{A4'}$$

$$0 \leq \langle \lambda^1_{ij}, x^1_i - x^1_j \rangle \tag{A5'}$$

$$0 \leq \langle \lambda^2_{ij}, x^2_i - x^2_j \rangle \tag{A6'}$$

$$u^1_{ii} \geq u^1_{ij} \tag{A7}$$

$$u^2_{ii} \geq u^2_{ij} \tag{A8}$$

holds for $i, j \in N$ with $i \neq j$. Again, it is obvious that numbers and vectors which satisfy (A1)-(A4), (A5), (A6'), (A7), and (A8) also satisfy (A1')-(A6') and (A7)-(A8). For the other direction, suppose that $\langle \lambda^2_{ii}, x^1_i - x^2_i \rangle > 0$. Then, by positive scalar multiplication on λ^2_{ii}, we can make the left hand side as large as we wish, in order to satisfy (A1). Notice that this alteration of λ^2_{ii} does not affect other inequalities. If $\langle \lambda^2_{ii}, x^1_i - x^2_i \rangle \leq 0$, we have $u^1_{ii} - u^2_{ii} \leq 0$ by (A1'). Then, again by multiplying positive scalar on λ^2_{ii} (for example, multiply $1/k$ for sufficiently large $k > 0$), we have (A1) valid. In either cases, (A1') implies (A1). By similar way we can show (A2'), (A3'), and (A4') implies (A2), (A3), and (A4) respectively.

Now we are ready to eliminate numbers u^t_{ij}. The previous system of inequalities is equivalent to the following: there exists vectors $\lambda^1_{ii}, \lambda^1_{ij}, \lambda^2_{ii}, \lambda^2_{ij}$ for $ij \in N$ such that

$$\langle \lambda^2_{ii}, x^1_i - x^2_i \rangle \leq (<) 0 \implies \langle \lambda^1_{ii}, x^2_i - x^1_i \rangle \geq (>) 0 \tag{A1"}$$

$$\langle \lambda^2_{ii}, x^1_i - x^2_i \rangle \leq (<) 0 \implies \langle \lambda^1_{ij}, x^2_i - x^1_j \rangle \geq (>) 0 \tag{A2"}$$

$$\langle \lambda^1_{ii}, x^2_i - x^1_i \rangle \leq (<) 0 \implies \langle \lambda^2_{ij}, x^1_i - x^2_j \rangle \geq (>) 0 \tag{A3"}$$

$$\langle \lambda^2_{ij}, x^1_i - x^2_j \rangle \leq (<) 0 \implies \langle \lambda^1_{ij}, x^2_i - x^1_j \rangle \geq (>) 0 \tag{A4"}$$

$$0 \leq \langle \lambda^1_{ij}, x^1_i - x^1_j \rangle \tag{A5'}$$

$$0 \leq \langle \lambda^2_{ij}, x^2_i - x^2_j \rangle \tag{A6'}$$

holds for $i, j \in N$ with $i \neq j$. Indeed, (A1') and (A2') imply (A1"), and (A1'), (A4'), and (A7) imply (A2"). (A3") and (A4") are similarly derived. For the other direction, suppose $\lambda^1_{ii}, \lambda^1_{ij}, \lambda^2_{ii}, \lambda^2_{ij}$ satisfy (A1")-(A4"), (A5') and (A6'). We divide three cases.

[Case 1: $\langle \lambda_{ii}^2, x_i^1 - x_i^2 \rangle < 0$.] In this case, we have $\langle \lambda_{ii}^1, x_i^2 - x_i^1 \rangle > 0$ and $\langle \lambda_{ij}^1, x_i^2 - x_j^1 \rangle > 0$ by (A1") and (A2"). Therefore (A2') and (A4') become redundant. If $\langle \lambda_{ij}^2, x_i^1 - x_j^2 \rangle > 0$, (A3') also becomes redundant, hence numbers $u_{ii}^1, u_{ij}^1, u_{ii}^2, u_{ij}^2$ which satisfy $u_{ij}^1 \le u_{ii}^1 < u_{ii}^2$ and $u_{ij}^2 \le u_{ii}^2$ satisfy (A1')-(A6'), (A7), and (A8). If $\langle \lambda_{ij}^2, x_i^1 - x_j^2 \rangle \le (<)0$, we need add the relation $u_{ii}^1 \le (<) u_{ij}^2$, so, for example, $u_{ij}^1 \le u_{ii}^1 < u_{ij}^2 \le u_{ii}^2$ satisfy (A1')-(A6'), (A7), and (A8).

