Existence and uniqueness of viscosity solutions of an integro-differential equation arising in option pricing*

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Abstract. We prove the existence and uniqueness of the viscosity solution of an integro-differential equation (IDE) arising in the pricing of American-style multi-asset options in a multivariate Ornstein-Uhlenbeck-type stochastic volatility model. We prove an extended version of the maximum principle of Crandall and Ishii [Differential Integral Equations **3** (1990)], and use it to prove the comparison theorem.

Key words. Viscosity solutions, option pricing, multivariate stochastic volatility model with jumps, maximum principle

AMS subject classifications. 35B50, 35D40, 35Q91, 49L25, 91G20

1. Introduction.

Option prices can often be characterized as the solutions of an associated partial differential equation (PDE). Black and Scholes [9] model the stock price as a geometric Brownian motion and relate the price of a European-style option to a parabolic PDE with constant coefficients. One common extension of this model that has been proposed in the literature is to make the volatility of the stock price a stochastic process. In the continuous case, one can consider a stochastic volatility of the form:

$$\frac{dS_t}{S_t} = \sqrt{Y_t}dB_t^1$$
$$dY_t = \beta(Y_t)dt + \eta(Y_t)dB_t^2,$$

where S is the stock price, and B^1 and B^2 are two correlated Brownian motions. For instance, Heston [18] proposes to take η as the square-root function, whereas Hull and White [19] take η of the form $\eta(y) = \xi y$, $\xi > 0$. The (discounted) option price u can then be shown to be a solution to a PDE of the form

$$-\frac{\partial u}{\partial t} + \frac{1}{2}y\frac{\partial u}{\partial x} - \beta(y)\frac{\partial u}{\partial y} - \frac{1}{2}y\frac{\partial^2 u}{\partial x^2} - \rho\sqrt{y}\eta(y)\frac{\partial^2 u}{\partial x\partial y} - \frac{1}{2}\eta(y)^2\frac{\partial^2 u}{\partial y^2} = 0,$$

with ρ the correlation coefficient between the two Brownian motions, and x the log of S. The existence and uniqueness of solutions to this PDE do not follow from classical theory which typically assumes stronger regularity of the coefficients. Ekström, and Tysk [14] give weaker conditions on the coefficients of the associated stochastic differential equations and boundary conditions of the PDE that insure that the option price is the unique solution of the associated

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PDE. Heath and Schweizer [17] provide other sufficient conditions to a more general class of PDEs used in financial modelling that go beyond standard PDE results.

In their seminal paper, Barndorff-Nielsen and Shephard [5] introduced continuous-time non-Gaussian Ornstein-Uhlenbeck-type processes to model stochastic volatility with jumps. The model is now widely used in financial mathematics due to its ability to capture stylized features of financial time series such as heavy-tailed distribution, long-range dependence and negative correlation between volatility and asset prices. In this paper, we study the existence and uniqueness of viscosity solutions of an integro-differential equation arising in the pricing of options in a multivariate version of this model.

The stochastic volatility model of Barndorff-Nielsen and Shephard [5] has been extensively studied in the literature. Benth et al. [8] solve a classical portfolio optimization problem in this setting with the use of the Hamilton-Jacobi-Bellman differential equation associated to this control problem. Nicolato and Vernados [26] obtain probabilistic representations of European-type option prices with structure preserving martingale measures. Benth et al. [7] use these non-Gaussian Ornstein-Uhlenbeck-type processes to model electricity prices and provide option pricing formulas based on Fourier transforms. Pigorsch and Stelzer [29], [30] provide a multivariate extension of the non-Gaussian Ornstein-Uhlenbeck-type volatility processes of [5]. Muhle-Karbe et al. [25] use Fourier methods to compute prices of multi-asset options in this multivariate extension with leverage. We revisit this option pricing problem for the case of American-style multi-asset options in this multivariate stochastic volatility setting from the perspective of viscosity solutions of integro-differential equations. Viscosity solutions have been used extensively in the mathematical finance literature and allow the use of numerical methods to compute prices and solutions to control problems in many financial models. In the option pricing case, notable early uses of viscosity solutions include the nonlinear Black-Scholes equation of Barles and Soner [3], and the utility indifference equations of Davis et al. [13] that both arise in markets with transaction costs. Cont and Voltchkova [10] provide a rigorous treatment of the existence and uniqueness of viscosity solutions of the option pricing integro-differential equations in exponential Lévy models.

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ is defined a d-dimensional Brownian motion B and an $\mathbb{S}_d^+$-valued Lévy process Z, independent of B, with Lévy measure ν taking values in $\mathbb{S}_d^+ \setminus \{0\}$. The Lévy process Z satisfies $Z_t - Z_s \in \mathbb{S}_d^+$, for all $0 \leq s < t$. It is commonly referred to as a matrix subordinator and satisfies $\int_{\mathbb{S}_d^+} (\|z\| \wedge 1)\nu(dz) < \infty$ (cf. [4, Proposition 3.1]). The multivariate non-Gaussian Ornstein-Uhlenbeck-type volatility model ([6], [29], [30]) is defined as follows. Consider d risky assets for which the discounted prices are given under a structure preserving risk-neutral measure (cf. [26]) by the following stochastic processes:

$$
\begin{aligned}
\frac{dS_t^i}{S_t^i} &= \sum_{j=1}^{d} r_{ij}(Y_t) dB_t^j \\
dY_t &= (AY_t + Y_t A^*)dt + dZ_t, \qquad (1.1)
\end{aligned}
$$

in which $r : \mathbb{S}_d^+ \to M_d(\mathbb{R})$ satisfies $r(y)r(y)^* = y$ for all $y \in \mathbb{S}_d^+$. Here, $M_d(\mathbb{R})$ denotes the set of $d \times d$ real matrices and $\mathbb{S}_d^+$ is the set of positive semi-definite $d \times d$ matrices. Different factorizations r are possible, but [6, Proposition 2.2] shows that it does not affect the distribution of (S, Y). We take $r(y)$ as the square-root of y, i.e. the unique symmetric positive semi-definite matrix that satisfies $r(y)^2 = y$. We refer to Y as the variance process. It satisfies $Y_t \in \mathbb{S}_d^+$ for all $t \geq 0$.

We further assume that for some constant $\lambda > 1$, the measure ν satisfies

$$\int_{\mathbb{S}_d^+} (\|z\| + \|z\|^\lambda)\nu(dz) < \infty. \tag{1.2}$$

As noted above, the integrability of the function $\|z\|$ is not really an assumption since the process Z is a subordinator. According to [4, Lemma 3.1], the process Z has the following representation in terms of its associated Poisson random measure N:

$$dZ_t = b_0 dt + \int_{\mathbb{S}_d^+} zN(dt, dz)$$

in which $b_0 \in \mathbb{S}_d^+$. Due to (1.2), $dZ_t = \Big(b_0 + \int_{\mathbb{S}_d^+} z\nu(dz)\Big)dt + \int_{\mathbb{S}_d^+} z\tilde{N}(dt, dz)$ with $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$, the compensated jump martingale measure of N. Consider the change of variable $X_t^i = \log(S_t^i)$. Then,

$$dX_t^i = \sum_{j=1}^d r_{ij}(Y_t)dB_t^j - \tfrac{1}{2}Y_t^{ii}dt \quad (i \leq d).$$

In the vectorial notation, this can be written as

$$dX_t = r(Y_t)dB_t - \tfrac{1}{2}\pi(Y_t)dt, \tag{1.3}$$

where $\pi(Y)$ denotes the d-dimensional vector $(Y_t^{11}, \dots, Y_t^{dd})$.

We consider a general American-style derivative product on multiple assets with payoff function $h : \mathbb{R}^d \to \mathbb{R}$ and maturity T. For example, the payoff of an index put option is of the form

$$h(x) = \max\{K - \sum_i w_i \exp(x_i), 0\},$$

for some $K, w_i > 0$, $i \leq d$. In probabilistic terms, for each initial state (x, y), the price of the option is given by the following stopping time problem:

$$\sup_{\tau \in \mathcal{T}_T} \mathbf{E} h(X_\tau^{x,y}, Y_\tau^y, \tau). \tag{1.4}$$

In the above expression, $\mathcal{T}_T$ is the set of stopping times τ with value less or equal to T, Y^y is the process given by (1.1) with $Y_0^y = y$ and $X^{x,y}$ is the process defined by (1.3) with $X_0^{x,y} = x$ and h is a general payoff function on $\mathbb{R}^d \times \mathbb{S}_d^+ \times [0, T]$.

Let $Q_T = \mathbb{R}^d \times \mathbb{S}_d^+ \times [0, T)$, and $\overline{Q}_T = \mathbb{R}^d \times \mathbb{S}_d^+ \times [0, T]$, the closure of Q_T. Until the end of section 6, we are mostly concerned with the case where h is a bounded and Lipschitz continuous function on $\overline{Q}_T$ and, in the last section, we generalize our results to the case where h is a continuous function on $\overline{Q}_T$ having a polynomial growth[1].

[1]Although, the polynomial growth assumption on h excludes the case of call options, it is well known that for non-dividend paying assets, American call option prices are equal to their European counterpart. For practical purposes, we can therefore approximate u_0 arbitrarily well by a sequence of functions which are unique solutions of the IDE continuous function on $\overline{Q}_T$.

To investigate this problem, we introduce the value function u_0 on $\overline{Q}_T$:

$$u_0(x,y,t) = \sup_{\tau \in \mathcal{T}_{T-t}} \mathbf{E} h(X^{x,y}_\tau, Y^y_\tau, t+\tau) \quad \text{for } (x,y,t) \in \overline{Q}_{T-t}, \tag{1.5}$$

where $\mathcal{T}_{T-t}$ is the set of all stopping times τ such that $0 \le \tau \le T-t$.

Our goal is to show that the following integro-differential equation has a unique viscosity solution given by u_0, the price of the option of (1.5):

$$\min\{\mathcal{M}u, u-h\} = 0 \quad \text{on } Q_T \tag{1.6}$$

with terminal condition $u(x,y,T) = h(x,y,T)$ for $(x,y) \in \mathbb{R}^d \times \mathbb{S}^+_d$. In the above equation,

$$\mathcal{M}\phi := -\frac{\partial}{\partial t}\phi - L\phi - J\phi,$$

and the operators L and J are given by

$$L\phi(x,y,t) = \tfrac{1}{2}\langle y, D^2_x\phi(x,y,t)\rangle - \tfrac{1}{2}\langle \pi(y), D_x\phi(x,y,t)\rangle + \langle Ay + yA^* + b_0, D_y\phi(x,y,t)\rangle,$$
$$J\phi(x,y,t) = \int_{\mathbb{S}^+_d} (\phi(x,y+z,t) - \phi(x,y,t))\,\nu(dz).$$

In terms of PDE theory, (1.6) is a kind of obstacle problem with obstacle h.

We are concerned with viscosity solutions of (1.6) on Q_T having at most a polynomial growth of order $\kappa \ge 0$, that is, functions $f : Q_T \to \mathbb{R}$ satisfying

$$\sup_{(x,y,t)\in Q_T} \frac{|f(x,y,t)|}{(1+|x|+\|y\|)^\kappa} < \infty.$$

The space of such functions f is denoted by $\mathcal{V}_\kappa$. With a slight abuse of notation, we sometimes write $f \in \mathcal{V}_\kappa$ for $f : \overline{Q}_T \to \mathbb{R}$ if its restriction to Q_T is in $\mathcal{V}_\kappa$.

The main mathematical difficulty is the comparison principle. In the univariate case, Roch [31] showed the uniqueness of the solution under the additional assumption that $u(x,0,t) = h(x)$. However, this is generally a restrictive condition, and it is not satisfied in most financial applications. Pham [28] obtains a comparison principle for a related integro-differential equation of a stochastic control problem in which the second-order coefficient is of the form $\sigma(x,t;\alpha)\sigma(x,t;\alpha)^*$ with σ globally Lipschitz in x. The lack of this Lipschitz condition, as a function of (x,y), in the present case makes the problem more challenging mathematically. In particular, to give a rigorous proof of the comparison principle we present a straightforward extension of the maximum principle for semicontinuous functions of Crandall and Ishii [11]. There have already been substantial contributions (see [22, 2] among others) to the maximum principle for semicontinuous functions, which can be applied to integro-differential equations.

As mentioned above, the main difficulty in the proof of the comparison theorem comes from the lack of the standard Lipschitz condition on the coefficient y of the second-order term in the PDE (the first term of operator L). Precisely, the coefficient y is factorized as $y = (\sqrt{y})^2$ and $\sqrt{y}$ is not Lipschitz continuous on $\mathbb{S}^+_d$. To deal with this difficulty, we make a clear distinction of the two variables x and y and take advantage of the form $\langle y, D_x^2 u\rangle$ of the second-order term, where the coefficient y does not depend on x.

In section 2 and section 3, we prove that u_0 is a continuous viscosity solution of (1.6). In section 4, we present a new version of the maximum principle, and in section 5 the solution is shown to be unique by proving the comparison principle for viscosity solutions of (1.6).

Throughout the paper, we adopt the following notation. For $A, B \in M_d(\mathbb{R})$, A^* denotes the transpose of A, $\mathrm{Tr}(A)$ is the trace of A, $\langle A, B\rangle = \mathrm{Tr}(AB^*)$ is the inner product, $\|A\| = \sqrt{\langle A, A\rangle}$ is the associated norm and $|A| = \max_{\xi\in\mathbb{R}^d, |\xi|=1}\langle A\xi, \xi\rangle$ is the operator norm. A remark is that if $A \in \mathbb{S}_d$ and μ_i, $i = 1, \dots, d$, are the eigenvalues of A, then $\|A\| = \sqrt{\sum_i \mu_i^2}$ and $|A| = \max_i |\mu_i|$. Hence, $|A| \le \|A\| \le \sqrt{d}|A|$ for $A \in \mathbb{S}^d$. For vectors $x, y \in \mathbb{R}^d$, $\langle x, y\rangle = x^* y$. $\mathrm{USC}(U)$ and $\mathrm{LSC}(U)$ denote the sets of upper semicontinuous and lower semicontinuous functions on a set U.

2. Continuity of the solution.

We begin with the continuity of the function u_0. For this, we need the following lemma.

Lemma 2.1. *Let $(x, y) \in \mathbb{R}^d \times \mathbb{S}_d^+$ and $\tau, \tau' \in \mathcal{T}_T$ such that $\tau \le \tau' \le \tau + \epsilon$ for some constant $\epsilon > 0$. There exists a constant $C > 0$, independent of $x, y, \tau, \tau', \epsilon$, such that*

$$\begin{aligned}
\mathbf{E} \sup_{0\le s\le T} \|Y_s^y\|^\lambda &\le C(1 + \|y\|)^\lambda, \\
\mathbf{E} \sup_{0\le s\le T} |X_s^{x,y}|^\lambda &\le C(1 + |x| + \|y\|)^\lambda, \\
\mathbf{E} \sup_{\tau\le s\le\tau'} \|Y_s^y - Y_\tau^y\|^\lambda &\le C(1 + \|y\|)^\lambda \epsilon, \\
\mathbf{E} \sup_{\tau\le s\le\tau'} |X_s^{x,y} - X_\tau^{x,y}| &\le C(1 + \|y\|)\sqrt{\epsilon}.
\end{aligned}$$

Proof. In this proof, C is a positive constant that changes from line to line, but only depends on T, λ, ν, $\|A\|$, b_0 and d.

Since $Z_T - Z_s \in \mathbb{S}_d^+$, we deduce that $|Z_T| \ge |Z_s|$ and hence $\sqrt{d}\|Z_T\| \ge \|Z_s\|$ for all $0 \le s \le T$. Indeed, we have for any unit vector $\xi \in \mathbb{R}^d$,

$$\langle Z_T\xi, \xi\rangle = \langle Z_s\xi, \xi\rangle + \langle (Z_T - Z_s)\xi, \xi\rangle \ge \langle Z_s\xi, \xi\rangle,$$

which yields the above inequality. By [32, Theorem 25.3], we have $\mathbf{E}\,\|Z_T\|^\lambda \le C$ for some constant $C > 0$ and hence,

$$\mathbf{E} \sup_{0\le s\le T} \|Z_s\|^\lambda \le C \quad \text{for any } 0 \le s \le T. \tag{2.1}$$

We know from (1.1) that for any $\tau \le s \le T$,

$$\begin{aligned}
Y_s^y - Y_\tau^y &= \int_\tau^s (AY_u^y + Y_u^y A^*)du + Z_s - Z_\tau \\
&= \int_\tau^s (A(Y_u^y - Y_\tau^y) + (Y_u^y - Y_\tau^y)A^*)du + (s - \tau)(AY_\tau^y + Y_\tau^y A^*) + Z_s - Z_\tau,
\end{aligned}$$

and moreover,

$$\|Y_s^y - Y_\tau^y\|^\lambda \le C\left(\|Y_\tau^y\|^\lambda + \sup_{\tau\le t\le T}\|Z_t - Z_\tau\|^\lambda + \int_\tau^s \|Y_u^y - Y_\tau^y\|^\lambda du\right).$$

By Gronwall's inequality,

$$\|Y_s^y - Y_\tau^y\|^\lambda \le C(\|Y_\tau^y\|^\lambda + \sup_{\tau\le t\le T}\|Z_t - Z_\tau\|^\lambda)(s-\tau) \quad \text{for all } \tau \le s \le T, \tag{2.2}$$

which, with choice $\tau = 0$ and (2.1), implies

$$\mathbf{E}\sup_{0\le s\le T}\|Y_s^y\|^\lambda \le C(1+\|y\|)^\lambda.$$

Hence, from (2.2),

$$\mathbf{E}\sup_{\tau\le s\le \tau'}\|Y_s^y - Y_\tau^y\|^\lambda \le C(1+\|y\|)^\lambda\epsilon.$$

Now, by Burkholder's inequality,

$$\begin{aligned}
\mathbf{E}\sup_{0\le s\le T}|X_s^{x,y}|^\lambda &\le C\left(|x|^\lambda + \mathbf{E}\sup_{0\le s\le T}\left|\int_0^s \pi(Y_u^y)du\right|^\lambda + \mathbf{E}\sup_{0\le s\le T}\left|\int_0^s r(Y_u^y)dB_u\right|^\lambda\right)\\
&= C\left(|x|^\lambda + \mathbf{E}\left(\int_0^T |\pi(Y_u^y)|du\right)^\lambda + \mathbf{E}\left(\int_0^T \operatorname{Tr} Y_u^y du\right)^{\lambda/2}\right)\\
&\le C\left(1+|x|^\lambda + \mathbf{E}\left(\int_0^T \|Y_u^y\|du\right)^\lambda\right)\\
&\le C\left(1+|x|^\lambda + \mathbf{E}\sup_{0\le s\le T}\|Y_s^y\|^\lambda\right) \le C(1+|x|+\|y\|)^\lambda.
\end{aligned}$$

Next, observe that

$$\begin{aligned}
\mathbf{E}\sup_{\tau\le s\le\tau'}|X_s^{x,y} - X_\tau^{x,y}| &\le \tfrac{1}{2}\mathbf{E}\sup_{\tau\le s\le\tau'}\left|\int_\tau^s \pi(Y_u^y)du\right| + \mathbf{E}\sup_{\tau\le s\le\tau'}\left|\int_\tau^s r(Y_u^y)dB_u\right|\\
&\le C\mathbf{E}\left|\int_\tau^{\tau'}\|Y_u^y\|du\right| + C\mathbf{E}\sqrt{\int_\tau^{\tau'}\operatorname{Tr}(Y_u^y)du}
\end{aligned}$$

also by Burkholder's inequality. Consequently,

$$\begin{aligned}
\mathbf{E}\sup_{\tau\le s\le\tau'}|X_s^{x,y} - X_\tau^{x,y}| &\le C\mathbf{E}\sup_{\tau\le s\le\tau'}\|Y_s^y\|(\tau'-\tau) + C\sqrt{\mathbf{E}(\tau'-\tau)\sup_{\tau\le u\le\tau'}\|Y_u^y\|}\\
&\le C(1+\|y\|)\sqrt{\epsilon}.
\end{aligned}$$

■

Proposition 2.2. *Assume that the function h is bounded and Lipschitz continuous on $\overline{Q}_T$. The value function u_0 of (1.4) is continuous on $\overline{Q}_T$, belongs to $\mathcal{V}_0$ and the terminal condition $u_0(x,y,T) = h(x,y,T)$ is satisfied. Furthermore, $u_0 \ge h$ on $\overline{Q}_T$.*

Proof. We start by noting that, by definition (1.4), u_0 is bounded on $\overline{Q}_T$, that is, $u_0 \in \mathcal{V}_0$, that $u_0(x,y,T) = h(x,y,T)$ is satisfied and that $u_0 \geq h$ on $\overline{Q}_T$. Next, we show the continuity of u_0 with respect to (x,y), uniformly in t. Joint continuity will then follow once it is shown that $u_0(x,y,t') \to u_0(x,y,t)$ as $t' \to t$, for all (x,y).

