Higher-order Graph Cuts

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Labeling problem

- Assign a label to each pixel
- Pixel $v \in V \rightarrow$ Label $X_v \in L$
Energy minimization

$E(X) = \sum_{v} \lambda |Y_v - X_v| + \sum_{(u,v)} \kappa |X_u - X_v|$

• Globally optimizeable using graph cuts

Greig, Porteous and Seheult ’89
First-order (pairwise) energy

Good (Low Energy)  Bad (High Energy)

12 Bad
40 Good

12 Bad
40 Good
Higher-order energy

Good (Low Energy) vs. Bad (High Energy):

- Better (Lower Energy) vs. Worse (Higher Energy):
  - A
  - B
  - C
  - D

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Higher-order energy

Better (Lower Energy)  Worse (Higher Energy)

A: 15
B: 12
C: 1
D: 0

A: 16
B: 6
C: 5
D: 1

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Higher-order energy

\[ E(X) = \sum_{c \in C} f_c(X_c) \]

\[ = \sum_{v \in V} g_v(X_v) + \sum_{(u,v) \in E} h_{uv}(X_u, X_v) \]

\[ + \sum_{(u,v,s,t)} k_{uvst}(X_u, X_v, X_s, X_t) \]

Better (Lower Energy)  ↔  Worse (Higher Energy)
Functions of binary variables

• Pseudo-Boolean function (PBF)
  • Function of binary (0 or 1) variables
  • Can always write it uniquely as a polynomial

• One variable $x$: $E_0(1-x) + E_1x$

• Two variables $x, y$:
  $E_{00}(1-x)(1\bar{y}) + E_{01}(1-x)y + E_{10}x(1\bar{y}) + E_{11}xy$

• Three variables $x, y, z$:
  $E_{000}(1-x)(1-y)(1-z) + E_{001}(1-x)(1-y)z + \cdots + E_{111}xyz$

• $n^{th}$ order binary MRF = $(n + 1)^{th}$ degree PBF
Reducing higher-order energy

- Convert any higher-order binary energy

\[ E(X) = E(X_1, ..., X_n) = \sum_{c \in C} f_c(X_c) \]

into an equivalent first-order energy

\[ \tilde{E}(X) = \tilde{E}(X_1, ..., X_n, ..., X_m) = \sum g_v(X_v) + \sum h_{uv}(X_u, X_v) \]

- Adds variables

- More than 2 labels → Fusion moves

Ishikawa CVPR 2009, PAMI 2011

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New!

- Convert many higher-order binary energies

\[ E(X) = E(X_1, ..., X_n) = \sum_{c \in \mathcal{C}} f_c(X_c) \]

into an equivalent first-order energy

\[ \tilde{E}(X) = \tilde{E}(X_1, ..., X_n) = \sum g_v(X_v) + \sum h_{uv}(X_u, X_v) \]

- Without adding variables

Example: a cubic term

- $\varphi(x, y, z):$ cubic (3rd degree)

$$
\varphi(0,0,0) = a, \; \varphi(1,0,0) = b, \; \varphi(0,1,0) = c, \; \varphi(1,1,0) = d,
\varphi(0,0,1) = e, \; \varphi(1,0,1) = f, \; \varphi(0,1,1) = g, \; \varphi(1,1,1) = h
$$

- $\varphi(x, y, z) = a(1-x)(1-y)(1-z) + bx(1-y)(1-z)
  + c(1-x)y(1-z) + dxy(1-z) + e(1-x)(1-y)z
  + fx(1-y)z + g(1-x)yz + hxyz$

- $xyz$ coefficient: $s = -a + b + c - d + e - f - g + h$

- Define new function:

$$
\varphi'(x, y, z) = \begin{cases} 
\varphi(x, y, z) & \text{when } (x, y, z) \neq (0,0,0) \\
\varphi(0,0,0) + s & \text{when } (x, y, z) = (0,0,0) 
\end{cases}
$$

i.e., value is added $s$ only when $(x, y, z) = (0,0,0)$
Example: a cubic term

• Define new function:

\[ \varphi'(x, y, z) = \begin{cases} 
\varphi(x, y, z) & \text{when } (x, y, z) \neq (0,0,0) \\
\varphi(0,0,0) + s & \text{when } (x, y, z) = (0,0,0) 
\end{cases} \]

i.e., value is added \( s \) only when \( (x, y, z) = (0,0,0) \)

• New \( xyz \) coefficient (replace \( a \) with \( a + s \)):

\[ s' = -(a + s) + b + c - d + e - f - g + h = s - s = 0 \]

• So \( \varphi' \) is now quadratic (2\(^{nd}\) degree)

• Similarly, we can reduce the degree by changing one of the 8 possible values

• But \( \varphi \) and \( \varphi' \) are different functions!
When can we do this?

\[ \varphi'(0,0,0) = a + s, \ \varphi'(1,0,0) = b, \ \varphi'(0,1,0) = c, \ \varphi'(1,1,0) = d, \]
\[ \varphi'(0,0,1) = e, \ \varphi'(1,0,1) = f, \ \varphi'(0,1,1) = g, \ \varphi'(1,1,1) = h \]

- Different only when \((x, y, z) = (0,0,0)\)
- Suppose \(\varphi\) is a potential in \(E(X) = \sum f_c(X_c)\)
  - \(x, y, z\) are three of the variables in \(X\)
- If
  - \(s > 0\), and
  - For minimizer \(X\) of \(E(X)\), \((x, y, z) \neq (0,0,0)\),

then, we can replace \(\varphi\) with \(\varphi'\) without changing the minimizer.
**ELC**

- **Excludable Local Configuration (ELC)**
  - A (usually) locally-testable sufficient condition for local configuration \((x, y, z)\) to be not part of global minimizer
  - “Excludable as a part of global minimizer”

- **ELC may not exist**

- **May take time to find**

- **Approximation**
  - Just use the local configuration \((x, y, z)\) with the largest value \(\varphi(x, y, z)\)
Experiment: 4\textsuperscript{th} deg. FoE denoising

Original

Noise added

4\textsuperscript{th} degree
Experiment: 4\textsuperscript{th} deg. FoE denoising

Number of variables after conversion

- ELC+HOCHR
- ELC+Fix et. al
- Approx. ELC
- HOCHR
- Fix et. al

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Experiment: 4th deg. FoE denoising

Energy

Iterations

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Energy

Time (sec.)

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% pixels labeled

ELC Approx.

ELC+HOCR

Fix et al.

HOCR

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Experiment: 4\textsuperscript{th} deg. FoE denoising

- **ELC exists:**
  - 3\textsuperscript{rd} degree: 96.12%
  - 4\textsuperscript{th} degree: 99.60%

- **Approximation:**
  - Guessed configuration is in fact an ELC
    - 3\textsuperscript{rd} degree: 88% of the time
    - 4\textsuperscript{th} degree: 97% of the time
  - Even if it is not an ELC, it is not part of maximizer
    - 3\textsuperscript{rd} degree: 99.98%
    - 4\textsuperscript{th} degree: 99.997%
  - 99.99988% of (labeled) pixels correctly labeled
Conclusion

• Higher-order energy minimization
  • Unary, Pairwise, Triple, ....
  • Binary labels: reduce to first order
    • Then use graph cuts
  • Multiple labels: use Fusion Move

• Reducing to first order
  • Before: add variables
  • New: no variables added
    • Faster
    • Much less memory

• Code available:
  http://www.f.waseda.jp/hfs/software.html