[Case 2: $\langle \lambda_{ii}^2, x_i^1 - x_i^2 \rangle = 0$.] In this case, we have $\langle \lambda_{ii}^1, x_i^2 - x_i^1 \rangle \ge 0$ and $\langle \lambda_{ij}^1, x_i^2 - x_j^1 \rangle \ge 0$ by (A1") and (A2"). If the either of these (or both) hold by equality, we never have $\langle \lambda_{ij}^2, x_i^1 - x_j^2 \rangle < 0$ by (A3") and (A4"), so, for example, $u_{ii}^1 = u_{ij}^1 = u_{ii}^2 = u_{ij}^2$ satisfy (A1')-(A6'), (A7), and (A8). If $\langle \lambda_{ii}^1, x_i^2 - x_i^1 \rangle > 0$ and $\langle \lambda_{ij}^1, x_i^2 - x_j^1 >\ge 0$, we only need to care when $\langle \lambda_{ij}^2, x_i^1 - x_j^2 \rangle < 0$, in which case, it is sufficient to define numbers as $u_{ij}^1 \le u_{ii}^1 < u_{ij}^2 \le u_{ii}^2$.

[Case 3: $\langle \lambda_{ii}^2, x_i^1 - x_i^2 \rangle > 0$.] Notice that in this case, (A1') becomes redundant. We further divide two subcases.

(Case 3-1: $\langle \lambda_{ii}^1, x_i^2 - x_i^1 \rangle \le 0$) We have $\langle \lambda_{ij}^2, x_i^1 - x_i^2 \rangle \ge 0$ by (A3") If $\langle \lambda_{ij}^2, x_i^1 - x_i^2 \rangle > 0$, (A3') becomes redundant as well. Consequently, numbers which satisfy $u_{ij}^2 \le u_{ii}^2 < u_{ij}^1 \le u_{ii}^1$ satisfy (A1')-(A6'), (A7), and (A8) irrespective of the sign of $\langle \lambda_{ij}^1, x_i^2 - x_j^1 \rangle$. If $\langle \lambda_{ij}^2, x_i^1 - x_i^2 \rangle = 0$, $u_{ij}^2 = u_{ii}^2 = u_{ij}^1 = u_{ii}^1$ satisfy (A1')-(A6'), (A7), and (A8) (notice that $\langle \lambda_{ij}^1, x_i^2 - x_j^1 \rangle < 0$ never happen by (A4")).

(Case 3-2: $\langle \lambda_{ii}^1, x_i^2 - x_i^1 \rangle > 0$) in this case (A2') becomes innocent. If $\langle \lambda_{ij}^2, x_i^1 - x_i^2 \rangle > 0$, we need to do is to define numbers which satisfy (A4'), (A7), and (A8), which is easily accomplished. If $\langle \lambda_{ij}^2, x_i^1 - x_i^2 \rangle > 0$, numbers $u_{ij}^2 = u_{ii}^2 = u_{ij}^1 = u_{ii}^1$ satisfy (A1')-(A6'), (A7), and (A8) (notice that $\langle \lambda_{ij}^1, x_i^2 - x_j^1 \rangle < 0$ never happen by (A4")). Finally, if $\langle \lambda_{ij}^2, x_i^1 - x_i^2 \rangle < 0$, we have $\langle \lambda_{ij}^1, x_i^2 - x_j^1 \rangle > 0$ by (A4"), which makes (A4') innocent. Therefore we need to do is to define numbers which satisfy (A3'), (A7) and (A8), which is again easily accomplished.

Now we show that (A1")-(A4"), (A5'), and (A6') are equivalent to WARNE, that is

$$x_i^t \not\gg x_j^t, \text{ for } t = 1,2 \tag{W1}$$

$$\begin{aligned}&\max\{\langle \lambda, x_i^1 - x_j^2 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x_i^2 - x_j^2 \rangle \ge 0\} \le (<)0 \\ &\Longrightarrow \max\{\langle \lambda, x_i^2 - x_j^1 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x_i^1 - x_j^1 \rangle \ge 0\} \ge (>)0 \end{aligned}\tag{W2}$$