Let $x, x' \in \mathbb{R}^d$, $y, y' \in \mathbb{S}_d^+$, $\Delta x := x' - x$, $\Delta y := y' - y$ and $M_t^{y,y'} := \int_0^t (r(Y_s^{y'}) - r(Y_s^y))dB_s$, so that

$$Y_t^{y'} - Y_t^y = \Delta y + \int_0^t \left(A(Y_s^{y'} - Y_s^y) + (Y_s^{y'} - Y_s^y)A^* \right) ds,$$

$$X_t^{x',y'} - X_t^{x,y} = \Delta x - \tfrac{1}{2} \int_0^t \pi(Y_s^{y'} - Y_s^y)ds + M_t^{y,y'}.$$

The former yields together with Gronwall's inequality

$$\|Y_t^{y'} - Y_t^y\| \leq C\|\Delta y\|$$

for some constant $C > 0$. It is well-known (see e.g. [33, Eq. (3.2)]) that $\|r(y_1) - r(y_2)\|^2 \leq \sqrt{d}\|y_1 - y_2\|$ for all $y_1, y_2 \in \mathbb{S}_d^+$. Therefore,

$$\mathbf{E}|M_\tau^{y,y'}|^2 \leq \mathbf{E}\left(\int_0^T \|r(Y_s^{y'}) - r(Y_s^y)\|^2 ds \right) \leq C\mathbf{E}\left(\int_0^T \|Y_s^{y'} - Y_s^y\| ds \right) \leq C\|\Delta y\|.$$

From the Lipschitz condition of h, we find that

$$\begin{aligned}
\left|u_0(x',y',t) - u_0(x,y,t)\right| &\leq C \sup_{\tau \in \mathcal{T}_{T-t}} \mathbf{E}\Big(|X_\tau^{x',y'} - X_\tau^{x,y}| + \|Y_\tau^{y'} - Y_\tau^y\| \Big) \\
&\leq C\Big(|\Delta x| + \|\Delta y\| + \sup_{\tau \in \mathcal{T}_{T-t}} \mathbf{E}|M_\tau^{y,y'}| \Big) \\
&\leq C(|\Delta x| + \|\Delta y\| + \sqrt{\|\Delta y\|}).
\end{aligned}$$

We now show continuity with respect to time for fixed (x,y). Let $0 \leq t \leq t' \leq T$. Take $\tau \in \mathcal{T}_{T-t}$ and define $\tau' = \tau \wedge (T - t')$. Then, note that $\tau' \in \mathcal{T}_{T-t'}$ and $\tau' \leq \tau \leq \tau + t' - t$, and compute

$$\begin{aligned}
&\mathbf{E}h(X_\tau^{x,y}, Y_\tau^y, t+\tau) = \mathbf{E}h(X_{\tau'}^{x,y}, Y_{\tau'}^y, t'+\tau') + \mathbf{E}\left(h(X_\tau^{x,y}, Y_\tau^y, t+\tau) - h(X_{\tau'}^{x,y}, Y_{\tau'}^y, t'+\tau') \right) \\
&\quad \leq u_0(x,y,t') + \mathbf{E}\left| h(X_\tau^{x,y}, Y_\tau^y, t+\tau) - h(X_{\tau'}^{x,y}, Y_{\tau'}^y, t'+\tau') \right| \\
&\quad \leq u_0(x,y,t') + \mathbf{E} \sup_{\tau' \leq s \leq \tau' + t' - t} \left| h(X_s^{x,y}, Y_s^y, t+s) - h(X_{\tau'}^{x,y}, Y_{\tau'}^y, t'+\tau') \right|.
\end{aligned}$$

From this inequality and the fact that $u_0(x,y,t') \leq u_0(x,y,t)$, we readily find that

$$\begin{aligned}
|u_0(x,y,t) - u_0(x,y,t')| &\leq C\mathbf{E} \sup_{\tau' \leq s \leq \tau' + t' - t} \left(|X_s^{x,y} - X_{\tau'}^{x,y}| + \|Y_s^y - Y_{\tau'}\| + |t' - t| + |s - \tau'| \right) \\
&\leq C(1 + \|y\|)\sqrt{t' - t} \vee (t' - t)^{1/\lambda}
\end{aligned}$$

by Lemma 2.1. ∎

3. Viscosity Solutions. Our notion of viscosity solution depends on the constant $\lambda > 1$ in the integrability condition (1.2).

Lemma 3.1. *Let $u \in \mathcal{V}_\lambda$, $\phi \in \mathrm{C}^1(Q_T)$ and $(x, y, t) \in Q_T$. Assume that $u - \phi$ attains a global minimum at $(x, y, t) \in Q_T$. Then the function $z \mapsto u(x, y+z, t) - u(x, y, t)$ (respectively, $z \mapsto \phi(x, y+z, t) - \phi(x, y, t)$) is bounded from below (from above) by a function on $\mathbb{S}_d^+$ which is integrable with respect to ν.*

A main consequence of the above lemma is that the integrals $Ju(x, y, t) \in \mathbb{R} \cup \{+\infty\}$ and $J\phi(x, y, t) \in \mathbb{R} \cup \{-\infty\}$ make sense as extended real numbers.

We remark that, in the above lemma, if, instead, $u - \phi$ attains a global maximum at (x, y, t), then the conclusion is: the function $z \mapsto u(x, y + z, t) - u(x, y, t)$ (respectively, $z \mapsto \phi(x, y + z, t) - \phi(x, y, t)$) is bounded from above (from below) by an integrable function on K with respect to ν. To see this, we simply observe that $-(u-\phi)$ attains a global minimum at (x, y, t) and apply the lemma above to $-u$ and $-\phi$.

Proof. By the C^1-regularity of ϕ, there is a constant $C_1 > 0$ such that for any $z \in \mathbb{S}_d^+$, if $\|z\| \le 1$, then

$$|\phi(x, y + z, t) - \phi(x, y, t)| \le C_1\|z\|. \tag{3.1}$$

Since (x, y, t) is a minimum point of $u - \phi$, we have for all $z \in \mathbb{S}_d^+$,

$$(u - \phi)(x, y + z, t) \ge (u - \phi)(x, y, t),$$

which reads

$$u(x, y + z, t) - u(x, y, t) \ge \phi(x, y + z, t) - \phi(x, y, t). \tag{3.2}$$

Since $u \in \mathcal{V}_\lambda$, we have

$$|u(x, y + z, t) - u(x, y, t)| \le C_2(1 + \|z\|^\lambda) + |u(x, y, t)| \le C_3(1 + \|z\|^\lambda) \tag{3.3}$$

for all $z \in \mathbb{S}_d^+$ and some positive constants C_2, C_3. Combining the last two inequalities yields

$$\phi(x, y + z, t) - \phi(x, y, t) \le C_3(1 + \|z\|^\lambda) \quad \text{for all } z \in \mathbb{S}_d^+.$$

From this and (3.1), we get for all $z \in \mathbb{S}_d^+$,

$$\phi(x, y + z, t) - \phi(x, y, t) \le \begin{cases} C_1\|z\| & \text{if } \|z\| \le 1, \\ C_3(1 + \|z\|^\lambda) & \text{otherwise.} \end{cases}$$

If $f : \mathbb{S}_d^+ \to \mathbb{R}$ is the function given by the right side of the above inequality, then f is integrable with respect to ν and $\phi(x, y + z, t) - \phi(x, y, t) \le f(z)$ for all $z \in \mathbb{S}_d^+$.

Similarly, we find by (3.1), (3.2) and (3.3) that for all $z \in \mathbb{S}_d^+$,

$$\begin{aligned} u(x, y + z, t) - u(x, y, t) &\ge \max\{\phi(x, y + z, t) - \phi(x, y, t), -C_3(1 + \|z\|^\lambda)\} \\ &\ge \begin{cases} -C_1\|z\| & \text{if } \|z\| \le 1, \\ -C_3(1 + \|z\|^\lambda) & \text{otherwise,} \end{cases} \end{aligned}$$

which shows that $u(x, y + z, t) - u(x, y, t) \ge -f(z)$ for all $z \in \mathbb{S}_d^+$, where $-f$ is integrable on $\mathbb{S}_d^+$ with respect to ν. ■

In the above proof and in what follows, it is important to see that, for any $\phi \in C^1(Q_T)$ and $(x,y,t) \in Q_T$, the function $z \mapsto \phi(x,y+z,t) - \phi(x,y,t)$ is integrable with ν on every compact subset of $\mathbb{S}_d^+$. Moreover, if $\phi \in \mathcal{V}_\lambda$, then the function $z \mapsto \phi(x,y+z,t) - \phi(x,y,t)$ is integrable on $\mathbb{S}_d^+$ with ν.

The definition of viscosity solutions of (1.6) is as follows:

Definition 3.2. *Let $u \in \mathrm{LSC}(Q_T) \cap \mathcal{V}_\lambda$ (respectively, $u \in \mathrm{USC}(Q_T) \cap \mathcal{V}_\lambda$). We call u a viscosity supersolution (subsolution) of* (1.6) *if*

$$\min\{\mathcal{M}\phi(x,y,t), u(x,y,t) - h(x,y,t)\} \geq 0\ (\leq 0), \tag{3.4}$$

whenever $\phi \in \mathrm{C}^2(Q_T)$ and $u - \phi$ attains a global minimum (maximum) at $(x,y,t) \in Q_T$.

It is convenient to state the viscosity property pointwise: given a point $(x,y,t) \in Q_T$, we say that u is a *viscosity supersolution (subsolution) of* (1.6) *at* (x,y,t) if the conditions in the above definition are satisfied for the fixed (x,y,t).

We remark that in the above definition of viscosity supersolutions, the left side of (3.4) takes a finite value owing to Lemma 3.1. On the other hand, in the definition of viscosity subsolutions, the left side of (3.4) takes a finite value or the value $-\infty$ (see the remark after Lemma 3.1).

Fix $\theta \in Q_T$ and $\kappa \geq 0$. In view of the integro-differential character of (1.6), we introduce the function space $\mathcal{W}_\kappa(\theta)$ as the set of functions $\phi \in C(Q_T)$ such that $\phi \in \mathcal{V}_\kappa$ and ϕ is a C^2-function in a neighborhood of θ. We say that a sequence $\{\phi_j\} \subset \mathcal{W}_\kappa(\theta)$ converges to $\phi \in \mathcal{W}_\kappa(\theta)$ if $\{\phi_j\}$ converges to ϕ in $\mathrm{C}(Q_T)$ and in $\mathrm{C}^2(K)$ for some neighborhood K of θ, and $|\phi_j| \leq g$ on Q_T for all $j \in \mathbb{N}$ and some $g \in \mathcal{V}_\kappa$. Note that if $\phi \in \mathcal{W}_\kappa(\theta)$, then $\phi \in \mathcal{W}_\kappa(\zeta)$ for all ζ in a neighborhood of θ. Similarly, if $\{\phi_j\}$ converges to $\phi \in \mathcal{W}_\kappa(\theta)$, then $\{\phi_j\}$ converges to ϕ in $\mathcal{W}_\kappa(\zeta)$ for all ζ in a neighborhood of θ.

Lemma 3.3. *Let $\theta \in Q_T$, $0 \leq \kappa \leq \lambda$, $\phi \in \mathcal{W}_\kappa(\theta)$, $\{\phi_j\}_{j\in\mathbb{N}} \subset \mathcal{W}_\kappa(\theta)$ and $\{\theta_j\}_{j\in\mathbb{N}} \subset Q_T$. Assume that $\lim_{j\to\infty} \theta_j = \theta$ and $\{\phi_j\}_{j\in\mathbb{N}}$ converges to ϕ in $\mathcal{W}_\kappa(\theta)$. Then*

$$\lim_{j\to\infty} \mathcal{M}\phi_j(\theta_j) = \mathcal{M}\phi(\theta).$$

Notice that $\mathcal{W}_\kappa(\theta) \subset \mathcal{V}_\kappa$ for all $\theta \in Q_T$ and that $\mathcal{M}\phi_j(\theta_j)$ makes sense in the above lemma when j is large enough.

Proof. For some small $r > 0$, we have the C^2-convergence of $\{\phi_j\}$ to ϕ on the set $B_r(\theta)$, where $B_r(\theta)$ is the ball in Q_T of radius r centered at θ. We write $\theta = (x,y,t)$ and $\theta_j = (x_j,y_j,t_j)$ for $j \in \mathbb{N}$. For $j \in \mathbb{N}$ sufficiently large, we have $\theta_j \in B_{r/2}(\theta)$ and for some $C > 0$ uniform in j,

$$|\phi_j(x_j,y_j+z,t_j) - \phi_j(x_j,y_j,t_j)| \leq C\|z\| \quad \text{if } \|z\| \leq r/2.$$

Since $|\phi_j| \leq g$ on Q_T for some $g \in \mathcal{V}_\kappa$, we may assume that for all $j \in \mathbb{N}$ and $z \in \mathbb{S}_d^+$,

$$|\phi_j(x_j,y_j+z,t_j) - \phi_j(x_j,y_j,t_j)| \leq |g(x_j,y_j+z,t_j)| + |g(x_j,y_j,t_j| \leq C(1+\|z\|)^\lambda.$$

It is clear that the functions $z \mapsto \phi_j(x_j,y_j+z,t_j) - \phi_j(x_j,y_j,t_j)$ converge to the function $z \mapsto \phi(x,y+z,t) - \phi(x,y,t)$ pointwise as $j \to \infty$. The dominated convergence theorem thus assures that

$$\lim_{j\to\infty} J\phi_j(\theta_j) = J\phi(\theta).$$

Furthermore, the C^2-convergence of $\{\phi_j\}$ on $B_r(\theta)$ implies readily that

$$\lim_{j\to\infty} (-\partial_t\phi_j(\theta_j) - L\phi_j(\theta_j)) = -\partial_t\phi(\theta) - L\phi(\theta),$$

which completes the proof. ■

Lemma 3.4. *Let $\theta \in Q_T$, $0 \le \kappa \le \lambda$ and $u \in \mathrm{LSC}(Q_T) \cap \mathcal{V}_\kappa$ (respectively, $u \in \mathrm{USC}(Q_T) \cap \mathcal{V}_\kappa$). Then u is a viscosity supersolution (subsolution) of (1.6) at θ if and only if*

$$\min\{\mathcal{M}\phi(\theta), u(\theta) - h(\theta)\} \ge 0 \ (\le 0), \tag{3.5}$$

whenever $\phi \in \mathcal{W}_\kappa(\theta)$ and $u - \phi$ attains a global minimum (maximum) at $\theta \in Q_T$.

Proof. We treat only the case of viscosity supersolution, and leave it to the reader to check the other case. We first prove the "if" part. Let u be a supersolution of (1.6) at $\theta \in Q_T$. Let $\phi \in \mathcal{W}_\kappa(\theta)$ be such that $u - \phi$ has a global minimum at θ. Noting that $\mathcal{M}(\psi + C) = \mathcal{M}\psi$ for any $\psi \in C^2(Q_T) \cap \mathcal{V}_\lambda$ and $C \in \mathbb{R}$ and adding a constant to ϕ if necessary, we may assume that $u(\theta) = \phi(\theta)$, a consequence of which is that $u \ge \phi$ on Q_T. Let $B_r(\theta) \subset Q_T$ be the ball of radius $r > 0$ with center at θ such that $\phi \in C^2(B_r(\theta))$. Select a cut-off function $f \in C^2(Q_T)$ so that $0 \le f \le 1$ on Q_T, $f = 0$ on $B_{r/2}(\theta)$ and $f = 1$ on $Q_T \setminus B_r(\theta)$. For each $j \in \mathbb{N}$, we select $\phi_j \in C^2(Q_T)$ so that $|\phi - \phi_j| \le 1/j$ on Q_T. Setting $\psi_j = f(\phi_j - j^{-1}) + (1 - f)\phi$, we note that $\psi_j \in C^2(Q_T)$, $\phi - j^{-1} \le \psi_j \le \phi$ on Q_T and $\psi_j = \phi$ on $B_{r/2}(\theta)$. Hence, we find that $u - \psi_j$ attains a global minimum at θ and also that the sequence $\{\psi_j\}$ converges to ϕ in $\mathcal{W}_\kappa(\theta)$. Since u is a viscosity supersolution of (1.6) at θ, we have

$$\min\{\mathcal{M}\psi_j(\theta), u(\theta) - h(\theta)\} \ge 0,$$

which yields thanks to Lemma 3.3 that $\min\{\mathcal{M}\phi(\theta), u(\theta) - h(\theta)\} \ge 0$.

Next, we prove the "only if" part and thus assume that u satisfies the condition given in Lemma 3.4. Fix any $\phi \in C^2(Q_T)$. Assume that $u - \phi$ has a global minimum at θ. As before we may assume that $u \ge \phi$ on Q_T and $u(\theta) = \phi(\theta)$. We choose a function $g \in C(Q_T) \cap \mathcal{V}_\kappa$ so that $u > g$ on Q_T. Define $\psi \in C(Q_T)$ by $\psi = \phi \vee g$ (the pointwise maximum of ϕ and g). Note that $u \ge \psi \ge g$ on Q_T, $\phi = \psi$ in a neighborhood of θ and $u(\theta) = \psi(\theta)$. In particular, $\psi \in \mathcal{W}_\kappa(\theta)$ and $\min(u - \psi) = (u - \psi)(\theta)$. Hence, by assumption, we have

$$\min\{\mathcal{M}\psi(\theta), (u - h)(\theta)\} \ge 0.$$

It is clear that $-\partial_t\psi(\theta) - L\psi(\theta) = -\partial_t\phi(\theta) - L\phi(\theta)$. Since $\psi \ge \phi$ on Q_T and $\psi(\theta) = \phi(\theta)$, if we write $\theta = (x, y, t)$, we have for all $z \in \mathbb{S}_d^+$,

$$\phi(x, y + z, t) - \psi(x, y, t) \ge \phi(x, y + z, t) - \phi(x, y, t),$$

which implies that $J\psi(\theta) \ge J\phi(\theta)$. Thus, we have $\min\{\mathcal{M}\phi(\theta), (u - h)(\theta)\} \ge 0$, which concludes the proof. ■

We remark that the above lemma is valid with $\mathcal{W}_\kappa(\theta) \cap \mathrm{C}^2(Q_T)$ in place of $\mathcal{W}_\kappa(\theta)$. To check this, we say temporarily that $\mathrm{C}[\theta, \mathcal{W}_\kappa(\theta)]$ (respectively, $\mathrm{C}[\theta, \mathcal{W}_\kappa(\theta) \cap \mathrm{C}^2(Q_T)]$) holds if the condition (condition with $\mathcal{W}_\kappa(\theta) \cap \mathrm{C}^2(Q_T)$ in place of $\mathcal{W}_\kappa(\theta)$) stated after "only if" in Lemma 3.4 holds. Since $\mathcal{W}_\kappa(\theta) \cap \mathrm{C}^2(Q_T) \subset \mathcal{W}_\kappa(\theta)$ it is clear that $\mathrm{C}[\theta, \mathcal{W}_\kappa(\theta)]$ implies $\mathrm{C}[\theta, \mathcal{W}_\kappa(\theta) \cap \mathrm{C}^2(Q_T)]$, while the proof of "if part" in the above proof shows that $\mathrm{C}[\theta, \mathcal{W}_\kappa(\theta) \cap \mathrm{C}^2(Q_T)]$ implies $\mathrm{C}[\theta, \mathcal{W}_\kappa(\theta)]$. Notice that $\mathrm{C}^2(Q_T) \cap \mathcal{V}_\kappa = \mathrm{C}^2(Q_T) \cap \mathcal{W}_\kappa(\theta)$ for all $\theta \in Q_T$.

We introduce $F : \mathbb{S}_d^+ \times (\mathbb{R}^d \times M_d(\mathbb{R}) \times \mathbb{R}) \times \mathbb{S}_n \to \mathbb{R}$ (with $n := d + d^2 + 1$) defined by

$$F(y, p, \mathcal{X}) = -p_3 - \tfrac{1}{2}\langle y, \mathcal{X}_1 \rangle + \tfrac{1}{2}\langle \pi(y), p_1 \rangle - \langle Ay + yA^* + b_0, p_2 \rangle,$$

where

$$\mathcal{X} = \begin{pmatrix} \mathcal{X}_1 & \mathcal{X}_2 \\ \mathcal{X}_2^* & \mathcal{X}_3 \end{pmatrix}, \text{ with } \mathcal{X}_1 \in \mathbb{S}_d, \mathcal{X}_2 \in M_{d\times(d^2+1)}, \mathcal{X}_3 \in \mathbb{S}_{d^2+1},$$

and

$$p = (p_1, p_2, p_3), \text{ with } p_1 \in \mathbb{R}_d, p_2 \in M_d(\mathbb{R}), p_3 \in \mathbb{R}.$$

With this notation, we have

$$-\partial_t \phi(x, y, t) - L\phi(x, y, t) = F(y, D\phi(x, y, t), D^2\phi(x, y, t)).$$

Definition 3.2 is equivalent to

Definition 3.5. *Any $u \in \mathrm{LSC}(Q_T) \cap \mathcal{V}_\lambda$ (respectively, $u \in \mathrm{USC}(Q_T) \cap \mathcal{V}_\lambda$) is a viscosity supersolution (subsolution) of (1.6) if*

$$\min\{F(y, p, \mathcal{X}) - Ju(x, y, t), (u - h)(x, y, t)\} \geq 0 \; (\leq 0), \tag{3.6}$$

whenever $(p, \mathcal{X}) \in J^{2,-}u(x, y, t)$ $\left((p, \mathcal{X}) \in J^{2,+}u(x, y, t)\right)$, $(x, y, t) \in Q_T$.

Proof of equivalence of definitions. Assume that $u \in \mathrm{LSC}(Q_T) \cap \mathcal{V}_\lambda$ is a viscosity supersolution of (1.6) in the sense of Definition 3.5. Let $\theta = (x, y, t) \in Q_T$ and $\phi \in \mathrm{C}^2(Q_T)$, and assume that $u - \phi$ takes a global minimum at θ. As in the proof of Lemma 3.1, we have for all $z \in \mathbb{S}_d^+$,

$$u(x, y + z, t) - u(x, y, t) \geq \phi(x, y + z, t) - \phi(x, y, t),$$

and hence, in view of Lemma 3.1,

$$Ju(\theta) \geq J\phi(\theta).$$

Note as well that $(D\phi(\theta), D^2\phi(\theta)) \in J^{2,-}u(x, y, t)$. Thus, by Definition 3.5 we have

$$\begin{aligned} 0 &\leq \min\{F(y, D\phi(\theta), D^2\phi(\theta)) - Ju(\theta), \, (u - h)(\theta)\} \\ &\leq \min\{-\partial_t\phi(\theta) - L\phi(\theta) - J\phi(\theta), \, (u - h)(\theta)\}, \end{aligned}$$

which ensures that u is a viscosity supersolution of (1.6) in the sense of Definition 3.2.

Next, we assume that $u \in \mathrm{LSC}(Q_T) \cap \mathcal{V}_\lambda$ is a viscosity supersolution of (1.6) in the sense of Definition 3.2. Let $\theta = (x, y, t) \in Q_T$ and $(p, \mathcal{X}) \in J^{2,-}u(\theta)$. As is well-known, there exists a function $\phi \in \mathrm{C}^2(Q_T)$ such that $D\phi(\theta) = p$, $D^2\phi(\theta) = \mathcal{X}$, and $u - \phi$ attains a global minimum at θ. We may assume that the minimum value is 0, so that $u(\theta) = \phi(\theta)$ and $u \geq \phi$

on Q_T. Since $u \in \mathrm{LSC}(Q_T) \cap \mathcal{V}_\lambda$, there exists an increasing sequence $\{\psi_j\}_{j\in\mathbb{N}} \in \mathrm{C}(Q_T) \cap \mathcal{V}_\lambda$ such that $u > \psi_j$ on Q_T and $\lim_{j\to\infty} \psi_j(\zeta) = u(\zeta)$ for all $\zeta \in Q_T$. Set $\phi_j = \phi \vee \psi_j$, and note that $u - \phi_j$ has a global minimum at θ, $\phi = \phi_j$ in a neighborhood of θ, which may depend on j, and $\psi \in \mathcal{W}_\lambda(\theta)$. By Lemma 3.4, we find that $\min\{\mathcal{M}\phi_j(\theta), (u-h)(\theta)\} \geq 0$. Obviously, we have $F(y, D\phi_j(\theta), D^2\phi_j(\theta)) = F(y, p, \mathcal{X})$. Since the function $z \mapsto \phi_1(x, y+z, t) - \phi_1(x, y, t)$ is integrable with nu and $\phi_j(x, y+z, t) - \phi_j(x, y, t) \uparrow u(x, y+z, t) - u(x, y, t)$ as $j \to \infty$, we find by the monotone convergence theorem that, as $j \to \infty$, $J\phi_j(\theta) \uparrow J\phi(\theta)$. Hence, we get $\min\{\mathcal{M}\phi(\theta), (u-h)(\theta)\} \geq 0$. Thus, we conclude that u is a viscosity supersolution of (1.6) in the sense of Definition 3.5.

The proof concerning the subsolution property parallels the above, which we skip here. ■

The existence proof of the viscosity solution is based on the following observation which states that it is never optimal to stop the process before the Snell envelope $u(X_s, Y_s, t-s)$ reaches the payoff $h(X_s, Y_s, t+s)$. If existence of a solution of IDE (1.6) is the only goal, one may apply Perron's method. However, we want to characterize u_0 as a solution to this IDE.

Proposition 3.6. *Assume that the function h is bounded and Lipschitz continuous on $\overline{Q}_T$. Let $(x_0, y_0, t_0) \in Q_T$. Define the process U by $U_t = u_0(X_t^{x_0,y_0}, Y_t^{y_0}, t_0+t)$ and set*

$$\tau_0 = \inf\{0 \leq t \leq T - t_0 : U_t = h(X_t^{x_0,y_0}, Y_t^{y_0}, t_0+t)\}. \tag{3.7}$$

Then:
(i) *U is a supermartingale on $[0, T-t_0]$.*
(ii) *U is a martingale on $[0, \tau_0]$.*

Note that, since $u_0(x, y, T) = h(x, y, T)$ by Proposition 2.2, τ_0 in the above proposition is less or equal to $T - t_0$.

Proof. Since (X, Y) satisfies the strong Markov property, it follows from [16, Theorem 3.4] that the process U is identified with the Snell envelope of $h(X_t, Y_t, t_0+t)$, given by

$$\operatorname*{ess\,sup}_{\tau\geq t} \mathbf{E}(h(X_\tau, Y_\tau, t_0+\tau)|\mathcal{F}_t).$$

Note that (X, Y) is quasi-left-continuous since it is the solution of an SDE with respect to a Lévy process. Therefore, $h(X_t, Y_t, t_0+t)$ is quasi-left-continuous since h is continuous. By [21, Proposition I.2.26], quasi-left-continuity is equivalent to left-continuity over stopping times since (X, Y) is càdlàg. We can therefore apply [27, Theorem 2.2] (see also [24, 15]), and conclude that (i) and (ii) are valid. ■

We can now state and prove the main theorem of the section.

Theorem 3.7. *Assume that the function h is bounded and Lipschitz continuous on $\overline{Q}_T$. Then, u_0 is a viscosity solution of IDE (1.6).*

Proof. By Proposition 2.2, u_0 is continuous and in $\mathcal{V}_0 \subset \mathcal{V}_\lambda$. Let $\theta_0 = (x_0, y_0, t_0) \in Q_T$ and $\phi \in \mathcal{V}_\lambda \cap \mathrm{C}^2(Q_T)$. We check the viscosity super and subsolution property of u_0 at θ_0 based on Lemma 3.4 together with the remark next to it.

Let τ_0 be the stopping time defined by (3.7),

$$\tau_1 = \inf\{0 \leq t \leq T - t_0 : |X_t - x_0| + \|Y_t - y_0\| + |t| > \delta\},$$

and $\tau = \tau_0 \wedge \tau_1$. To simplify the notation, we write $X_t = X_t^{x_0,y_0}$ and $Y_t = Y_t^{y_0}$. We claim that

$$\mathbf{E}\,\phi(X_\tau, Y_\tau, t_0+\tau) - \phi(\theta_0) = -\,\mathbf{E}\int_0^\tau \mathcal{M}\phi(X_s, Y_{s-}, t_0+s)ds. \tag{3.8}$$

We note that

$$\begin{aligned} dY_t &= \Big(AY_t + Y_tA^* + b_0 + \int_{\mathbb{S}_d^+, \|z\|<1} z\nu(dz)\Big)dt \\ &\quad + \int_{\mathbb{S}_d^+, \|z\|<1} z\tilde{N}(dz,dt) + \int_{\mathbb{S}_d^+, \|z\|\geq 1} zN(dz,dt). \end{aligned}$$

Then, by [1, Theorem 4.4.7], it follows that

$$\begin{aligned} \phi(X_\tau,&Y_\tau, t_0+\tau) - \phi(\theta_0) \\ &= \int_0^\tau \Big(\partial_t\phi(X_s, Y_{s-}, t_0+s) + \tfrac{1}{2}\langle D_x^2\phi(X_s, Y_{s-}, t_0-s), Y_{s-}\rangle \\ &\qquad - \tfrac{1}{2}\langle D_x\phi(X_s, Y_{s-}, t_0+s), \pi(Y_{s-})\rangle \\ &\qquad + \Big\langle D_y\phi(X_s, Y_{s-}, t_0+s), AY_{s-} + Y_{s-}A^* + b_0 - \int_{\mathbb{S}_d^+, \|z\|<1} z\nu(dz)\Big\rangle\Big)ds \\ &\quad + \int_0^\tau \Big\langle D_x\phi(X_s, Y_{s-}, t_0+s)), r(Y_{s-})\Big\rangle dB_s \\ &\quad + \int_0^\tau \int_{\mathbb{S}_d^+, \|z\|\geq 1} (\phi(X_s, Y_{s-}+z, t_0+s) - \phi(X_s, Y_{s-}, t_0+s))N(dz,ds) \\ &\quad + \int_0^\tau \int_{\mathbb{S}_d^+, \|z\|<1} (\phi(X_s, Y_{s-}+z, t_0+s) - \phi(X_s, Y_{s-}, t_0+s))\tilde{N}(dz,ds) \\ &\quad + \int_0^\tau \int_{\mathbb{S}_d^+, \|z\|<1} \Big(\phi(X_s, Y_{s-}+z, t_0+s) - \phi(X_s, Y_{s-}, t_0+s) \\ &\qquad - \langle D_y\phi(X_s, Y_{s-}, t_0+s), z\rangle\Big)\nu(dz)ds, \end{aligned}$$

Since the two terms involving the integrators dB_s and $\tilde{N}(dz,ds)$ above are martingales, we get after cancellation of terms involving $\int_{\mathbb{S}_d^+, \|z\|<1} z\nu(dz)$,

$$\begin{aligned} \mathbf{E}\phi(X_\tau,&Y_\tau, t_0+\tau) - \phi(\theta_0) \\ &= \mathbf{E}\Big(\int_0^\tau (\partial_t\phi + L\phi)(X_s, Y_s, t_0+s)ds \\ &\qquad + \int_0^\tau \int_{\mathbb{S}_d^+} (\phi(X_s, Y_{s-}+z, t_0+s) - \phi(X_s, Y_{s-}, t_0+s))\nu(dz)ds \\ &\qquad + \int_0^\tau \int_{\mathbb{S}_d^+, \|z\|\geq 1} (\phi(X_s, Y_{s-}+z, t_0+s) - \phi(X_s, Y_{s-}, t_0+s))\tilde{N}(dz,ds)\Big), \end{aligned}$$

and, again by the martingale property of the last term in the above,

$$\mathbf{E}\phi(X_\tau, Y_\tau, t_0+\tau) - \phi(\theta_0) = -\,\mathbf{E}\int_0^\tau \mathcal{M}\phi(X_s, Y_{s-}, t_0+s)ds,$$

proving (3.8).

(1) Subsolution property:

To prove the subsolution property, assume that $u_0 - \phi$ attains a global maximum at θ_0 and $u_0(\theta) = \theta_0$. We argue by contradiction to prove that $\min\{\mathcal{M}\phi(\theta_0), (u_0 - h)(\theta_0)\} \le 0$. We thus suppose that $\mathcal{M}\phi(\theta_0) > 0$ and $(u_0 - h)(\theta_0) > 0$, and will get a contradiction.

Note by Lemma 3.3 that the function $\mathcal{M}\phi$ is continuous at θ_0. Then, there exists $0 < \delta < T - t_0$ such that

$$\mathcal{M}\phi(\theta) > \delta \quad \text{for all } \theta \in B_\delta(\theta_0), \tag{3.9}$$

where $B_\delta(\theta_0) = \{\theta = (x, y, t) \in Q_T : |x - x_0| + \|y - y_0\| + |t - t_0| \le \delta\}$. Since the process $(X_s, Y_s, t_0 + s)$ is càdlàg, we have $\tau_0 > 0$ and $\tau_1 > 0$ a.s. and also we find by (ii) of Proposition 3.6 and Doob's optional sampling theorem that $u_0(X_{s\wedge\tau}, Y_{s\wedge\tau}, t_0 + s\wedge\tau))$ is a martingale.

Note that $(X_t, Y_{t-}, t_0 + t) \in B_\delta(\theta_0)$ for all $t \in [0, \tau]$ a.s., that, by the martingale property,

$$u_0(\theta_0) = \mathbf{E}\, u_0(X_\tau, Y_\tau, t_0 + \tau),$$

and that $u_0(\theta) - u_0(\theta_0) \le \phi(\theta) - \phi(\theta_0)$ for all $\theta \in Q_T$. Hence, we deduce by (3.8) and (3.9) that

$$\begin{aligned} 0 &= \mathbf{E}\, u_0(X_\tau, Y_\tau, t_0 + \tau) - u_0(\theta_0) \\ &\le \mathbf{E}\, \phi(X_\tau, Y_\tau, t_0 + \tau) - \phi(\theta_0) \\ &= -\mathbf{E} \int_0^\tau \mathcal{M}\phi(X_s, Y_{s-}, t_0 + s) ds \le -\delta\, \mathbf{E}\, \tau. \end{aligned}$$

This implies that $\tau = 0$ a.s. On the other hand, we have $\tau > 0$ a.s. by definition. This is a contradiction, which proves that u_0 is a viscosity solution of (1.6) at θ_0.