for $t = 1,2$ and $i, j \in N$. Suppose that there exist vectors λ_{ij}^t $(t = 1,2,\ i,j \in N)$ such that (A1")-(A4"), (A5'), and (A6') are true. It is obvious that (W1) holds. Note that by (A5') and (A6'), the constraint sets of the maximization problem in both sides of (W2) are non-empty. Assume that $\max\{\langle \lambda, x_i^1 - x_j^2 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x_i^2 - x_j^2 \rangle \ge 0\} \le (<)0$. Then we must have $\langle \lambda_{ij}^2, x_i^1 - x_j^2 \rangle \le (<)0$, which in turn $\langle \lambda_{ij}^1, x_i^2 - x_j^1 \rangle \ge (>)0$ by (A4"). By a scalar multiplication, we can think of $\lambda_{ij}^1 \in \Delta_+(L)$ without affecting this inequality. Therefore we have

$$\max\{\langle \lambda, x_i^2 - x_j^1 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x_i^1 - x_j^1 \rangle \ge 0\} \ge \langle \lambda_{ij}^1, x_i^2 - x_j^1 \rangle \ge (>)0. \tag{4.12}$$

For the other direction, suppose that WARNE is true. Note that by (W1), $\{\lambda \in \Delta_+(L) \mid \langle \lambda, x_i^1 - x_j^2 \rangle\} \ne \emptyset$ and $\{\lambda \in \Delta_+(L) \mid \langle \lambda, x_i^2 - x_j^1 \rangle\} \ne \emptyset$. It is obvious that these sets are compact, hence we can take $\lambda_{ij}^2 \in \operatorname{argmax}\{\langle \lambda, x_i^1 - x_j^2 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x_i^2 - x_j^2 \rangle \ge 0\}$ and $\lambda_{ij}^1 \in \operatorname{argmax}\{\langle \lambda, x_i^2 - x_j^1 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x_i^1 - x_j^1 \rangle \ge 0\}$. Notice that, by this specification, (A5') and (A6') are always satisfied. We divide two cases.

[Case 1: $\langle \lambda_{ij}^2, x_i^1 - x_j^2 \rangle > 0$.] In this case, (A3") and (A4") become innocent. If $\langle \lambda_{ij}^1, x_i^2 - x_j^1 \rangle > 0$, (A2") also becomes redundant. Then, by letting $\lambda_{ii}^1 = \lambda_{ii}^2$, (A1") holds. If $\langle \lambda_{ij}^1, x_i^2 - x_j^1 \rangle \le (<)0$, we have $\langle \lambda_{ij}^1 x_i^1 - x_i^2 \ge (>)0$. Indeed, [2]

$$0 \le (<) \langle \lambda_{ij}^1, x_i^1 - x_j^1 \rangle - \langle \lambda_{ij}^1, x_i^2 - x_j^1 \rangle = \langle \lambda_{ij}^1, x_i^1 - x_i^2. \tag{4.13}$$

[2] Recall that $\lambda_{ij}^1 \in \operatorname{argmax}\{\langle \lambda, x_i^2 - x_j^1 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x_i^1 - x_j^1 \rangle \ge 0\}$

Therefore by setting $\lambda^2_{ii} = \lambda^1_{ij}$, (A2") is satisfied, and set $\lambda^1_{ii} = \lambda^2_{ii}$, so (A1") holds.

[Case 2: $\langle \lambda^2_{ij}, x^1_i - x^2_j \rangle \leq (<)0$.] By (W2), we have $\langle \lambda^1_{ij}, x^2_i - x^1_j \rangle \geq (>)0$, so (A4") is satisfied. Set $\lambda^1_{ii} = \lambda^2_{ij}$, so (A3") is satisfied. Indeed, [3]

$$0 \leq (<)\langle \lambda^2_{ij}, x^2_i - x^2_j \rangle - \langle \lambda^2_{ij}, x^1_i - x^2_j \rangle = \langle \lambda^2_{ij}, x^2_i - x^1_i. \tag{4.14}$$

If $\langle \lambda^1_{ij}, x^2_i - x^1_j \rangle > 0$, setting $\lambda^2_{ii} = \lambda^1_{ii}$ guarantees (A1") (notice that in this case, (A2") becomes redundant). If $\langle \lambda^1_{ij}, x^2_i - x^1_j \rangle = 0$, we have that

$$0 \leq \langle \lambda^1_{ij}, x^1_i - x^1_j \rangle - \langle \lambda^1_{ij}, x^2_i - x^1_j \rangle = \langle \lambda^1_{ij}, x^1_i - x^2_j \rangle. \tag{4.15}$$

Therefore, by setting $\lambda^2_{ii} = \lambda^1_{ij}$, (A2") is satisfied. Notice that these specification of λ^1_{ii} and λ^2_{ii} guarantee (A1") since $\langle \lambda^1_{ii}, x^1_i - x^2_i \rangle \geq 0$ and $\langle \lambda^2_{ii}, x^2_i - x^1_i \rangle \geq 0$.