(2) Supersolution property:

To prove the supersolution property, assume that $u_0 - \phi$ attains a global minimum at θ_0 and $u_0(\theta_0) = \phi(\theta_0)$. As noted before, we know by the definition of u_0 that $u_0 \ge h$ in $\overline{Q}_T$.

We note by (i) of Proposition 3.6 and Doob's optional sampling theorem that for any $\tau \in \mathcal{T}_{T-t_0}$,

$$u_0(\theta_0) \ge \mathbf{E}\, u_0(X_\tau, Y_\tau, t_0 + \tau). \tag{3.10}$$

It remains to prove that $\mathcal{M}\phi(\theta_0) \ge 0$. To show this, we suppose to the contrary that $\mathcal{M}\phi(\theta_0) < 0$. We follow the argument above for the supersolution property, and we choose a constant $0 < \delta < T - t_0$ so that $\mathcal{M}\phi(\theta) < -\delta$ for all $\theta \in B_\delta(\theta_0)$. Then, using (3.10), we get

$$\begin{aligned} 0 \ge \mathbf{E}\, u_0(X_{\tau_1}, Y_{\tau_1}, t_0 + \tau_1) - u_0(\theta_0) &\ge \mathbf{E}\, \phi(X_{\tau_1}, Y_{\tau_1}, t_0 + \tau_1) - \phi(\theta_0) \\ &= -\mathbf{E} \int_0^{\tau_1} \mathcal{M}\phi(X_s, Y_{s-}, t_0 + s) ds \ge \delta\, \mathbf{E}\, \tau_1, \end{aligned}$$

which implies $\tau_1 = 0$ a.s. This is a contradiction and the proof is complete. ∎

4. An Invariance Property and Maximum Principle.

The following invariance property states that a classical subsolution (resp. supersolution) of (1.6) is also a viscosity subsolution (resp. supersolution) of the same IDE.

Proposition 4.1. *Let $v \in \mathrm{C}^2(Q_T) \cap \mathcal{V}_\lambda$ be a classical subsolution (resp. supersolution) of (1.6). Then v is also a viscosity subsolution (resp. supersolution) of (1.6).*

Proof. We only prove the case of a subsolution. Assume $v \in \mathrm{C}^2(Q_T) \cap \mathcal{V}_\lambda$ is a classical subsolution of (1.6). Let $\phi \in \mathrm{C}^2(Q_T) \cap \mathcal{V}_\lambda$ and assume that $v - \phi$ has a maximum at (x, y, t). Then, we have

$$v(x, y+z, t) - v(x, y, t) \le \phi(x, y+z, t) - \phi(x, y, t) \quad \text{for all } z \in \mathbb{S}_d^+,$$

which readily yields

$$Jv(x, y, t) \le J\phi(x, y, t).$$

If $y \in \operatorname{int} \mathbb{S}_d^+$ and $t > 0$, then, as usual, we have

$$Dv(x, y, t) = D\phi(x, y, t) \quad \text{and} \quad D_x^2 v(x, y, t) \le D_x^2 \phi(x, y, t).$$

In general, we note that for a small $\delta > 0$,

$$(x + \xi, e^{sA} y e^{sA^*}, t + u) \in Q_T \quad \text{for all } (\xi, s) \in \mathbb{R}^d \times \mathbb{R} \times [0, \delta],$$

the function

$$(\xi, s, u) \mapsto (v - \phi)(x + \xi, e^{sA} y e^{sA^*}, t + u)$$

achives its maximum on $\mathbb{R}^d \times \mathbb{R} \times [0, \delta]$ at $(0, 0, 0)$, and hence

$$\begin{cases} 0 = D_x(v - \phi)(x, y, t), \qquad 0 \ge D_x^2(v - \phi)(x, y, t), \\ 0 = \frac{d}{ds}(v - \phi)(x, e^{sA} y e^{sA^*}, t)\Big|_{s=0} = \langle D_y(v - \phi)(x, y, t), Ay + yA^* \rangle, \\ 0 \ge (v - \phi)_t(x, y, t). \end{cases}$$

From these together, we get

$$\mathcal{M}v(x, y, t) \ge \mathcal{M}\phi(x, y, t),$$

and conclude that

$$\min\{\mathcal{M}\phi(x, y, t), v(x, y, t) - h(x, y, t)\} \le 0. \qquad \blacksquare$$

The comparison principle is based on the following maximum principle, which we state in general terms due to its wider applicability potential and separate interest. Theorem 4.2 below can be seen as an extension of the maximum principle for semicontinuous functions found in [11], [12]. Thus, our result is based on the classical work due to Jensen [23], Ishii [20] and others (see [12] for the development of the theory of the maximum principle and viscosity solutions). The theorem below makes a similar claim to the maximum principles [22, Theorem 4.9] and [2, Lemma 1], but its statement is less involved, and it might be more user friendly. It is nothing but a straightforward extension of [11, Theorem 1] to the generality of applicable to integro-PDEs.

For later convenience, we introduce the notation: for any $\mathcal{A} \in \mathbb{S}_m$, with $m \in \mathbb{N}$, and $\epsilon > 0$, we write

$$\mathcal{A}_\epsilon = \mathcal{A} + \epsilon \mathcal{A}^2, \quad \lambda = \frac{1}{\epsilon} + |\mathcal{A}| \quad \text{and} \quad E_\lambda = \lambda I_m, \tag{4.1}$$

where

$$|\mathcal{A}| = \max_{\xi \in \mathbb{R}^m,\, |\xi|=1} \langle \mathcal{A}\xi, \xi \rangle.$$

Theorem 4.2. *Let U, V be locally compact subsets of $\mathbb{R}^n$, $n \geq 1$. Fix $\hat{\theta} \in U$, $\hat{\zeta} \in V$, $u \in \mathrm{USC}(U), v \in \mathrm{USC}(V)$ and $\varphi \in \mathrm{C}^2(U \times V)$. Define $w : U \times V \to \mathbb{R}$ by $w(\theta, \zeta) = u(\theta) + v(\zeta)$. Assume*

$$\max_{U \times V}(w - \varphi) = (w - \varphi)(\hat{\theta}, \hat{\zeta}). \tag{4.2}$$

Let $\epsilon > 0$ and W be a compact neighborhood of $(\hat{\theta}, \hat{\zeta})$, relative to $U \times V$. Let $\hat{p} = D_\theta \varphi(\hat{\theta}, \hat{\zeta}), \hat{q} = D_\zeta \varphi(\hat{\theta}, \hat{\zeta}), \mathcal{A} = D^2\varphi(\hat{\theta}, \hat{\zeta})$. Define $\mathcal{A}_\epsilon$, λ and E_λ by formula (4.1), with $m = 2n$. Select a function $\varphi_\epsilon \in \mathrm{C}^2(U \times V)$ so that

$$\begin{cases} \varphi \leq \varphi_\epsilon \ \text{ on } U \times V, \\ \varphi_\epsilon(\hat{\theta}, \hat{\zeta}) = \varphi(\hat{\theta}, \hat{\zeta}), \\ D\varphi_\epsilon(\hat{\theta}, \hat{\zeta}) = D\varphi(\hat{\theta}, \hat{\zeta}) \\ D^2\varphi_\epsilon(\hat{\theta}, \hat{\zeta}) = \mathcal{A}_\epsilon. \end{cases} \tag{4.3}$$

Then, there exist sequences $\{(\theta_j, \zeta_j)\} \subset U \times V, \{(\mathcal{X}_j, \mathcal{Y}_j)\} \subset \mathbb{S}_n \times \mathbb{S}_n$, and $\{\varphi_j\} \subset \mathrm{C}^2(U \times V)$ such that the following conditions hold for all $j \in \mathbb{N}$:

$$\begin{cases} \lim\limits_{k\to\infty} (\theta_k, \zeta_k) = (\hat{\theta}, \hat{\zeta}), \\ \max\limits_{U \times V}(w - \varphi_j) = (w - \varphi_j)(\theta_j, \zeta_j), \\ (D_\theta \varphi_j(\theta_j, \zeta_j), \mathcal{X}_j) \in J^{2,+}u(\theta_j), \qquad (D_\zeta \varphi_j(\theta_j, \zeta_j), \mathcal{Y}_j) \in J^{2,+}v(\zeta_j), \\ -E_\lambda \leq \begin{pmatrix} \mathcal{X}_j & 0 \\ 0 & \mathcal{Y}_j \end{pmatrix} \leq D^2\varphi_j(\theta_j, \zeta_j), \\ \varphi_j = \varphi_\epsilon \ \text{ on } (U \times V) \setminus W, \qquad \lim\limits_{k\to\infty} \varphi_k = \varphi_\epsilon \ \text{ in } \mathrm{C}^2(U \times V). \end{cases} \tag{4.4}$$

In the theorem above, a possible choice of φ_ϵ is the function

$$\varphi_\epsilon(\theta, \zeta) = \varphi(\theta, \zeta) + \frac{\epsilon}{2}|\mathcal{A}(\theta - \hat{\theta}, \zeta - \hat{\zeta})|^2.$$

In what follows we fix a function $\chi^m \in \mathrm{C}^2(\mathbb{R}^m)$, where the superscript "m" indicates the dimension of the space $\mathbb{R}^m$, such that

$$0 \leq \chi^m(x) \leq 1 \quad \text{for all } \ x \in \mathbb{R}^m,$$
$$\chi^m(x) = \begin{cases} 1 & \text{if } \ x \in B_{1/4}, \\ 0 & \text{if } \ x \in \mathbb{R}^m \setminus B_{1/2}. \end{cases}$$

For any $r > 0$, we set

$$\chi_r^m(x) = \chi^m(x/r) \quad \text{for} \quad x \in \mathbb{R}^m,$$

so that $\chi_r^m(x) = 1$ if $x \in B_{r/4}$ and $\chi_r^m(x) = 0$ if $x \in \mathbb{R}^m \setminus B_{r/2}$. Here and later, $B_r = B_r^m$ and $\overline{B}_r = \overline{B}_r^m$ denotes the open and closed balls of $\mathbb{R}^m$ with radius r and center at the origin, respectively.

Lemma 4.3. *Let $R > 0$ and $f \in \mathrm{C}^2(\overline{B}_R)$. For $r \in (0, R)$ set $f_r(x) = \chi_r^m(x)f(x)$ for $x \in \overline{B}_R$. Assume that $f(0) = 0$ and $Df(0) = 0$.*

(i) We have

$$\sup_{r\in(0,\,R)} \|f_r\|_{\mathrm{C}^2(\overline{B}_R)} < \infty.$$

(ii) Assume, in addition, that $D^2 f(0) = 0$. Then

$$\lim_{r\to 0+} \|f_r\|_{\mathrm{C}^2(\overline{B}_R)} = 0.$$

Proof. By differentiation, we get

$$\begin{aligned} Df_r(x) &= r^{-1}D\chi^m(x/r)f(x) + \chi^m(x/r)Df(x), \\ D^2 f_r(x) &= r^{-2}D^2\chi^m(x/r)f(x) + r^{-1}D\chi^m(x/r)\otimes Df(x) + r^{-1}Df(x)\otimes D\chi^m(x/r) \\ &\quad + \chi^m(x/r)D^2 f(x), \end{aligned}$$

where, for $v = (v_1, \ldots, v_m), w = (w_1, \ldots, w_m) \in \mathbb{R}^m$, $v\otimes w$ denotes the $m\times m$ matrix with $v_i w_j$ as its (i, j) entry. Also, by the assumption that $f(0) = 0$ and $Df(0) = 0$, we have for all $x \in \overline{B}_r$,

$$|Df(x)| \le \|D^2 f\|_{\mathrm{C}(\overline{B}_r)} r \quad \text{and} \quad |f(x)| \le \|Df\|_{\mathrm{C}(\overline{B}_r)} r \le \|D^2 f\|_{\mathrm{C}(\overline{B}_r)} r^2.$$

Combining these, we obtain for all $x \in \overline{B}_r$,

$$\begin{aligned} |f_r(x)| &\le \|f\|_{\mathrm{C}(\overline{B}_r)}, \\ |Df_r(x)| &\le (\|D\chi^m\|_{\mathrm{C}(\overline{B}_1)} + 1)\|Df\|_{\mathrm{C}(\overline{B}_r)}, \\ \|D^2 f_r(x)\| &\le (\|D^2\chi^m\|_{\mathrm{C}(\overline{B}_1)} + 2\|D\chi^m\|_{\mathrm{C}(\overline{B}_1)} + 1)\|D^2 f\|_{\mathrm{C}(\overline{B}_r)}. \end{aligned}$$

Noting that $f_r(x) = 0$ for all $x \in \mathbb{R}^m \setminus B_r$, we conclude that

$$\|f_r\|_{\mathrm{C}^2(\overline{B}_R)} = \|f_r\|_{\mathrm{C}^2(\overline{B}_r)} \le C_0\|\chi^m\|_{\mathrm{C}^2(\overline{B}_1)}\|f\|_{\mathrm{C}^2(\overline{B}_r)}$$

for some absolute constant $C_0 > 0$. From this, the assertion (i) follows since $\|f\|_{\mathrm{C}^2(\overline{B}_r)} \le \|f\|_{\mathrm{C}^2(\overline{B}_R)}$, and, also, the assertion (ii) follows since $\lim_{r\to 0+} \|f\|_{\mathrm{C}^2(\overline{B}_r)} = 0$. ∎

Proof of Theorem 4.2. We follow the streamline of the proof of [11, Proposition 2].

1. First of all, we organize the situation to make the proof simple. We may assume by replacing u and v by the functions

$$\theta \mapsto u(\theta) - u(\hat\theta) - \langle \hat p, \theta - \hat\theta\rangle \quad \text{and} \quad \theta \mapsto v(\theta) - v(\hat\zeta) - \langle \hat q, \theta - \hat\zeta\rangle,$$

respectively, as well as φ by the function

$$(\theta,\zeta)\mapsto\varphi(\theta,\zeta)-\varphi(\hat\theta,\hat\zeta)-\langle\hat p,\theta-\hat\theta\rangle-\langle\hat q,\zeta-\hat\zeta\rangle$$

that

$$u(\hat\theta)=v(\hat\zeta)=0,\quad \varphi(\hat\theta,\hat\zeta)=0\quad\text{and}\quad \hat p=\hat q=0,$$

Furthermore, we may assume by translation that

$$\hat\theta=\hat\zeta=0.$$

Let $\delta>0$. We introduce functions $u_\delta\in\mathrm{USC}(U)$ and $v_\delta\in\mathrm{USC}(V)$ by

$$u_\delta(\theta)=u(\theta)-\frac{\delta}{2}|\theta|^2\quad\text{and}\quad v_\delta(\theta)=v(\theta)-\frac{\delta}{2}|\theta|^2.$$

We set $w_\delta(\theta,\zeta)=u_\delta(\theta)+v_\delta(\zeta)$ for $(\theta,\zeta)\in U\times V$.

We may assume by choosing $r\in(0,\ \delta\wedge R)$ small enough, so that φ is defined on $\overline B_r=\overline B_r^{2n}$, as a C^2 function, and so is the function φ_ϵ. We may replace r by a smaller $r=r(\delta)>0$, in view of the Taylor theorem, so that for all $(\theta,\zeta)\in\overline B_r\cap(U\times V)$,

$$\varphi(\theta,\zeta)\le\frac12\langle\mathcal{A}(\theta,\zeta),(\theta,\zeta)\rangle+\frac{\delta}{2}(|\theta|^2+|\zeta|^2)$$

and this inequality is strict if $(\theta,\zeta)\ne(0,0)$.

Now, we note that the function

$$w_\delta(\theta,\zeta)-\frac12\langle\mathcal{A}(\theta,\zeta),(\theta,\zeta)\rangle$$

attains a strict maximum value 0 at the origin $(0,0)$ over the set $\overline B_r\cap(U\times V)$. By using the Schwarz inequality, we compute that for all $\xi,\ \eta\in\mathbb{R}^n$,

$$\begin{aligned}\langle\mathcal{A}(\theta,\zeta),(\theta,\zeta)\rangle&=\langle\mathcal{A}(\xi,\eta)+\mathcal{A}(\theta-\xi,\zeta-\eta),(\xi,\eta)+(\theta-\xi,\zeta-\eta)\rangle\\&=\langle\mathcal{A}(\xi,\eta),(\xi,\eta)\rangle+2\langle\mathcal{A}(\xi,\eta),(\theta-\xi,\zeta-\eta)\rangle\\&\quad+\langle\mathcal{A}(\theta-\xi,\zeta-\eta),(\theta-\xi,\zeta-\eta)\rangle\\&\le\langle\mathcal{A}(\xi,\eta),(\xi,\eta)\rangle+2|\mathcal{A}(\xi,\eta)||(\theta-\xi,\zeta-\eta)|+|\mathcal{A}||(\theta-\xi,\zeta-\eta)|^2\\&\le\langle\mathcal{A}(\xi,\eta),(\xi,\eta)\rangle+\epsilon|\mathcal{A}(\xi,\eta)|^2+\left(\frac{1}{\epsilon}+|\mathcal{A}|\right)|(\theta-\xi,\zeta-\eta)|^2\\&\le\langle\mathcal{A}_\epsilon(\xi,\eta),(\xi,\eta)\rangle+\lambda(|\theta-\xi|^2+|\zeta-\eta|^2).\end{aligned}\tag{4.5}$$

Hence, we get

$$w_\delta(\theta,\zeta)-\frac{\lambda}{2}(|\theta-\xi|^2+|\zeta-\eta|^2)-\frac12\langle\mathcal{A}_\epsilon(\xi,\eta),(\xi,\eta)\rangle\le0\tag{4.6}$$

for all $(\theta,\zeta)\in\overline B_r\cap(U\times V)$ and $(\xi,\eta)\in\mathbb{R}^{2n}$, and this inequality is strict if $(\theta,\zeta)\ne(0,0)$.