Appendix B: proof of Theorem 4.2

By the similar procedure employed to prove Theorem 3.2, we can easily show that solvability of the inequality system (4.1)-(4.3) is equivalent to the following; there exist vectors $\lambda^1, \lambda^2, \lambda^t_{ij}$ ($t = 1, 2$, $i, j \in 1, 2$ with $i \neq j$) such that

$$\langle \lambda^2, x^1_i - x^2_i \rangle \leq (<)0 \implies \langle \lambda^1, x^2_i - x^1_i \rangle \geq (>)0 \tag{B1"}$$

$$\langle \lambda^2, x^1_i - x^2_i \rangle \leq (<)0 \implies \langle \lambda^1_{ij}, x^2_i - x^1_j \rangle \geq (>)0 \tag{B2"}$$

$$\langle \lambda^1, x^2_i - x^1_i \rangle \leq (<)0 \implies \langle \lambda^2_{ij}, x^1_i - x^2_j \rangle \geq (>)0 \tag{B3"}$$

$$\langle \lambda^2_{ij}, x^1_i - x^2_j \rangle \leq (<)0 \implies \langle \lambda^1_{ij}, x^2_i - x^1_j \rangle \geq (>)0 \tag{A4"}$$

$$0 \leq \langle \lambda^1_{ij}, x^1_i - x^1_j \rangle \tag{A5'}$$

$$0 \leq \langle \lambda^2_{ij}, x^2_i - x^2_j \rangle \tag{A6'}$$

Now we show that (B1")-(B3"), (A4"), (A5'), and (A6') are equivalent to WARNE, that is

$$x^t_i \not\gg x^t_j, \text{ for } t = 1, 2 \tag{W1}$$

$$\max\{\langle \lambda, x^1_i - x^2_j \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x^2_i - x^2_j \rangle \geq 0\} \leq (<)0$$

$$\implies \max\{\langle \lambda, x^2_i - x^1_j \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x^1_i - x^1_j \rangle \geq 0\} \geq (>)0 \tag{W2}$$

for $t = 1, 2$ and $i, j = 1, 2$. It is obvious that (B1")-(B3"), (A4"), (A5'), and (A6') imply WARNE.[4]

For the other direction, suppose that WARNE is true. As in the proof of Theorem 3.2, take $\lambda^2_{ij} \in \text{argmax}\{\langle \lambda, x^1_i - x^2_j \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x^2_i - x^2_j \rangle \geq 0\}$ and $\lambda^1_{ij} \in \text{argmax}\{\langle \lambda, x^2_i - x^1_j \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x^1_i - x^1_j \rangle \geq 0\}$. Notice that, by this specification, (A5') and (A6') are always satisfied, and (W2) guarantees that (A4") holds. Therefore what remains to do is to specify λ^1 and λ^2 so that (B1")-(B3") are satisfied for both agent 1 and agent 2. We divide cases.

[3] Recall that $\lambda^2_{ij} \in \text{argmax}\{\langle \lambda, x^1_i - x^2_j \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x^2_i - x^2_j \rangle \geq 0\}$

[4] Notice that in the proof of Theorem 3.2, we have in fact used that (A4"), (A5'), and (A6') to show (W1) and (W2).

[Case 1:]

$$\langle \lambda^1_{12}, x^2_1 - x^1_2 \rangle > 0,\ \langle \lambda^2_{12}, x^1_1 - x^2_2 \rangle > 0$$
$$\langle \lambda^1_{21}, x^2_2 - x^1_1 \rangle > 0,\ \langle \lambda^2_{21}, x^1_2 - x^2_1 \rangle > 0$$

In this case, any λ^1 and λ^2 with $\lambda^1 = \lambda^2$ can be taken to satisfy (B1")-(B3") (notice that (B2") and (B3") are redundant).

[Case 2:]

$$\langle \lambda^1_{12}, x^2_1 - x^1_2 \rangle > 0,\ \langle \lambda^2_{12}, x^1_1 - x^2_2 \rangle > 0$$
$$\langle \lambda^1_{21}, x^2_2 - x^1_1 \rangle \leq 0,\ \langle \lambda^2_{21}, x^1_2 - x^2_1 \rangle > 0$$

Note that in this case, (B2") and (B3") for agent 1, and (B3") for agent 2 are redundant. Set $\lambda^2 = \lambda^1_{21}$, so we have

$$0 \leq \langle \lambda^2, x^1_1 - x^2_2 \rangle + \langle \lambda^1_{12}, x^1_2 - x^1_1 \rangle = \langle \lambda^2, x^1_2 - x^2_2 \rangle \tag{4.16}$$

with strict inequality when $\langle \lambda^1_{21}, x^2_2 - x^1_1 \rangle < 0$. Therefore (B2") for agent 2 is satisfied. Finally, set $\lambda^1 = \lambda^1$, so (B1") for both agent 1 and agent 2 hold.

[Case 3:]

$$\langle \lambda^1_{12}, x^2_1 - x^1_2 \rangle > 0,\ \langle \lambda^2_{12}, x^1_1 - x^2_2 \rangle > 0$$
$$\langle \lambda^1_{21}, x^2_2 - x^1_1 \rangle > 0,\ \langle \lambda^2_{21}, x^1_2 - x^2_1 \rangle \leq 0$$

This case is essentially the same as the second case, so we omit the detail.

[Case 4:]

$$\langle \lambda^1_{12}, x^2_1 - x^1_2 \rangle > 0,\ \langle \lambda^2_{12}, x^1_1 - x^2_2 \rangle > 0$$
$$\langle \lambda^1_{21}, x^2_2 - x^1_1 \rangle = 0,\ \langle \lambda^2_{21}, x^1_2 - x^2_1 \rangle = 0$$

In this case, neither $x^2_2 \ll x^1_2$ nor $x^2_2 \gg x^1_2$ happen. Indeed, suppose that $x^2_2 \ll x^1_2$. Then we have

$$\langle \lambda^2_{21}, x^2_2 - x^1_2 \rangle + \langle \lambda^2_{21}, x^1_2 - x^2_1 \rangle = \langle \lambda^2_{21}, x^2_2 - x^2_1 \rangle < 0. \tag{4.17}$$

This contradicts with $\langle \lambda^2_{21}, x^2_2 - x^1_2 \rangle \geq 0.$[5] $x^2_2 \gg x^1_2$ leads similar contradiction. Therefore we conclude that neither $x^2_2 \ll x^1_2$ nor $x^2_2 \gg x^1_2$ happen. Consequently, we can find $\lambda_\perp \in \Delta_+(L)$ such that $\langle \lambda_\perp, x^1_2 - x^2_2 \rangle = 0$. Set vectors $\lambda^1 = \lambda^2 = \lambda_\perp$, so (B1")-(B3") are satisfied.

[Case 5:]

$$\langle \lambda^1_{12}, x^2_1 - x^1_2 \rangle \leq 0,\ \langle \lambda^2_{12}, x^1_1 - x^2_2 \rangle > 0$$
$$\langle \lambda^1_{21}, x^2_2 - x^1_1 \rangle > 0,\ \langle \lambda^2_{21}, x^1_2 - x^2_1 \rangle \leq 0$$

Note that in this case, (B3") for agent 1 and (B2") for agent 2 are redundant. Note also that we have

$$0 \leq \langle \lambda^1_{12}, x^1_2 - x^2_1 \rangle + \langle \lambda^1_{12}, x^1_1 - x^1_2 \rangle = \langle \lambda^1_{12}, x^1_1 - x^2_1 \rangle \tag{4.18}$$

[5] Rcall that $\lambda^2_{21} \in \mathrm{argmax}\{\langle \lambda, x^1_2 - x^2_1 \rangle \mid \lambda \in \Delta_+(L),\ \langle \lambda, x^2_2 - x^2_1 \rangle \geq 0\}$.

with strict inequality when $\langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle < 0$, and

$$0 \leq \langle \lambda_{21}^2, x_1^2 - x_2^1 \rangle + \langle \lambda_{21}^2, x_2^2 - x_1^2 \rangle = \langle \lambda_{21}^2, x_2^2 - x_2^1 \rangle \tag{4.19}$$

with strict inequality when $\langle \lambda_{21}^2, x_2^1 - x_1^2 \rangle < 0$. We further divide three cases.