2. We define $u_{\delta,\lambda}, v_{\delta,\lambda} : \mathbb{R}^n \to \mathbb{R}$ by

$$u_{\delta,\lambda}(\xi) = \max_{\theta\in\overline{B}_{r/2}\cap U}(u_\delta(\theta) - \frac{\lambda}{2}|\theta-\xi|^2) \quad \text{and} \quad v_{\delta,\lambda}(\xi) = \max_{\theta\in\overline{B}_{r/2}\cap V}(v_\delta(\theta) - \frac{\lambda}{2}|\theta-\xi|^2).$$

The functions $u_{\delta,\lambda}, v_{\delta,\lambda} : \mathbb{R}^n \to \mathbb{R}$ are the λ–sup-convolutions of u_δ, v_δ, respectively. Noting that $B^n_{r/2} \times B^n_{r/2} \subset B^{2n}_r$, we see that the above formulas define real-valued functions and that for all $(\xi,\eta) \in \mathbb{R}^{2n}$,

$$u_{\delta,\lambda}(\xi) + v_{\delta,\lambda}(\eta) - \frac{1}{2}\langle \mathcal{A}_\epsilon(\xi,\eta),(\xi,\eta)\rangle \le 0.$$

It is easy to see that $u_{\delta,\lambda}(0) = v_{\delta,\lambda}(0) = 0$. Accordingly, the function

$$u_{\delta,\lambda}(\xi) + v_{\delta,\lambda}(\eta) - \frac{1}{2}\langle \mathcal{A}_\epsilon(\xi,\eta),(\xi,\eta)\rangle$$

takes the maximum 0 at $(0,0)$ over $\mathbb{R}^{2n}$. We define the function $\Phi : \mathbb{R}^{2n} \to \mathbb{R}$ by

$$\Phi(\xi,\eta) = u_{\delta,\lambda}(\xi) + v_{\delta,\lambda}(\eta) - \frac{1}{4}(|\xi|^4 + |\eta|^4) - \frac{1}{2}\langle \mathcal{A}_\epsilon(\xi,\eta),(\xi,\eta)\rangle$$

and observe that the function Φ has a strict maximum at $(0,0)$ over $\mathbb{R}^{2n}$. For notational convenience, we put

$$f_\epsilon(\xi,\eta) = \frac{1}{2}\langle \mathcal{A}_\epsilon(\xi,\eta),(\xi,\eta)\rangle \quad \text{for} \quad (\xi,\eta) \in \mathbb{R}^{2n}.$$

Since the functions $u_{\delta,\lambda}(\xi) + (\lambda/2)|\xi|^2$ and $v_{\delta,\lambda}(\xi) + (\lambda/2)|\xi|^2$ are convex, as in [11], we see that there exist sequences $\{(\xi_k,\eta_k)\}_{k\in\mathbb{N}} \subset B_1 \times B_1$ and $\{(p_k,q_k)\}_{k\in\mathbb{N}} \subset \mathbb{R}^{2n}$ such that

$$\lim_{k\to\infty}(\xi_k,\eta_k) = \lim_{k\to\infty}(p_k,q_k) = (0,0),$$

and such that, for any $k \in \mathbb{N}$, if we set $\Phi_k(\xi,\eta) = \Phi(\xi,\eta) - \langle p_k,\xi\rangle - \langle q_k,\eta\rangle$ for $(\xi,\eta) \in \mathbb{R}^{2n}$, then

$$\max_{\overline{B}_1\times\overline{B}_1} \Phi_k = \Phi_k(\xi_k,\eta_k), \tag{4.7}$$

$$J^2\Phi_k(\xi_k,\eta_k) := J^{2,+}\Phi_k(\xi_k,\eta_k) \cap J^{2,-}\Phi_k(\xi_k,\eta_k) \ne \emptyset. \tag{4.8}$$

The latter of the above says that Φ_k has a second-order differential at (ξ_k,η_k).

3. By the definition of $u_{\delta,\lambda}$ and $v_{\delta,\lambda}$, there are points $\theta_k \in \overline{B}_{r/2} \cap U$ and $\zeta_k \in \overline{B}_{r/2} \cap V$ such that

$$\begin{aligned} u_{\delta,\lambda}(\xi_k) &= u_\delta(\theta_k) - \frac{\lambda}{2}|\theta_k - \xi_k|^2, \\ v_{\delta,\lambda}(\eta_k) &= v_\delta(\zeta_k) - \frac{\lambda}{2}|\zeta_k - \eta_k|^2. \end{aligned}$$

We intend to show that

$$\begin{cases} \lim_{k\to\infty}\theta_k = \lim_{k\to\infty}\zeta_k = 0, \\ \lim_{k\to\infty}u_\delta(\theta_k) = u_\delta(0) = 0, \\ \lim_{k\to\infty}v_\delta(\zeta_k) = v_\delta(0) = 0. \end{cases} \tag{4.9}$$

By (4.7), we get

$$\Phi_k(\xi_k, \eta_k) \geq \Phi_k(0,0) = u_{\delta,\lambda}(0) + v_{\delta,\lambda}(0) \geq u_\delta(0) + v_\delta(0) = 0.$$

Hence, we have

$$\begin{aligned} 0 \leq \Phi_k(0,0) \leq & u_\delta(\theta_k) + v_\delta(\zeta_k) - \frac{\lambda}{2}(|\theta_k - \xi_k|^2 + |\zeta_k - \eta_k|^2) \\ & - \frac{1}{4}(|\xi_k|^4 + |\eta_k|^4) - f_\epsilon(\xi_k, \eta_k) - \langle p_k, \xi_k \rangle - \langle q_k, \eta_k \rangle. \end{aligned}$$

For any convergent subsequence $\{(\theta_{k_m}, \zeta_{k_m})\}_{m\in\mathbb{N}}$ of $\{(\theta_k, \zeta_k)\}_{k\in\mathbb{N}}$, setting

$$\lim_{m\to\infty} (\theta_{k_m}, \zeta_{k_m}) = (\bar{\theta}, \bar{\zeta})$$

and noting that w_δ is upper semicontinuous at $(0,0)$, from the inequality above, we get

$$\frac{\lambda}{2}(|\bar{\theta}|^2 + |\bar{\zeta}|^2) \leq \liminf_{m\to\infty} w_\delta(\theta_{k_m}, \zeta_{k_m}) \leq \limsup_{m\to\infty} w_\delta(\theta_{k_m}, \zeta_{k_m}) \leq w_\delta(\bar{\theta}, \bar{\zeta}).$$

Since the inequality (4.6) is strict if $(\theta, \zeta) \neq (0,0)$, the above inequality ensures that $(\bar{\theta}, \bar{\zeta}) = (0,0)$, and moreover,

$$\liminf_{m\to\infty} w_\delta(\theta_{k_m}, \zeta_{k_m}) = \limsup_{m\to\infty} w_\delta(\theta_{k_m}, \zeta_{k_m}) = w_\delta(0,0) = 0.$$

This observation combined with a simple argument by contradiction assures that

$$\lim_{k\to\infty} w_\delta(\theta_k, \zeta_k) = 0.$$

Combining this with the fact that $\limsup_{k\to\infty} u_\delta(\theta_k) \leq 0$ and $\limsup_{k\to\infty} v_\delta(\zeta_k) \leq 0$, we conclude that

$$\lim_{k\to\infty} u_\delta(\theta_k) = 0 = u_\delta(0) \quad \text{and} \quad \lim_{k\to\infty} v_\delta(\zeta_k) = 0 = v_\delta(0),$$

and that (4.9) is valid.

4. Towards the end of the proof, we convert the conditions (4.7) and (4.8) into those at the points (θ_k, ζ_k) with an appropriate choice of functions φ_k. Replacing $r = r(\delta)$ by a smaller number and relabeling the sequence

$$\{(\xi_k, \eta_k, p_k, q_k, \theta_k, \zeta_k)\}_{k\in\mathbb{N}},$$

we may assume that $0 < r < 1$ and $\{\theta_k, \xi_k, \zeta_k, \eta_k\} \subset B_{r/4}$ for all $k \in \mathbb{N}$. Consequently, if $\theta, \zeta \in B_{r/2}$, then $\theta - \theta_k + \xi_k$, $\zeta - \zeta_k + \eta_k \in B_1$ for all $k \in \mathbb{N}$. Hence, by (4.7), we have for all $(\theta, \zeta) \in (\overline{B}_{r/2} \times \overline{B}_{r/2}) \cap (U \times V)$,

$$\Phi_k(\theta - \theta_k + \xi_k, \zeta - \zeta_k + \eta_k) \leq \Phi_k(\xi_k, \eta_k). \tag{4.10}$$

We define the function $\psi_k^\delta \in \mathrm{C}^\infty(\mathbb{R}^{2n})$ by

$$\begin{aligned}\psi_k^\delta(\theta,\zeta) &= \frac{\delta}{2}\left(|\theta|^2+|\zeta|^2\right)+\frac{\lambda}{2}\left(|\theta_k-\xi_k|^2+|\zeta_k-\eta_k|^2\right)\\ &\quad+\frac{1}{4}\left(|\theta-\theta_k+\xi_k|^4+|\zeta-\zeta_k+\eta_k|^4\right)+f_\epsilon(\theta-\theta_k+\xi_k,\zeta-\zeta_k+\eta_k)\\ &\quad+\langle p_k,\theta-\theta_k+\xi_k\rangle+\langle q_k,\zeta-\zeta_k+\eta_k\rangle.\end{aligned}$$

By this definition and the choice of θ_k and ζ_k, we have

$$w(\theta_k,\zeta_k)-\psi_k^\delta(\theta_k,\zeta_k)=\Phi_k(\xi_k,\eta_k).$$

By the definition of $u_{\delta,\lambda}$ and $v_{\delta,\lambda}$, we have for all $(\theta,\zeta)\in(\overline{B}_{r/2}\times\overline{B}_{r/2})\cap(U\times V)$,

$$\begin{aligned}u_{\delta,\lambda}(\theta-\theta_k+\xi_k) &\geq u(\theta)-\frac{\delta}{2}|\theta|^2-\frac{\lambda}{2}|\theta_k-\xi_k|^2,\\ v_{\delta,\lambda}(\zeta-\zeta_k+\eta_k) &\geq v(\zeta)-\frac{\delta}{2}|\zeta|^2-\frac{\lambda}{2}|\zeta_k-\eta_k|^2,\end{aligned}\tag{4.11}$$

and hence,

$$w(\theta,\zeta)-\psi_k^\delta(\theta,\zeta)\leq\Phi_k(\theta-\theta_k+\xi_k,\zeta-\zeta_k+\eta_k).$$

Combining this with (4.10) and (4.11) yields

$$w(\theta,\zeta)-\psi_k^\delta(\theta,\zeta)\leq w(\theta_k,\zeta_k)-\psi_k^\delta(\theta_k,\zeta_k)\tag{4.12}$$

for all $(\theta,\zeta)\in(\overline{B}_{r/2}\times\overline{B}_{r/2})\cap(U\times V)$.

By assumption, we have for all $(\theta,\zeta)\in U\times V$,

$$w(\theta,\zeta)-\varphi(\theta,\zeta)\leq w(0,0)-\varphi(0,0)=0,$$

which implies that

$$w(\theta,\zeta)-\varphi_\epsilon(\theta,\zeta)\leq 0.$$

From this and (4.12), setting

$$\varphi_k^\delta(\theta,\zeta)=(1-\chi_r^{2n}(\theta,\zeta))\varphi_\epsilon(\theta,\zeta)+\chi_r^{2n}(\theta,\zeta)(\psi_k^\delta(\theta,\zeta)+w(\theta_k,\zeta_k)-\psi_k^\delta(\theta_k,\zeta_k)),$$

where the function χ_r^{2n} is chosen as in Lemma 4.3, we get

$$(w-\varphi_k^\delta)(\theta,\zeta)\leq 0\quad\text{ for all }\ (\theta,\zeta)\in U\times V.$$

Since $\theta_k,\zeta_k\in B_{r/4}$ and $\chi_r^{2n}(\theta_k,\zeta_k)=1$, we have $(w-\varphi_k^\delta)(\theta_k,\zeta_k)=0$ and therefore

$$\max_{U\times V}(w-\varphi_k^\delta)=(w-\varphi_k^\delta)(\theta_k,\zeta_k).$$

It is easily seen that

$$\lim_{k\to\infty}\varphi_k^\delta=(1-\chi_r^{2n})\varphi_\epsilon+\chi_r^{2n}\psi^\delta\quad\text{ in }\ \mathrm{C}^2(U\times V),\tag{4.13}$$

where $\psi^\delta \in \mathrm{C}^2(U \times V)$ is the function defined as

$$\psi^\delta(\theta,\zeta) = \frac{\delta}{2}(|\theta|^2 + |\zeta|^2) + \frac{1}{4}(|\theta|^4 + |\zeta|^4) + f_\epsilon(\theta,\zeta).$$

Recall that $r = r(\delta) \in (0,\delta)$. Applying Lemma 4.3, we deduce that

$$\lim_{\delta\to 0+} \chi_r^{2n}(\psi^\delta - \varphi_\epsilon) = 0 \quad \text{in } \mathrm{C}^2(U \times V).$$

Thus, we have

$$\lim_{\delta\to 0+}((1-\chi_r^{2n})\varphi_\epsilon + \chi_r^{2n}\psi^\delta) = \varphi_\epsilon \quad \text{in } \mathrm{C}^2(U \times V). \tag{4.14}$$

According to (4.8), there exist $(p_k^{\delta,\lambda}, \mathcal{X}_k^{\delta,\lambda}) \in J^2 u_{\delta,\lambda}(\xi_k)$ and $(q_k^{\delta,\lambda}, \mathcal{Y}_k^{\delta,\lambda}) \in J^2 v_{\delta,\lambda}(\eta_k)$. Furthermore, by (4.7), we obtain

$$\begin{aligned}
p_k^{\delta,\lambda} &= p_k + D_\xi f_\epsilon(\xi_k,\eta_k) + |\xi_k|^2\xi_k,\\
q_k^{\delta,\lambda} &= q_k + D_\eta f_\epsilon(\xi_k,\eta_k) + |\eta_k|^2\eta_k,
\end{aligned}$$

$$\begin{pmatrix} \mathcal{X}_k^{\delta,\lambda} & 0 \\ 0 & \mathcal{Y}_k^{\delta,\lambda} \end{pmatrix} \le D^2 f_\epsilon(\xi_k,\eta_k) + \begin{pmatrix} |\xi_k|^2 I_n + 2\xi_k\otimes\xi_k & 0 \\ 0 & |\eta_k|^2 I_n + 2\eta_k\otimes\eta_k \end{pmatrix}. \tag{4.15}$$

As one of basic properties of sup-convolution (see e.g. [11, Lemma 4], [12, Lemma A.5]), we have

$$(p_k^{\delta,\lambda}, \mathcal{X}_k^{\delta,\lambda}) \in J^{2,+}u_\delta(\theta_k) \quad \text{and} \quad (q_k^{\delta,\lambda}, \mathcal{Y}_k^{\delta,\lambda}) \in J^{2,+}v_\delta(\zeta_k),$$

which yields

$$(p_k^{\delta,\lambda} + \delta\theta_k, \mathcal{X}_k^{\delta,\lambda} + \delta I_n) \in J^{2,+}u(\theta_k) \quad \text{and} \quad (q_k^{\delta,\lambda} + \delta\zeta_k, \mathcal{Y}_k^{\delta,\lambda} + \delta I_n) \in J^{2,+}v(\zeta_k). \tag{4.16}$$

Recalling that the functions $u_{\delta,\lambda}(\xi) + (\lambda/2)|\xi|^2$ and $v_{\delta,\lambda}(\xi) + (\lambda/2)|\xi|^2$ are convex, we see that

$$\mathcal{X}_k^{\delta,\lambda} \ge -\lambda I_n \quad \text{and} \quad \mathcal{Y}_k^{\delta,\lambda} \ge -\lambda I_n. \tag{4.17}$$

Noting that

$$\varphi_k^\delta(\theta,\zeta) = \psi_k^\delta(\theta,\zeta) + w(\theta_k,\zeta_k) - \psi_k^\delta(\theta_k,\zeta_k)$$

in a neighborhood of (θ_k,ζ_k), we see that

$$\begin{aligned}
D\varphi_k^\delta(\theta_k,\zeta_k) &= D\psi_k^\delta(\theta_k,\zeta_k) = \delta(\theta_k,\zeta_k) + (|\xi_k|^2\xi_k, |\eta_k|^2\eta_k) + Df_\epsilon(\xi_k,\eta_k) + (p_k,q_k)\\
&= (\delta\theta_k + p_k^{\delta,\lambda}, \delta\zeta_k + q_k^{\delta,\lambda}),\\
D^2\varphi_k^\delta(\theta_k,\zeta_k) &= \delta I_{2n} + D^2 f_\epsilon(\xi_k,\eta_k) + \begin{pmatrix} |\xi_k|^2 I_n + 2\xi_k\otimes\xi_k & 0 \\ 0 & |\eta_k|^2 I_n + 2\eta_k\otimes\eta_k \end{pmatrix}.
\end{aligned} \tag{4.18}$$

Henceforth we take care of the dependence on δ of p_k, q_k, ξ_k, η_k, θ_k, and ζ_k and write p_k^δ, q_k^δ, ξ_k^δ, η_k^δ, θ_k^δ, and ζ_k^δ for them, respectively. We fix a sequence $\{\delta_j\}$ of positive numbers converging to zero. For instance, we may set $\delta = 1/j$ for $j \in \mathbb{N}$. We choose a sequence $\{k_j\}$ of positive integers diverging to infinity so that

$$\lim_{j\to\infty}(p_{k_j}^{\delta_j}, q_{k_j}^{\delta_j}, \xi_{k_j}^{\delta_j}, \eta_{k_j}^{\delta_j}, \theta_{k_j}^{\delta_j}, \zeta_{k_j}^{\delta_j}) = (0,0,0,0,0,0) \tag{4.19}$$

Thanks to (4.13) and (4.14), we may assume that

$$\lim_{j\to\infty}\varphi_{k_j}^{\delta_j} = \varphi_\epsilon \quad \text{in } \mathrm{C}^2(U\times V). \tag{4.20}$$

With obvious abuse of notation, we set

$$\theta_j = \theta_{k_j}^{\delta_j}, \quad \zeta_j = \zeta_{k_j}^{\delta_j}, \quad \varphi_j = \varphi_{k_j}^{\delta_j}, \quad \mathcal{X}_j = \mathcal{X}_{k_j}^{\delta_j,\lambda} + \delta_j I_n, \quad \text{and} \quad \mathcal{Y}_j = \mathcal{Y}_{k_j}^{\delta_j,\lambda} + \delta_j I_n.$$

we see from (4.15), (4.17), (4.18) and (4.16) that

$$\begin{gathered} -\lambda I_{2n} \le \begin{pmatrix} \mathcal{X}_j & 0 \\ 0 & \mathcal{Y}_j \end{pmatrix} \le D^2\varphi_j(\theta_j,\zeta_j), \\ (D_\theta\varphi_j(\theta_j,\zeta_j),\mathcal{X}_j) \in J^{2,+}u(\theta_j) \quad \text{and} \quad (D_\zeta\varphi_j(\theta_j,\zeta_j),\mathcal{Y}_j) \in J^{2,+}v(\zeta_j). \end{gathered}$$

Finally, recalling (4.19) and (4.20), we conclude that the sequence $\{(\theta_j,\zeta_j,\varphi_j,\mathcal{X}_j,\mathcal{Y}_j)\}$ has all the required properties. ∎

In the proof of our comparison theorem below, we use the following variant of Theorem 4.2. Let n, U, V, φ be as in Theorem 4.2. We consider the situation where

$$\begin{cases} \text{there are } n_1, n_2 \in \mathbb{N},\ U_1, V_1 \subset \mathbb{R}^{n_1},\ U_2, V_2 \subset \mathbb{R}^{n_2},\ \varphi_1 \in \mathrm{C}^2(U_1\times V_1) \text{ and} \\ \varphi_2 \in \mathrm{C}^2(U_2\times V_2) \text{ such that} \\ \qquad n = n_1 + n_2, \quad U = U_1\times U_2, \quad V = V_1\times V_2, \\ \text{and for all } \theta = (\theta_1,\theta_2) \in U_1\times U_2,\ \zeta = (\zeta_1,\zeta_2) \in V_1\times V_2, \\ \qquad \varphi(\theta,\zeta) = \varphi_1(\theta_1,\zeta_1) + \varphi_2(\theta_2,\zeta_2). \end{cases} \tag{4.21}$$

Here and afterwards, with a little abuse of notation, we write $\theta_1 = (\theta_1,\dots,\theta_{n_1})$, $\theta_2 = (\theta_{n_1+1},\dots,\theta_n)$ for $\theta = (\theta_1,\dots,\theta_n) \in \mathbb{R}^n$ and so on.