(Case 5-1: $\langle \lambda_{21}^2, x_1^2 - x_1^1 \rangle < 0$ and $\langle \lambda_{12}^1, x_1^1 - x_1^2 \rangle = 0$) We have

$$0 < \langle \lambda_{21}^2, x_1^1 - x_1^2 \rangle + \langle \lambda_{21}^2, x_1^2 - x_2^1 \rangle = \langle \lambda_{21}^2, x_1^1 - x_2^1 \rangle. \tag{4.20}$$

Since $\lambda_{12}^1 \in \operatorname{argmax}\{\langle \lambda, x_1^2 - x_2^1 \rangle \mid \lambda \in \Delta_+(L), \langle \lambda, x_1^1 - x_2^1 \rangle \geq 0\}$, (4.20) implies that

$$\langle \lambda_{21}^2, x_1^2 - x_2^1 \rangle \leq \langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle. \tag{4.21}$$

We also have that

$$0 = \langle \lambda_{12}^1, x_1^1 - x_1^2 \rangle = \langle \lambda_{12}^1, x_2^1 - x_1^2 \rangle + \langle \lambda_{12}^1, x_1^1 - x_2^1 \rangle, \tag{4.22}$$

and $\langle \lambda_{12}^1, x_2^1 - x_1^2 \rangle \geq 0$ and $\langle \lambda_{12}^1, x_1^1 - x_2^1 \rangle \geq 0$. Therefore $\langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle = 0$, which in turn, by (4.21), $\langle \lambda_{21}^2, x_1^2 - x_2^1 \rangle \leq 0$. Since we have assumed that $\langle \lambda_{21}^2, x_2^1 - x_1^2 \rangle \leq 0$, we have

$$\langle \lambda_{21}^2, x_2^1 - x_1^2 \rangle = 0. \tag{4.23}$$

Set vectors $\lambda^1 = \lambda^2 = \lambda_{21}^2$. Since $\lambda^1 = \lambda^2$, (B1") for both agent 1 and agent 2 are satisfied. By (4), (B3") for agent 2 is satisfied. Finally, by (4.20) and (4.23), we have $\langle \lambda^2, x_1^1 - x_1^2 \rangle > 0$, so (B2") for agent 1 is satisfied.

(Case 5-2: $\langle \lambda_{21}^2, x_2^2 - x_2^1 \rangle = 0$ and $\langle \lambda_{12}^1, x_2^1 - x_2^2 \rangle < 0$) We have

$$0 < \langle \lambda_{12}^1, x_2^2 - x_2^1 \rangle + \langle \lambda_{12}^1, x_2^1 - x_1^2 \rangle = \langle \lambda_{12}^1, x_2^2 - x_1^2 \rangle. \tag{4.24}$$

Since $\lambda_{21}^2 \in \operatorname{argmax}\{\langle \lambda, x_2^1 - x_1^2 \rangle \mid \lambda \in \Delta_+(L), \langle \lambda, x_2^2 - x_1^2 \rangle \geq 0\}$, (4.24) implies that

$$\langle \lambda_{12}^1, x_2^1 - x_1^2 \rangle \leq \langle \lambda_{21}^2, x_2^1 - x_1^2 \rangle. \tag{4.25}$$

We also have that

$$0 = \langle \lambda_{21}^2, x_2^2 - x_2^1 \rangle = \langle \lambda_{21}^2, x_1^2 - x_2^1 \rangle + \langle \lambda_{21}^2, x_2^2 - x_1^2 \rangle, \tag{4.26}$$

and $\langle \lambda_{21}^2, x_1^2 - x_2^1 \rangle \geq 0$ and $\langle \lambda_{21}^2, x_2^2 - x_1^2 \rangle \geq 0$. Therefore $\langle \lambda_{21}^2, x_2^1 - x_1^2 \rangle = 0$, which in turn, by (4.25), $\langle \lambda_{12}^1, x_2^1 - x_1^2 \rangle \leq 0$. Since we have assumed that $\langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle \leq 0$, we have

$$\langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle = 0. \tag{4.27}$$

Set vectors $\lambda^1 = \lambda^2 = \lambda_{12}^1$. Since $\lambda^1 = \lambda^2$, (B1") for both agent 1 and agent 2 are satisfied. By (4.18), (B2") for agent 1 is satisfied. Finally, by (4.24) and (4.27), we have $\langle \lambda^1, x_2^2 - x_2^1 \rangle > 0$, so (B3") for agent 2 is satisfied.

(Case 5-3: other) Set vectors $\lambda^1 = \lambda_{21}^2$ and $\lambda^2 = \lambda_{12}^1$. (4.18) and (4) guarantee that (B2") for agent 1 and (B3") for agent 2 hold. (B1") for both agent 1 and agent 2 are satisfied since otherwise, the case falls either 5-1 or 5-2.