Define $\tilde\varphi_1, \tilde\varphi_2 \in \mathrm{C}^2(U\times V)$ by setting

$$\tilde\varphi_1(\theta,\zeta) = \varphi(\theta_1,\zeta_1) \quad \text{and} \quad \tilde\varphi_2(\theta,\zeta) = \varphi_2(\theta_2,\zeta_2) \tag{4.22}$$

for $\theta = (\theta_1,\theta_2) \in U_1\times U_2$ and $\tau = (\tau_1,\tau_2) \in U_2\times V_2$.

Given a pair $\epsilon = (\epsilon_1,\epsilon_2)$ of positive numbers and two matrices $\mathcal{A}_1$, $\mathcal{A}_2 \in \mathbb{S}_{2n}$, we set

$$\begin{cases} \mathcal{A}_\epsilon := \mathcal{A}_1 + \mathcal{A}_2 + \epsilon_1\mathcal{A}_1^2 + \epsilon_2\mathcal{A}_2^2, \\ \lambda_i := \dfrac{1}{\epsilon_i} + |\mathcal{A}_i| \quad \text{for } i = 1,2, \quad \lambda := (\lambda_1,\lambda_2), \\ E_\lambda := \lambda_1 I_{n_1} \oplus \lambda_2 I_{n_2} \oplus \lambda_1 I_{n_1} \oplus \lambda_2 I_{n_2} = \begin{pmatrix} \lambda_1 I_{n_1} & 0 & 0 & 0 \\ 0 & \lambda_2 I_{n_2} & 0 & 0 \\ 0 & 0 & \lambda_1 I_{n_1} & 0 \\ 0 & 0 & 0 & \lambda_2 I_{n_2} \end{pmatrix}. \end{cases} \tag{4.23}$$

Corollary 4.4. *Let U, V, u, v, φ, w, $\hat\theta$, $\hat\zeta$ and W be as in Theorem 4.2, and assume that (4.2) and (4.21) hold. Define $\tilde\varphi_i$, with $i = 1, 2$, by (4.22) and set $\mathcal{A}_i = D^2\tilde\varphi_i(\hat\theta, \hat\zeta)$ for $i = 1, 2$. Define $\mathcal{A}_\epsilon$, $\lambda = (\lambda_1, \lambda_2)$ and E_λ by (4.23), and select φ_ϵ so that all the conditions of (4.3) are satisfied. Then the same conclusion as Theorem 4.2 is valid with the current E_λ.*

Proof. We need only to follow the proof of Theorem 4.2, with minors changes. We here give a few details how to modify it to adapt to our proof.

We set

$$\mathcal{A} := D^2\varphi(\hat\theta, \hat\zeta) = \mathcal{A}_1 + \mathcal{A}_2,$$

use the notation: for $\theta = (\theta_1, \theta_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$

$$\tilde\theta_1 = (\theta_1, 0), \quad \tilde\theta_2 = (0, \theta_2) \in \mathbb{R}^n,$$

and note that for any $\theta = (\theta_1, \theta_2), \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and some $\xi = (\xi_1, \xi_2), \eta = (\eta_1, \eta_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$\mathcal{A}_i(\theta, \zeta) = \mathcal{A}_i(\tilde\theta_i, \tilde\zeta_i) = (\tilde\xi_i, \tilde\eta_i) \quad \text{for } i = 1, 2.$$

Moreover, we compute similarly to (4.5) that for $\theta = (\theta_1, \theta_2)$, $\zeta = (\zeta_1, \zeta_2)$, $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$\begin{aligned}
\langle \mathcal{A}(\theta, \zeta), (\theta, \zeta)\rangle &= \sum_{i=1,2} \langle \mathcal{A}_i(\tilde\theta_i, \tilde\zeta_i), (\tilde\theta_i, \tilde\zeta_i)\rangle \\
&\le \sum_{i=1,2} \langle \mathcal{A}_i(\tilde\xi_i, \tilde\eta_i), (\tilde\xi_i, \tilde\eta_i)\rangle + 2|\mathcal{A}_i(\tilde\xi_i, \tilde\eta_i)||(\tilde\theta_i - \tilde\xi_i, \tilde\zeta_i - \tilde\eta_i)| + |\mathcal{A}_i||(\tilde\theta_i - \tilde\xi_i, \tilde\zeta_i - \tilde\eta_i)|^2 \\
&\le \sum_{i=1,2} \langle \mathcal{A}_i(\tilde\xi_1, \tilde\eta_i), (\tilde\xi_i, \tilde\eta_i)\rangle + \epsilon_i|\mathcal{A}_i(\tilde\xi_i, \tilde\eta_i)|^2 + \left(\frac{1}{\epsilon_i} + |\mathcal{A}_i|\right)|(\tilde\theta_i - \tilde\xi_i, \tilde\zeta_i - \tilde\eta_i)|^2 \\
&\le \langle \mathcal{A}_\epsilon(\xi, \eta), (\xi, \eta)\rangle + \sum_{i=1,2} \lambda_i|(\tilde\theta_i - \tilde\xi_i, \tilde\zeta_i - \tilde\eta_i)|^2 \\
&= \langle \mathcal{A}_\epsilon(\xi, \eta), (\xi, \eta)\rangle + d_\lambda(\theta - \xi) + d_\lambda(\zeta - \eta),
\end{aligned}$$

where

$$d_\lambda(\xi) = d_\lambda(\xi_1, \xi_2) := \lambda_1|\xi_1|^2 + \lambda_2|\xi_2|^2.$$

The definition of $u_{\delta,\lambda}, v_{\delta,\lambda} : \mathbb{R}^n \to \mathbb{R}$ should be modified as follow:

$$u_{\delta,\lambda}(\xi) = \max_{\theta \in \overline{B}_{r/2} \cap U} \left(u_\delta(\theta) - \frac{1}{2}d_\lambda(\theta - \xi)\right) \quad \text{and} \quad v_{\delta,\lambda}(\xi) = \max_{\theta \in \overline{B}_{r/2} \cap V} \left(v_\delta(\theta) - \frac{1}{2}d_\lambda(\theta - \xi)\right).$$

After fixing ξ_k, η_k in the course of the proof, the choice of θ_k, ζ_k is done in the same spirit as in the proof of Theorem 4.2, to satisfy

$$u_{\delta,\lambda}(\xi_k) = u_\delta(\theta_k) - \frac{1}{2}d_\lambda(\theta_k - \xi_k) \quad \text{and} \quad v_{\delta,\lambda}(\eta_k) = v_\delta(\zeta_k) - \frac{1}{2}d_\lambda(\zeta_k - \eta_k).$$

We note that the functions $u_{\delta,\lambda}(\xi) + \frac{1}{2}d_\lambda(\xi)$ and $v_{\delta,\lambda}(\xi) + \frac{1}{2}d_\lambda(\xi)$ are convex on $\mathbb{R}^n$, which, instead of (4.17), yield

$$\mathcal{X}_k^{\delta,\lambda} \ge -\frac{1}{2}D^2 d_\lambda(\xi_k) = -\lambda_1 I_{n_1} \oplus \lambda_2 I_{n_2} \quad \text{and} \quad \mathcal{Y}_k^{\delta,\lambda} \ge -\frac{1}{2}D^2 d_\lambda(\eta_k) = -\lambda_1 I_{n_1} \oplus \lambda_2 I_{n_2}.$$

With these modifications, the proof goes parallel to that of Theorem 4.2. ■

5. Comparison Principle. The uniqueness of the solution of the IDE follows from the following comparison theorem:

Theorem 5.1. *Let $0 \le \kappa < \lambda$. Let $u \in \mathrm{USC}(\overline{Q}_T) \cap \mathcal{V}_\kappa$ and $v \in \mathrm{LSC}(\overline{Q}_T) \cap \mathcal{V}_\kappa$ be a subsolution and supersolution of (1.6), respectively. Assume that*

$$u(x,y,T) \le v(x,y,T) \qquad \text{for all } (x,y) \in \mathbb{R}^d \times \mathbb{S}_d^+. \tag{5.1}$$

Then,

$$u \le v \qquad \text{on } \overline{Q}_T. \tag{5.2}$$

As stated above, Theorem 5.1 has the following consequence.

Corollary 5.2. *If h is a bounded and Lipschitz continuous function on $\overline{Q}_T$, then the value function u_0 is in $\mathrm{C}(\overline{Q}_T) \cap \mathcal{V}_0$ and a unique viscosity solution of (1.6) satisfying the initial condition $u_0(x,y,0) = h(x,y,0)$. The uniqueness is valid among functions in $\mathcal{V}_\kappa$, with $0 \le \kappa < \lambda$.*

Proof. Every claims except the uniqueness are included in Proposition 2.2 and Theorem 3.7. The uniqueness claim is also immediate from Theorem 5.1. ∎

The following limiting lemma, which has a similar nature to Lemma 3.3, is useful in our proof of the theorem above.

Lemma 5.3. *Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence of Borel measurable functions on Q_T and $\{\theta_j\}_{j\in\mathbb{N}} \subset Q_T$. Set $\theta_j = (x_j, y_j, t_j) \in \mathbb{R}^d \times \mathbb{S}_d^+ \times \mathbb{R}$. Assume that $\{\theta_j\}_{j\in\mathbb{N}}$ converges to a point $\theta_0 \in Q_T$ and that there is a constant $C > 0$ such that*

$$|u_j(x,y,t)| \le C(1 + |x| + \|y\|)^\lambda \quad \text{for all } (x,y,t) \in Q_T,\ j \in \mathbb{N}.$$

Let $K \subset Q_T$ be a compact neighborhood of θ_0 and $\{\varphi_j\}_{j\in\mathbb{N}} \subset \mathrm{C}^1(K)$. Assume that for every $j \in \mathbb{N}$, $u_j - \varphi_j$ takes a global maximum (resp., minimum) on K at θ_j and that $\{\varphi_j\}_{j\in\mathbb{N}}$ is bounded in $\mathrm{C}^1(K)$. Then,

$$\int_{\mathbb{S}_d^+} \limsup_{j\to\infty} (u_j(x_j, y_j + z, t_j) - u_j(\theta_j))\nu(dz) \ge \limsup_{j\to\infty} Ju_j(\theta_j),$$
$$\left(\text{resp., } \int_{\mathbb{S}_d^+} \liminf_{j\to\infty} (u_j(x_j, y_j + z, t_j) - u_j(\theta_j))\nu(dz) \le \liminf_{j\to\infty} Ju_j(\theta_j).\right)$$

It should be noted that, in the above inequalities, it can be that $\limsup_{j\to\infty} Ju_j(\theta_j) = -\infty$, or $\liminf_{j\to\infty} Ju_j(\theta_j) = \infty$.

Proof. We only prove the claim which concerns "maximum". The other case can be treated similarly.

Note that for all $z \in \mathbb{S}_d^+$, $j \in \mathbb{N}$,

$$u_j(x_j, y_j + z, t_j) - u_j(\theta_j) \le C((1 + |x_j| + \|y_j + z\|)^\lambda + (1 + |x| + \|y_j\|)^\lambda \le C_1(1 + \|z\|)^\lambda,$$

where C_1 is a positive constant independent of j. Since the sequence $\{\theta_j\}$ is convergent to θ_0, we may choose $\delta > 0$ and $j_0 \in \mathbb{N}$ so that the δ-neighborhood of the set $\{\theta_j\}_{j>j_0}$ is contained in

K. Henceforth we are concerned with $j \in \mathbb{N}$ is larger than j_0. Since θ_j is a global maximum point of $u_j - \varphi_j$ on K, we have

$$u_j(x_j, y_j + z, t_j) - u_j(\theta_j) \le \varphi_j(x_j, y_j + z, t_j) - \varphi_j(\theta_j) \quad \text{for all } z \in \mathbb{S}_d^+, \text{ with } \|z\| \le \delta.$$

By assumption, the sequence $\{D\varphi_j\}_{j\in\mathbb{N}}$ is bounded in $\mathrm{C}(K, \mathbb{R}^{d+d^2+1})$, which ensures that there are constants $\delta > 0$ and $C_2 > 0$ such that

$$\varphi_j(x_j, y_j + z, t_j) - \varphi_j(\theta_j) \le C_2\|z\| \quad \text{for all } z \in \mathbb{S}_d^+, \text{ with } \|z\| \le \delta,\ j > j_0.$$

We define $f : \mathbb{S}_d^+ \to \mathbb{R}$ by

$$f(z) = \begin{cases} C_1(1 + \|z\|)^\lambda & \text{if } \|z\| > \delta, \\ C_2\|z\| & \text{otherwise.} \end{cases}$$

Noting that f is integrable with respect to the measure ν and that

$$u_j(x_j, y_j + z, t_j) - u_j(\theta_j) \le f(z) \quad \text{for all } z \in \mathbb{S}_d^+,$$

we deduce by Fatou's lemma that

$$\int_{\mathbb{S}_d^+} \limsup_{j\to\infty}(u_j(x_j, y_j + z, t_j) - u_j(\theta_j))\nu(dz) \ge \limsup_{j\to\infty} Ju_j(\theta_j),$$

which completes the proof. ■

Proof of Theorem 5.1. We divide our proof into four steps. In the first step, we arrange that $u - v$ takes a maximum at a point in Q_T.

1. We introduce functions $\rho : \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{R}$ and $f : \overline{Q}_T \to \mathbb{R}$ given by

$$\rho(x, y) = (1 + |x|^2 + \|y\|^2)^{\lambda/2}, \quad f(x, y, t) = \rho(x, y)e^{-Ct},$$

where $C > 0$ is a constant to be determined later. A simple computation shows that $\partial_t f = -Cf$ on Q_T and for all $(x, y) \in \mathbb{R}^d \times \mathbb{S}_d$ and some constant $C_0 > 0$,

$$|D_x\rho(x, y)| + \|D_x^2\rho(x, y)\| + \|D_y\rho(x, y)\| \le C_0\rho(x, y)(1 + |x| + \|y\|)^{-1}.$$

It is then easy to check that for some constant $C_1 > 0$,

$$Lf \le C_1 f \quad \text{on } Q_T.$$

Observe that for $(x, y) \in \mathbb{R}^d \times \mathbb{S}_d$ and $z \in \mathbb{S}_d$,

$$\rho(x, y + z) \le \rho(x, y) + \rho(0, z),$$

and, if $\|z\| < 1$, then for some constant $C_2 > 0$,

$$\rho(x, y + z) - \rho(x, y) \le \max_{\eta\in\mathbb{S}_d^+, \|\eta\|\le<1} \|D_y\rho(x, y + \eta)\|\|z\| \le C_2\rho(x, y)\|z\|.$$

From these, assuming $C_2 \geq 1$, we find that for all $(x, y) \in \mathbb{R}^d \times \mathbb{S}_d$,

$$\begin{aligned} J\rho(x,y) &\leq C_2\rho(x,y)\int_{\mathbb{S}_d^+, \|z\|<1} \|z\|\nu(dz) + \int_{\mathbb{S}_d^+, \|z\|\geq 1} \|\rho(0,z)\nu(dz) \\ &\leq C_2\rho(x,y)\Big(\int_{\mathbb{S}_d^+, \|z\|<1} \|z\|\nu(dz) + \int_{\mathbb{S}_d^+, \|z\|\geq 1} \|\rho(0,z)\nu(dz)\Big). \end{aligned}$$

Hence, for some $C_3 > 0$, we have

$$Jf \leq C_3 f \quad \text{on } Q_T.$$

Thus, we have

$$\mathcal{M}f \geq (C - C_1 - C_3)f \quad \text{on } Q_T.$$

We choose $C = C_1 + C_3$ so that

$$\mathcal{M}f \geq 0 \quad \text{on } Q_T.$$

For $\epsilon > 0$ we define u_ϵ on $\overline{Q}_T$ by

$$u_\epsilon(x,y,t) = u(x,y,t) - \epsilon f(x,y,t).$$

It is enough to show that for any $\epsilon > 0$,

$$u_\epsilon(x,y,t) \leq v(x,y,t) \quad \text{for all } (x,y,t) \in \overline{Q}_T.$$

Since f is a classical solution of

$$\mathcal{M}f \geq 0 \quad \text{on } Q_T$$

and $f > 0$ on Q_T, u_ϵ is a viscosity subsolution of

$$\min\{\mathcal{M}u_\epsilon, u_\epsilon - h\} = 0 \quad \text{on } Q_T.$$

Furthermore, there exist constants $\delta > 0$ and $C_4 > 0$ such that

$$u_\epsilon(x,y,t) \leq -\delta(|x| + \|y\|)^\lambda + C_4 \quad \text{for all } (x,y,t) \in \overline{Q}_T.$$

Thus, by replacing u by u_ϵ if needed, we may assume that for some constants $\delta > 0$ and $C > 0$,

$$u(x,y,t) \leq -\delta(|x|^2 + \|y\|^2) + C \quad \text{for all } (x,y,t) \in \overline{Q}_T. \tag{5.3}$$

Note however that we do not have $u \in \mathcal{V}_\kappa$ anymore and $u \in \mathcal{V}_\lambda$ instead.

If we set $u^\gamma(x,y,t) := u(x,y,t) + \gamma(t - T)$ with $\gamma > 0$, then $u^\gamma \leq u$ on $\overline{Q}_T$ and u^γ is a subsolution of

$$\min\{\mathcal{M}u^\gamma + \gamma, u^\gamma - h\} = 0 \text{ on } Q_T.$$

To show (5.2), it is enough to prove that for every $\gamma > 0$, $u^\gamma(x,y,t) \leq v(x,y,t)$ for all $(x,y,t) \in \overline{Q}_T$. Thus, we may henceforth assume, by replacing u by u^γ that u is a subsolution of

$$\min\{\mathcal{M}u + \gamma, u - h\} = 0 \text{ on } Q_T.$$

2. We start the contradiction argument to prove (5.2) and suppose that

$$\sup_{(x,y,t)\in\overline{Q}_T} u(x,y,t) - v(x,y,t) > 0. \tag{5.4}$$

Let $\alpha > 0, \beta > 0$, and consider the function

$$\Phi(x,y,t;\xi,\eta,\tau) := u(x,y,t) - v(\xi,\eta,\tau) - \alpha|x-\xi|^2 - \beta\|y-\eta\|^2 - \beta(t-\tau)^2$$

on $\overline{Q}_T \times \overline{Q}_T$. Taking into account of (5.3) and also the fact that $v \in \mathcal{V}$ and $u, -v \in \mathrm{USC}(\overline{Q}_T)$, the function Φ achieves a maximum. Let $(x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}, \xi_{\alpha\beta}, \eta_{\alpha\beta}, \tau_{\alpha\beta})$ be a maximum point of Φ. It is easily seen that as $(\alpha,\beta) \to (\infty,\infty)$, the points $(x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}, \xi_{\alpha\beta}, \eta_{\alpha\beta}, \tau_{\alpha\beta})$ stay bounded. Also, we have

$$\sup_{\alpha>1,\beta>1} (\alpha|x_{\alpha\beta}-\xi_{\alpha\beta}|^2 + \beta\|y_{\alpha\beta}-\eta_{\alpha\beta}\|^2 + \beta(t_{\alpha\beta}-\tau_{\alpha\beta})^2) < \infty.$$

Furthermore, for any sequence $\{(\alpha_k, \beta_k)\}$ such that

$$\lim_k \alpha_k = \infty, \quad \lim_k \beta_k = \infty,$$

there exists a subsequence such that, as $(\alpha,\beta) \to (\infty,\infty)$ along the subsequence,

$$(x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}, \xi_{\alpha\beta}, \eta_{\alpha\beta}, \tau_{\alpha\beta}) \to (x_0, y_0, t_0, x_0, y_0, t_0),$$

and moreover,

$$\begin{cases} \alpha|x_{\alpha\beta}-\xi_{\alpha\beta}|^2 + \beta\|y_{\alpha\beta}-\eta_{\alpha\beta}\|^2 + \beta(t_{\alpha\beta}-\tau_{\alpha\beta})^2 \to 0, \\ u(x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}) \to u(x_0,y_0,t_0), \\ v(\xi_{\alpha\beta}, \eta_{\alpha\beta}, \tau_{\alpha\beta}) \to v(x_0,y_0,t_0). \end{cases}$$

The last three claims on the convergence follow from the observation:

$$\begin{aligned} \max_{\overline{Q}_T}(u-v) &= \max_{(x,y,t)\in\overline{Q}_T} \Phi(x,y,t,x,y,t) \le \Phi(x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}, \xi_{\alpha\beta}, \eta_{\alpha\beta}, \tau_{\alpha\beta}) \\ &\le u((x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}) - v(\xi_{\alpha\beta}, \eta_{\alpha\beta}, \tau_{\alpha\beta}), \end{aligned}$$

and therefore,

$$\begin{aligned} \max_{\overline{Q}_T}(u-v) &\le \liminf_{(\alpha,\beta)\to(\infty,\infty)} \Phi((x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}, \xi_{\alpha\beta}, \eta_{\alpha\beta}, \tau_{\alpha\beta})) \\ &\le \limsup_{(\alpha,\beta)\to(\infty,\infty)} \Phi((x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}, \xi_{\alpha\beta}, \eta_{\alpha\beta}, \tau_{\alpha\beta})) \\ &\le u(x_0,y_0,t_0) - v(x_0,y_0,t_0) \le \max_{\overline{Q}_T}(u-v), \end{aligned}$$

where the liminf and limsup are taken along the subsequence selected above.

It now follows that (x_0, y_0, t_0) is a maximum point of $u - v$, which implies, together with (5.1) and (5.4), that $t_0 \neq T$. We may thus reselect sequences $\{\alpha_k\}_{k\in\mathbb{N}}$ and $\{\beta_m\}_{m\in\mathbb{N}}$ so that

$$(5.5)\quad \begin{cases} \lim\limits_{k\to\infty}\lim\limits_{m\to\infty}(x_{\alpha_k\beta_m}, y_{\alpha_k\beta_m}, t_{\alpha_k\beta_m}, \xi_{\alpha_k\beta_m}, \eta_{\alpha_k\beta_m}, \tau_{\alpha_k\beta_m}) = (x_0, y_0, t_0, x_0, y_0, t_0), \\ \lim\limits_{k\to\infty}\lim\limits_{m\to\infty}\alpha_k|x_{\alpha_k\beta_m} - \xi_{\alpha_k\beta_m}|^2 + \beta_m\|y_{\alpha_k\beta_m} - \eta_{\alpha_k\beta_m}\|^2 + \beta_m(t_{\alpha_k\beta_m} - \tau_{\alpha_k\beta_m})^2 = 0, \\ \lim\limits_{k\to\infty}\lim\limits_{m\to\infty}u(x_{\alpha_k\beta_m}, y_{\alpha_k\beta_m}, t_{\alpha_k\beta_m}) = u(x_0, y_0, t_0), \\ \lim\limits_{k\to\infty}\lim\limits_{m\to\infty}v(\xi_{\alpha_k\beta_m}, \eta_{\alpha_k\beta_m}, \tau_{\alpha_k\beta_m}) = v(x_0, y_0, t_0). \end{cases}$$

Furthermore, since $t_0 \neq T$, we may assume that for all α_k and β_m,

$$(5.6)\qquad (x_{\alpha_k,\beta_m}, y_{\alpha_k\beta_m}, t_{\alpha_k,\beta_m}) \in Q_T, \quad (\xi_{\alpha_k\beta_m}, \eta_{\alpha_k\beta_m}\tau_{\alpha_k\beta_m}) \in Q_T.$$

Also, since $u(x_0, y_0, t_0) > v(x_0, y_0, t_0)$ by (5.4), we may assume in view of (5.6) that for all α_k and β_m,

$$(5.7)\qquad (u - h)(x_{\alpha_k\beta_m}, y_{\alpha_k\beta_m}, t_{\alpha_k\beta_m}) > (v - h)(\xi_{\alpha_k\beta_m}, \eta_{\alpha_k\beta_m}, \tau_{\alpha_k\beta_m}).$$

We fix $k, m \in \mathbb{N}$, write $\alpha = \alpha_k$ and $\beta = \beta_m$ for notational simplicity, and intend to apply Corollary 4.4 to u, $-v$. For this, we set

$$\begin{cases} n = d + d^2 + 1, \quad n_1 = d, \quad n_2 = d^2 + 1, \\ U = V = Q_T, \quad U_1 = V_1 = \mathbb{R}^d, \quad U_2 = V_2 = \mathbb{S}_d^+ \times (0, T), \\ \hat\theta = (x_{\alpha,\beta}, y_{\alpha,\beta}, t_{\alpha,\beta}), \quad \hat\zeta = (\xi_{\alpha,\beta}, \eta_{\alpha,\beta}, \tau_{\alpha,\beta}). \end{cases}$$

Note that $\mathbb{S}_d^+$ is a locally compact subset of $\mathbb{R}^{d^2}$. Define the functions $\varphi, \tilde\varphi_i \in \mathrm{C}^2(U \times V)$, $\varphi_i \in \mathrm{C}^2(U_i \times V_i)$ for $i = 1, 2$ by

$$\begin{cases} \varphi(x, y, t, \xi, \eta, \tau) = \alpha|x - \xi|^2 + \beta(\|y - \eta\|^2 + (t - \tau)^2), \\ \varphi_1(x, , \xi) = \tilde\varphi_1(x, y, t, \xi, \eta, \tau) = \alpha|x - \xi|^2, \\ \varphi_2(y, t, \eta, \tau) = \tilde\varphi_2(x, y, t, \xi, \eta, \tau) = \beta(\|y - \eta\|^2 + (t - \tau)^2), \end{cases}$$

and set

$$\mathcal{A}_1 = D^2\tilde\varphi_1(\hat\theta, \hat\zeta), \quad \mathcal{A}_2 = D^2\tilde\varphi_2(\hat\theta, \hat\zeta), \quad \mathcal{A} = D^2\varphi(\hat\theta, \hat\zeta).$$

It is easy to check that

$$(5.8)\quad \begin{cases} \mathcal{A}_1 = 2\alpha\begin{pmatrix} I_d & 0 & -I_d & 0 \\ 0 & 0 & 0 & 0 \\ -I_d & 0 & I_d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{A}_2 = 2\beta\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I_{d^2+1} & 0 & -I_{d^2+1} \\ 0 & 0 & 0 & 0 \\ 0 & -I_{d^2+1} & 0 & I_{d^2+1} \end{pmatrix}, \\ \mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2, \quad \mathcal{A}^2 = \mathcal{A}_1^2 + \mathcal{A}_2^2 = 4\alpha\mathcal{A}_1 + 4\beta\mathcal{A}_2, \quad |\mathcal{A}_1| = 4\alpha, \quad |\mathcal{A}_2| = 4\beta. \end{cases}$$

We select

$$\epsilon_1 = \frac{1}{2\alpha}, \quad \epsilon_2 = \frac{1}{2\beta}, \quad \lambda_i = \frac{1}{\epsilon_i} + |\mathcal{A}_i| \quad \text{for } i = 1, 2,$$

note that

$$\lambda_1 = 6\alpha, \quad \lambda_2 = 6\beta,$$

and set $\epsilon = (\epsilon_1, \epsilon_2)$ and $\lambda = (\lambda_1, \lambda_2)$.

Noting that

$$\mathcal{A}_\epsilon = \mathcal{A} + \epsilon_1 \cdot 4\alpha\mathcal{A}_1 + \epsilon_2 \cdot 4\beta\mathcal{A}_2 = 3\mathcal{A}, \tag{5.9}$$

we define the function φ_ϵ on $Q_T \times Q_T$ by

$$\varphi_\epsilon(\theta, \zeta) = \varphi(\theta, \zeta) + 2\varphi(\theta - \hat{\theta}, \zeta - \hat{\zeta}),$$

and note that

$$D^2\varphi_\epsilon(\hat{\theta}, \hat{\zeta}) = 3\mathcal{A} = \mathcal{A}_\epsilon,$$

and moreover, all the conditions of (4.3) hold.

3. We are now ready to apply the maximum principle. Define $w \in \mathrm{USC}(Q_T \times Q_T)$ by $w(\theta, \zeta) = u(\theta) - v(\zeta)$, fix a compact neighborhood W of $(\hat{\theta}, \hat{\zeta})$ in $Q_T \times Q_T$ and invoke Corollary 4.4, to select sequences $\{(\theta_j, \zeta_j\} \subset Q_T \times Q_T, \{(\mathcal{X}_j, \mathcal{Y}_j)\} \subset \mathbb{S}_n \times \mathbb{S}_n$, and $\{\varphi_j\} \subset \mathrm{C}^2(Q_T \times Q_T)$ such that (4.4) holds, with $-v$ in place of v.

For any $j \in \mathbb{N}$, we recall by (4.4) that

$$\begin{aligned}
&\max_{Q_T \times Q_T} (w - \varphi_j) = (w - \varphi_j)(\theta_j, \zeta_j),\\
&(D_\theta\varphi_j(\theta_j, \zeta_j), \mathcal{X}_j) \in J^{2,+}u(\theta_j),\\
&-(D_\zeta\varphi_j(\theta_j, \zeta_j), \mathcal{Y}_j) \in J^{2,-}v(\zeta_j),
\end{aligned}$$

and write

$$\begin{aligned}
&\theta_j = (x_j, y_j, t_j) \in \mathbb{R}^d \times \mathbb{S}_d^+ \times (0, T), \quad \zeta_j = (\xi_j, \eta_j, \tau_j) \in \mathbb{R}^d \times \mathbb{S}_d^+ \times (0, T),\\
&D_\theta\varphi_j(\theta_j, \zeta_j) = p_j, \quad D_\zeta\varphi_j(\theta_j, \zeta_j) = q_j,
\end{aligned}$$

to obtain

$$\begin{cases} \min\{F(y_j, p_j, \mathcal{X}_j) - Ju(x_j, y_j, t_j) + \gamma,\, (u - h)(x_j, y_j, t_j)\} \le 0,\\ \min\{F(\eta_j, -q_j, -\mathcal{Y}_j) - Jv(\xi_j, \eta_j, \tau_j),\, (v - h)(\xi_j, \eta_j, \tau_j)\} \ge 0. \end{cases} \tag{5.10}$$

By (4.4), we also have

$$\lim_{j\to\infty} (\theta_j, \zeta_j) = (\hat{\theta}, \hat{\zeta}), \quad \lim_{j\to\infty} \varphi_j = \varphi_\epsilon \ \text{in} \ \mathrm{C}^2(U \times V), \tag{5.11}$$

and hence, by the semicontinuities of u, v, w,

$$\begin{aligned}
(w - \varphi_\epsilon)(\hat{\theta}, \hat{\zeta}) &\ge \limsup_{j\to\infty} u(\theta_j) - \liminf_{j\to\infty} v(\zeta_j) - \lim_{j\to\infty} \varphi_j(\theta_j, \zeta_j) \ge \limsup_{j\to\infty} (w - \varphi_j)(\theta_j, \zeta_j)\\
&\ge \liminf_{j\to\infty} (w - \varphi_j)(\theta_j, \zeta_j) \ge \liminf_{j\to\infty} \max_{Q_T \times Q_T} (w - \varphi_j).
\end{aligned}$$

Moreover, since

$$\max_{Q_T \times Q_T}(w - \varphi_j) \geq (w - \varphi_j)(\theta, \zeta) \quad \text{for all } \theta, \zeta \in Q_T,\ j \in \mathbb{N},$$

we infer that

$$\begin{cases} (w - \varphi_\epsilon)(\hat{\theta}, \hat{\zeta}) = \max\limits_{Q_T \times Q_T}(w - \varphi_\epsilon) = \lim\limits_{j \to \infty}(w - \varphi_j)(\theta_j, \zeta_j), \\ u(\hat{\theta}) = \lim\limits_{j\to\infty} u(\theta_j), \qquad v(\hat{\zeta}) = \lim\limits_{j\to\infty} v(\zeta_j). \end{cases} \tag{5.12}$$

These observations and (5.7) allow us to assume by relabeling θ_j, ζ_j and so on if necessary that for all $j \in \mathbb{N}$,

$$(u - h)(x_j, y_j, t_j) > (v - h)(\xi_j, \eta_j, \tau_j).$$

This and (5.10) together yield

$$F(y_j, p_j, \mathcal{X}_j) - Ju(x_j, y_j, t_j) + \gamma \leq 0 \leq F(\eta_j, -q_j, -\mathcal{Y}_j) - Jv(\xi_j, \eta_j, \tau_j) \quad \text{for all } j \in \mathbb{N}. \tag{5.13}$$

It follows from the inequalities above and the fact that the function $w - \varphi_j$ takes a maximum at (θ_j, ζ_j) that the functions $z \mapsto u(x_j, y_j + z, t_j) - u(x_j, y_j, t_j)$ and $z \mapsto v(\xi_j, \eta_j + z, \tau_j) - v(\xi_j, \eta_j, \tau_j)$ are integrable with respect to the measure ν.

Next, thanks to the inequality

$$-E_\lambda \leq \begin{pmatrix} \mathcal{X}_j & 0 \\ 0 & \mathcal{Y}_j \end{pmatrix} \leq D^2\varphi_j(\theta_j, \zeta_j),$$

together with the convergence

$$\lim_{j\to\infty}(\theta_j, \zeta_j) = (\hat{\theta}, \hat{\zeta}), \quad \lim_{j\to\infty}\varphi_j = \varphi_\epsilon \quad \text{in } \mathrm{C}^2(Q_T \times Q_T),$$

we see that the sequence $\{\mathcal{X}_j, \mathcal{Y}_j\}$ is bounded in $\mathbb{S}_n \times \mathbb{S}_n$ and, hence, we may assume by passing to a subsequence if necessary that for some $(\mathcal{X}_{\alpha\beta}, \mathcal{Y}_{\alpha\beta}) \in \mathbb{S}_n \times \mathbb{S}_n$,

$$\lim_{j\to\infty}(\mathcal{X}_j, \mathcal{Y}_j) = (\mathcal{X}_{\alpha\beta}, \mathcal{Y}_{\alpha\beta}).$$

Moreover, from the matrix inequality above, we get

$$-E_\lambda \leq \begin{pmatrix} \mathcal{X}_{\alpha\beta} & 0 \\ 0 & \mathcal{Y}_{\alpha\beta} \end{pmatrix} \leq D^2\varphi_\epsilon(\hat{\theta}, \hat{\zeta}) = \mathcal{A}_\epsilon. \tag{5.14}$$

Sending $j \to \infty$ and using Lemma 5.3, we get from (5.13)

$$F(y_{\alpha\beta}, p_{\alpha\beta}, \mathcal{X}_{\alpha\beta}) - Ju(x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}) + \gamma \leq 0 \leq F(\eta_{\alpha\beta}, -q_{\alpha\beta}, -\mathcal{Y}_{\alpha\beta}) - Jv(\xi_{\alpha\beta}, \eta_{\alpha\beta}, \tau_{\alpha\beta}), \tag{5.15}$$

where $p_{\alpha\beta} := D_\theta\varphi(x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}, \xi_{\alpha\beta}, \eta_{\alpha\beta}, s_{\alpha\beta})$ and $q_{\alpha\beta} := D_\zeta\varphi(x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}, \xi_{\alpha\beta}, \eta_{\alpha\beta}, s_{\alpha\beta})$. We remark that, in the application of Lemma 5.3 here, we have used the fact that for all $z \in \mathbb{S}_d^+$,

$$\begin{aligned} &\limsup_{j\to\infty}(u(x_j, y_j + z, t_j) - u(x_j, y_j, t_j)) \leq u(x_{\alpha\beta}, y_{\alpha\beta} + z, t_{\alpha\beta}) - u(x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}), \\ &\liminf_{j\to\infty}(v(\xi_j, \eta_j + z, \tau_j) - v(\xi_j, \eta_j, \tau_j)) \geq v(\xi_{\alpha\beta}, \eta_{\alpha\beta} + z, \tau_{\alpha\beta}) - v(\xi_{\alpha\beta}, \eta_{\alpha\beta}, \tau_{\alpha\beta}), \end{aligned}$$

which are consequences of (5.11), (5.12) and the upper semicontinuity of $u, -v$.