[Case 6:]

$$\langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle \leq 0,\ \langle \lambda_{12}^2, x_1^1 - x_2^2 \rangle > 0$$
$$\langle \lambda_{21}^1, x_2^2 - x_1^1 \rangle \leq 0,\ \langle \lambda_{21}^2, x_2^1 - x_1^2 \rangle > 0$$

In this case, (B3") for both agent 1 and agent 2 are redundant. Set $\lambda^2 \in \Delta_+(L)$ such that $\langle \lambda^2, x_1^1 - x_2^1 \rangle = 0$. Note that this is possible by (W1). Since $\lambda_{12}^1 \in \operatorname{argmax}\{\langle \lambda, x_1^2 - x_2^1 \rangle \mid \lambda \in \Delta_+(L), \langle \lambda, x_1^1 - x_2^1 \rangle \geq 0\}$, we have $\langle \lambda^2, x_1^2 - x_2^1 \rangle \leq \langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle \leq 0$. Consequently,

$$0 \leq \langle \lambda^2, x_2^1 - x_1^2 \rangle + \langle \lambda^2, x_1^1 - x_2^1 \rangle = \langle \lambda^2, x_1^1 - x_1^2 \rangle, \tag{4.28}$$

with strict inequality when $\langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle < 0$. Similary, we have $\langle \lambda^2, x_2^2 - x_1^1 \rangle \leq \langle \lambda_{21}^1, x_2^2 - x_1^1 \rangle \leq 0$, since $\lambda_{21}^1 \in \operatorname{argmax}\{\langle \lambda, x_2^2 - x_1^1 \rangle \mid \lambda \in \Delta_+(L), \langle \lambda, x_2^1 - x_1^1 \rangle \geq 0\}$. Therefore,

$$0 \leq \langle \lambda^2, x_1^1 - x_2^2 \rangle + \langle \lambda^2, x_2^1 - x_1^1 \rangle = \langle \lambda^2, x_2^1 - x_2^2 \rangle, \tag{4.29}$$

with strict inequality when $\langle \lambda_{21}^1, x_2^2 - x_1^1 \rangle < 0$. From (4.28) and (4.29), we see that (B2") for both agent 1 and agent 2 are satisfied. Set $\lambda^1 = \lambda^2$, so (B1") for both agents are also satisfied.

[Case 7:]

$$\langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle \leq 0,\ \langle \lambda_{12}^2, x_1^1 - x_2^2 \rangle > 0$$
$$\langle \lambda_{21}^1, x_2^2 - x_1^1 \rangle = 0,\ \langle \lambda_{21}^2, x_2^1 - x_1^2 \rangle = 0$$

Take vectors $\lambda_\perp^1$ and $\lambda_\perp^2$ such that $\langle \lambda_\perp^1, x_1^1 - x_2^1 \rangle = 0$ and $\langle \lambda_\perp^2, x_2^2 - x_1^2 \rangle = 0$. We have that

$$\langle \lambda_\perp^1, x_1^2 - x_2^1 \rangle \leq \langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle \leq 0, \tag{4.30}$$
$$\langle \lambda_\perp^1, x_2^2 - x_1^1 \rangle \leq \langle \lambda_{21}^1, x_2^2 - x_1^1 \rangle = 0, \tag{4.31}$$
$$\langle \lambda_\perp^2, x_2^1 - x_1^2 \rangle \leq \langle \lambda_{21}^2, x_2^1 - x_1^2 \rangle = 0. \tag{4.32}$$

Consequently, we have that

$$\langle \lambda_\perp^1, x_1^1 - x_1^2 \rangle \geq 0 \tag{4.33}$$

with strict inequality when $\langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle < 0$, and

$$\langle \lambda_\perp^1, x_2^1 - x_2^2 \rangle \geq 0,\ \langle \lambda_\perp^2, x_2^2 - x_2^1 \rangle \geq 0. \tag{4.34}$$

Therefore if $\langle \lambda_\perp^2, x_1^2 - x_1^1 \rangle \geq 0$, setting $\lambda^1 = \lambda_\perp^2$ and $\lambda^2 = \lambda_\perp^1$ guarantee that (B1")-(B3") are satisfied. If $\langle \lambda_\perp^2, x_1^2 - x_1^1 \rangle < 0$ we have that

$$0 > \langle \lambda_\perp^2, x_1^2 - x_1^1 \rangle + \langle \lambda_\perp^2, x_2^1 - x_1^2 \rangle = \langle \lambda_\perp^2, x_2^1 - x_1^1 \rangle. \tag{4.35}$$