4. We are going to show that (5.15), together with (5.14) and (5.5), yields a desired conclusion. We denote by $X_{\alpha\beta}$ and $Y_{\alpha\beta}$ the first $d \times d$ block of $\mathcal{X}_{\alpha\beta}$ and $\mathcal{Y}_{\alpha\beta}$, respectively. Computing the quadratic forms associated with the matrices appearing in (5.14) at $(\xi, 0, \eta, 0) \in \mathbb{R}^{2n}$, where $\xi, \eta \in \mathbb{R}^d$, we deduce from (5.14), (5.8) and (5.9) that

$$-6\alpha I_{2d} \leq \begin{pmatrix} X_{\alpha\beta} & 0 \\ 0 & Y_{\alpha\beta} \end{pmatrix} \leq 6\alpha \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}.$$

From this, we may assume by passing to a subsequence if necessary that for each $k \in \mathbb{N}$, as $m \to \infty$, $\{(X_{\alpha_k\beta_m}, Y_{\alpha_k\beta_m})\}$ converges to some (X_k, Y_k) in $\mathbb{S}_d \times \mathbb{S}_d$. From the inequality above, we get

$$-6\alpha_k I_{2d} \leq \begin{pmatrix} X_k & 0 \\ 0 & Y_k \end{pmatrix} \leq 6\alpha_k \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix},$$

which yields

$$X_k + Y_k \leq 0. \tag{5.16}$$

From (5.5), we see that for some $(\bar{x}_k, \bar{y}_k, \bar{t}_k) \in Q_T$, $\bar{\xi}_k \in \mathbb{R}^d$,

$$\begin{cases} \lim\limits_{m\to\infty} (x_{\alpha_k\beta_m}, y_{\alpha_k\beta_m}, t_{\alpha_k\beta_m}, \xi_{\alpha_k\beta_m}, \eta_{\alpha_k\beta_m}, \tau_{\alpha_k\beta_m}) = (\bar{x}_k, \bar{y}_k, \bar{t}_k, \bar{\xi}_k, \bar{y}_k, \bar{t}_k), \\ \lim\limits_{m\to\infty} \beta_m \|y_{\alpha_k\beta_m} - \eta_{\alpha_k\beta_m}\|^2 = 0. \end{cases} \tag{5.17}$$

Note that

$$\begin{aligned} p_{\alpha\beta} &= D_\theta \varphi(\hat{\theta}, \hat{\zeta}) = 2(\alpha(x_{\alpha\beta} - \xi_{\alpha\beta}), \beta(y_{\alpha\beta} - \eta_{\alpha\beta}), \beta(t_{\alpha\beta} - \tau_{\alpha\beta})), \\ q_{\alpha\beta} &= D_\zeta \varphi(\hat{\theta}, \hat{\zeta}) = -2(\alpha(x_{\alpha\beta} - \xi_{\alpha\beta}), \beta(y_{\alpha\beta} - \eta_{\alpha\beta}), \beta(t_{\alpha\beta} - \tau_{\alpha\beta})), \\ F(y_{\alpha\beta}, p_{\alpha\beta}, \mathcal{X}_{\alpha\beta}) &= -2\beta(t_{\alpha\beta} - \tau_{\alpha\beta}) - \frac{1}{2}\langle y_{\alpha\beta}, X_{\alpha\beta}\rangle - \alpha\langle \pi(y_{\alpha\beta}), x_{\alpha\beta} - \xi_{\alpha\beta}\rangle \\ &\quad - 2\beta\langle A y_{\alpha\beta} + y_{\alpha\beta} A^* + b_0, y_{\alpha\beta} - \eta_{\alpha\beta}\rangle, \\ F(\eta_{\alpha\beta}, -q_{\alpha\beta}, -\mathcal{Y}_{\alpha\beta}) &= -2\beta(t_{\alpha\beta} - \tau_{\alpha\beta}) - \frac{1}{2}\langle \eta_{\alpha\beta}, -Y_{\alpha\beta}\rangle - \alpha\langle \pi(\eta_{\alpha\beta}), x_{\alpha\beta} - \xi_{\alpha\beta}\rangle \\ &\quad - 2\beta\langle A\eta_{\alpha\beta} + \eta_{\alpha\beta} A^* + b_0, y_{\alpha\beta} - \eta_{\alpha\beta}\rangle. \end{aligned}$$

Combine the last two equalities above, to obtain

$$\begin{aligned} &F(\eta_{\alpha\beta}, -q_{\alpha\beta}, -\mathcal{Y}_{\alpha\beta}) - F(y_{\alpha\beta}, p_{\alpha\beta}, \mathcal{X}_{\alpha\beta}) \\ &\quad \leq \frac{1}{2}\langle y_{\alpha\beta}, X_{\alpha\beta}\rangle + \frac{1}{2}\langle \eta_{\alpha\beta}, Y_{\alpha\beta}\rangle + \alpha\langle \pi(y_{\alpha\beta} - \eta_{\alpha\beta}), x_{\alpha\beta} - \xi_{\alpha\beta}\rangle \\ &\qquad + 2\beta\langle A(y_{\alpha\beta} - \eta_{\alpha\beta}) + (y_{\alpha\beta} - \eta_{\alpha\beta})A^*, y_{\alpha\beta} - \eta_{\alpha\beta}\rangle \\ &\quad \leq \frac{1}{2}\langle y_{\alpha\beta}, X_{\alpha\beta}\rangle + \frac{1}{2}\langle \eta_{\alpha\beta}, Y_{\alpha\beta}\rangle + \alpha\langle \pi(y_{\alpha\beta} - \eta_{\alpha\beta}), x_{\alpha\beta} - \xi_{\alpha\beta}\rangle \\ &\qquad + 4\beta\|A\|\|y_{\alpha\beta} - \eta_{\alpha\beta}\|^2. \end{aligned} \tag{5.18}$$

We may regard φ as a function on $\mathbb{R}^d \times M_d(\mathbb{R}) \times \mathbb{R}$ having the property:

$$\varphi(\theta + \theta', \zeta + \theta') = \varphi(\theta, \zeta) \quad \text{for all } \theta, \theta', \zeta \in \mathbb{R}^d \times M_d(\mathbb{R}) \times \mathbb{R}.$$

The function $\varphi_\epsilon(\theta, \zeta) = \varphi(\theta, \zeta) + 2\varphi(\theta - \hat{\theta}, \zeta - \hat{\zeta})$ inherits the above invariance property. Now that for all $\theta, \zeta \in Q_T$,

$$u(\theta) - v(\zeta) \leq u(\hat{\theta}) - v(\hat{\zeta}) + \varphi_\epsilon(\theta, \zeta) - \varphi_\epsilon(\hat{\theta}, \hat{\zeta})$$

by (5.12), we obtain for all $z \in \mathbb{S}_d^+$,

$$u(x_{\alpha\beta}, y_{\alpha\beta} + z, t_{\alpha\beta}) - u(x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}) \leq v(\xi_{\alpha\beta}, \eta_{\alpha\beta} + z, \tau_{\alpha\beta}) - v(\xi_{\alpha\beta}, \eta_{\alpha\beta}, \tau_{\alpha\beta}),$$

which implies

$$Ju(x_{\alpha\beta}, y_{\alpha\beta}, t_{\alpha\beta}) \leq Jv(\xi_{\alpha\beta}, \eta_{\alpha\beta}, \tau_{\alpha\beta}).$$

Combining this, (5.17) and (5.18), we obtain

$$\gamma \leq \frac{1}{2}\langle y_{\alpha\beta}, X_{\alpha\beta}\rangle + \frac{1}{2}\langle \eta_{\alpha\beta}, Y_{\alpha\beta}\rangle + \alpha\langle \pi(y_{\alpha\beta} - \eta_{\alpha\beta}), x_{\alpha\beta} - \xi_{\alpha\beta}\rangle + 4\beta|A|\|y_{\alpha\beta} - \eta_{\alpha\beta}\|^2.$$

Since $\alpha = \alpha_k$ and $\beta = \beta_m$ in the above, sending $m \to \infty$ and using (5.17) and (5.16), we obtain from the above

$$\gamma \leq \frac{1}{2}\langle \bar{y}_k, X_k + Y_k\rangle \leq 0.$$

This is a contradiction, which completes the proof. ∎

6. Payoff function with polynomial growth.

In this section we extend our result stated in Corollary 5.2 to the case when the payoff function h has a polynomial growth of order less than λ and establish the following theorem.

Theorem 6.1. *Let $0 \leq \kappa < \lambda$. Assume that $h \in C(\overline{Q}_T) \cap \mathcal{V}_\kappa$. Then, u_0 given by (1.5) belongs to $\mathrm{C}(\overline{Q}_T) \cap \mathcal{V}_\kappa$ and is a unique viscosity solution of (1.6) satisfying $u_0(x, y, T) = h(x, y, T)$ for all $(x, y) \in \mathbb{R}^d \times \mathbb{S}_d^+$. The uniqueness holds in the class $\mathrm{C}(\overline{Q}_T) \cap \mathcal{V}_\kappa$.*

Before the proof, we give the following stability lemma, which is similar to the standard stability results [12, Sect. 6], [2, Sect. 3].

Lemma 6.2. *Let $\{v_j\}_{j\in\mathbb{N}}, \{g_j\}_{j\in\mathbb{N}} \subset \mathrm{C}(\overline{Q}_T) \cap \mathcal{V}_\lambda$. Assume that for some constant $C > 0$,*

$$|v_j(x, y, t)| \leq C(1 + |x| + \|y\|)^\lambda \quad \text{for all } (x, y, t) \in \overline{Q}_T, \tag{6.1}$$

and that for some functions $v, g \in \mathrm{C}(\overline{Q}_T) \cap \mathcal{V}_\lambda$, $\{v_j\}$ and $\{g_j\}$ converge to, respectively, v and g locally uniformly on $\overline{Q}_T$. Assume that for every $j \in \mathbb{N}$, v_j is a viscosity solution of (1.6), with g_j in place of h, that satisfies the terminal condition $v_j(x, y, T) = g_j(x, y, T)$ for all $(x, y) \in \mathbb{R}^d \times \mathbb{S}_d^+$. Then, v is a viscosity solution of (1.6), with g in place of h, satisfying $v(x, y, T) = g(x, y, T)$ for all $(x, y) \in \mathbb{R}^d \times \mathbb{S}_d^+$.

Proof. By the assumed convergence, we see immediately that $v(x,y,T) = g(x,y,T)$ for all $(x,y) \in \mathbb{R}^d \times \mathbb{S}_d^+$. We only need to prove the viscosity property of v. We present here the proof of the supersolution property of v, and leave it to the reader to check that v is a subsolution of (1.6) with $h = g$.

Fix $\theta_0 = (x_0, y_0, t_0) \in Q_T$ and let $(p, \mathcal{X}) \in J^{2,-}v(\theta_0)$. Fix a compact neighborhood $K \subset Q_T$ of θ_0 and choose a function $\phi \in \mathrm{C}^2(K)$ so that $v - \phi$ takes a strict minimum on K at θ_0 and $(p, \mathcal{X}) = (D\phi(\theta_0), D^2\phi(\theta_0))$. (The existence of such a function is a standard observation.) Let $\theta_j = (x_j, y_j, t_j) \in K$ be a minimum point of $v_j - \phi$ on K. Since $\{v_j\}$ converges to v in $\mathrm{C}(K)$, we see that $\lim_{j\to\infty} \theta_j = \theta_0$. By relabelling the sequence $\{(v_j, g_j, \theta_j)\}$ if necessary, we may assume that for every $j \in \mathbb{N}$, θ_j is an interior point of K.

Noting that $(D\phi(\theta_j), D^2\phi(\theta_j)) \in J^{2,-}v_j(\theta_j)$ and invoking Definition 3.5, we get

$$\min\{F(y_j, D\phi(\theta_j), D^2\phi(\theta_j)) - Jv_j(\theta_j), (v_j - g_j)(\theta_j)\} \ge 0 \quad \text{for all } j \in \mathbb{N}.$$

It is clear that

$$\lim_{j\to\infty} F(y_j, D\phi(\theta_j), D^2\phi(\theta_j)) = F(y_0, D\phi(\theta_0), D^2\phi(\theta_0)) = F(y_0, p, \mathcal{X}),$$

and $\lim_{j\to\infty}(v_j - g_j)(\theta_j) = (v - g)(\theta_0)$. By Lemma 5.3, we find that

$$\begin{aligned}\liminf_{j\to\infty} Jv_j(\theta_j) &\ge \int_{\mathbb{S}_d^+} \liminf_{j\to\infty} (v_j(x_j, y_j + z, t_j) - v_j(x_j, y_j, t_j))\nu(dz) \\ &= \int_{\mathbb{S}_d^+} (v(x_0, y_0 + z, t_0) - v_j(x_0, y_0, t_0))\nu(dz) = Jv(\theta_0).\end{aligned}$$

Thus, we obtain

$$\min\{F(y_0, p, \mathcal{X}) - Jv(\theta_0), (v - g)(\theta_0)\} \ge 0.$$

In light of Definition 3.5, we deduce that v is a viscosity supersolution of (1.6) with $h = g$. ∎

Proof of Theorem 6.1. Once it is proved that $u_0 \in C(\overline{Q}_T) \cap \mathcal{V}_\kappa$ and u_0 is a viscosity solution of (1.6) satisfying the terminal condition, then the uniqueness is immediate from Theorem 5.1.

To show that $u_0 \in \mathcal{V}_\kappa$, we recall some of the results in Lemma 2.1: there is a constant $C_0 > 0$ such that

$$\mathbf{E} \sup_{0\le s\le T} (1 + |X_s^{x,y}| + \|Y_s^y\|)^\lambda \le C_0(1 + |x| + \|y\|)^\lambda \quad \text{for all } (x,y) \in \mathbb{R}^d \times \mathbb{S}_d^+. \tag{6.2}$$

By assumption, we have for some constant $C_1 > 0$,

$$|h(x,y,t)| \le C_1(1 + |x| + \|y\|)^\kappa \quad \text{for all } (x,y,t) \in \overline{Q}_T. \tag{6.3}$$

It is then straightforward to see that

$$|u_0(x,y,t)| \le C_1 \sup_{\tau\in\mathcal{T}_{T-t}} \mathbf{E}(1 + |X_\tau^{x,y}| + \|Y_\tau^y\|)^\kappa \le C_0C_1(1 + |x| + \|y\|)^\kappa \quad \text{for all } (x,y,t) \in \overline{Q}_T.$$

By the definition of u_0, it is clear that $u_0(x,y,T) = h(x,y,T)$ for all $(x,y) \in \mathbb{R}^d \times \mathbb{S}_d^+$.

Next, we prove that $u_0 \in \mathrm{C}(\overline{Q}_T)$. For this, we select a sequence of bounded Lipschitz continuous functions $h_j \in C(\overline{Q}_T)$, with $j \in \mathbb{N}$, such that for some constant $C_2 > 0$,

$$h_j(x,y,t) \to h(x,y,t) \text{ locally uniformly on } \overline{Q}_T, \tag{6.4}$$

$$|h_j(x,y,t)| \le C_2(1+|x|+\|y\|)^\kappa \quad \text{for all } (x,y,t) \in \overline{Q}_T. \tag{6.5}$$

We define $u_j : \overline{Q}_T \to \mathbb{R}$ by

$$u_j(x,y,t) = \sup_{\tau \in \mathcal{T}_{T-t}} \mathbf{E}\, h_j(X_\tau^{x,y}, Y_\tau^y, t+\tau).$$

By Corollary 5.2, we know that u_j is in $\mathcal{V}_0 \cap \mathrm{C}(\overline{Q}_T)$ and a unique viscosity solution of (1.6) satisfying $u_j(x,y,T) = h_j(x,y,T)$ for $(x,y) \in \mathbb{R}^d \times \mathbb{S}_d^+$.

We show that for any compact subset K of $\mathbb{R}^d \times \mathbb{S}_d^+$, as $j \to \infty$,

$$u_j(x,y,t) \to u_0(x,y,t) \quad \text{uniformly on } K \times [0,\, T]. \tag{6.6}$$

This convergence assertion proves that $u_0 \in \mathrm{C}(\overline{Q}_T)$.

To check the above convergence of $\{u_j\}$, fix any compact $K \subset \mathbb{R}^d \times \mathbb{S}_d^+$. Define $g_j : \overline{Q}_T \to \mathbb{R}$ by $g_j = h_j - h$. By (6.3), (6.5), setting $C_3 = C_1 + C_2$, we have

$$|g_j(x,y,t)| \le C_3(1+|x|+\|y\|)^\kappa \quad \text{for all } (x,y,t) \in \overline{Q}_T.$$

By (6.2), there is a constant $C_K > 0$ such that

$$\mathbf{E} \sup_{0 \le t \le T} (1 + |X_t^{x,y}| + \|Y_t^y\|)^\lambda \le C_K \quad \text{for all } (x,y) \in K.$$

The above two inequalities imply that the family of random variables $g_j(X_\tau^{x,y}, Y_\tau^y, t+\tau)$, with $(x,y) \in K$ and $\tau \in \mathcal{T}_T$, is uniformly integrable. Thus, by the inequality

$$|\,\mathbf{E}\, h_j(X_\tau^{x,y}, Y_\tau^y, t+\tau) - \mathbf{E}\, h(X_\tau^{x,y}, Y_\tau^y, t+\tau)| \le \mathbf{E}\, |g_j(X_\tau^{x,y}, Y_\tau^y, t+\tau)|$$

and (6.4), we conclude the required convergence.

Now, Lemma 6.2 combined with (6.6) assures that $u_0 \in \mathrm{C}(\overline{Q}_T)$ is a viscosity solution of (1.6) with the terminal condition h. The proof is complete. ■

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