Therefore $\langle \lambda_\perp^2, x_1^2 - x_2^1 \rangle \leq \langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle \leq 0$. By combining this with (4.32), we obtain $\langle \lambda_\perp^2, x_1^2 - x_2^1 \rangle = 0$. Consequently, we have that

$$\langle \lambda_\perp^2, x_1^2 - x_2^1 \rangle + \langle \lambda_\perp^2, x_2^2 - x_1^2 \rangle = \langle \lambda_\perp^2, x_2^2 - x_2^1 \rangle = 0. \tag{4.36}$$

Set $\lambda^1 = \lambda^2 = \lambda_\perp^2$, so (B1") for both agent 1 and agent 2 are satisfied. Since we have assumed $\langle \lambda_\perp^2, x_1^2 - x_1^1 \rangle < 0$, (B2") for agent 1 is satisfied. Finally by (4.36), (B2") and (B3") for agent 2 are satisfied.

[Case 8:]

$$\langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle = 0,\ \langle \lambda_{12}^2, x_1^1 - x_2^2 \rangle = 0$$
$$\langle \lambda_{21}^1, x_2^2 - x_1^1 \rangle = 0,\ \langle \lambda_{21}^2, x_2^1 - x_1^2 \rangle = 0$$

Take $\lambda_\perp^1$ and $\lambda_\perp^2$ as in the Case 7. We have that

$$\langle \lambda_\perp^1, x_1^2 - x_2^1 \rangle \leq \langle \lambda_{12}^1, x_1^2 - x_2^1 \rangle = 0, \tag{4.37}$$

$$\langle \lambda_\perp^1, x_2^2 - x_1^1 \rangle \leq \langle \lambda_{21}^1, x_2^2 - x_1^1 \rangle = 0, \tag{4.38}$$

$$\langle \lambda_\perp^2, x_1^1 - x_2^2 \rangle \leq \langle \lambda_{12}^2, x_1^1 - x_2^2 \rangle = 0, \tag{4.39}$$

$$\langle \lambda_\perp^2, x_2^1 - x_1^2 \rangle \leq \langle \lambda_{21}^2, x_2^1 - x_1^2 \rangle = 0. \tag{4.40}$$

Combining these relations with the assumption that $\langle \lambda_\perp^1, x_1^1 - x_2^1 \rangle = 0$ and $\langle \lambda_\perp^2, x_2^2 - x_1^2 \rangle = 0$, we obtain

$$\langle \lambda_\perp^1, x_1^1 - x_1^2 \rangle \geq 0, \ \langle \lambda_\perp^1, x_2^1 - x_2^2 \rangle \geq 0, \tag{4.41}$$

$$\langle \lambda_\perp^2, x_1^2 - x_1^1 \rangle \geq 0, \ \langle \lambda_\perp^2, x_2^2 - x_2^1 \rangle \geq 0. \tag{4.42}$$

Therefore setting vectors $\lambda^1 = \lambda_\perp^2$ and $\lambda_\perp^2$ guarantees that (B1")-(B3") are satisfied.

References

[1] S. Afriat. The construction of utility functions from expenditure data. *International Economic Review*, 8:67–77, 1967.

[2] Y. Agatsuma. Testable implications of the core in TU market games. *Journal of Mathematical Economics*, 64:23–29, 2016.

[3] D. J. Brown and R. Matzkin. Testable restrictions on the equilibrium manifold. *Econometrica*, 64:1249–1262, 1996.

[4] A. Carvajal, I. Ray, and S. Snyder. Equilibrium behavior in markets and games: testable restrictions and identification. *Journal of Mathematical Economics*, 40:1–40, 2004.

[5] C. P. Chambers and F. Echenique. *Revealed Preference Theory.* Cambridge University Press, 2016.

[6] D. K. Foley. Resource allocation in the public sector. *Yale Economic Essays*, 7:73–76, 1964.

[7] S. K. Snyder. Testable restrictions of Pareto optimal public good provision. *Journal of Public Economics*, 71:97–119, 1999.

[8] S. K. Snyder. Nonparametric testable restrictions of household behavior. *Southern Economic Journal*, 67:171–185, 2000.

[9] L.-G. Svensson. Large indivisibles: An analysis with respect to price equilibrium and fairness. *Econometrica*, 51:939–954, 1983.

[10] H. Varian. Equity, envy, and efficiency. *Journal of Economic Theory*, 9:63–91, 1974.