

# Asymptotic solutions of Hamilton-Jacobi equations in Euclidean $n$ space

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**Abstract.** We study the asymptotic behavior of the viscosity solution of the Cauchy problem for the Hamilton-Jacobi equation  $u_t + \alpha x \cdot Du + H(Du) = f(x)$  in  $\mathbf{R}^n \times (0, \infty)$ , where  $\alpha$  is a positive constant and  $H$  is a convex function on  $\mathbf{R}^n$ , and establish a convergence result for the viscosity solution  $u(x, t)$  as  $t \rightarrow \infty$ .

## 1. Introduction and the main result

Recently there has been a great interest on the asymptotic behavior of viscosity solutions of the Cauchy problem for Hamilton-Jacobi equations or viscous Hamilton-Jacobi equations. Among others Fathi [F2] has first established a fairly general convergence result for the Hamilton-Jacobi equation

$$(1.1) \quad u_t(x, t) + H(x, Du(x, t)) = 0$$

on a compact manifold  $\mathcal{M}$  with smooth strictly convex Hamiltonian  $H$ . Associated with this problem is the stationary partial differential equation (PDE for short)

$$(1.2) \quad c + H(x, Dv) = 0 \quad \text{in } \mathcal{M},$$

where the unknown is the pair of a constant  $c \in \mathbf{R}$  and a solution  $v$  of (1.2). Here and in what follows we adapt the notion of viscosity solution to that of weak solution for first order PDE. It is known (see [LPV]) that a constant  $c$  for which (1.2) has a viscosity solution  $v$  is uniquely determined. The result obtained in [F2] is loosely stated

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as follows: for any viscosity solution  $u$  of (1.1) there is a viscosity solution  $v$  of (1.2) such that  $u(x, t) - ct \rightarrow v(x)$  uniformly on  $\mathcal{M}$  as  $t \rightarrow \infty$ . His approach to this asymptotic problem is based on the weak KAM theorem [F1] and especially on Aubry-Mather sets. A PDE approach to the same asymptotic problem has been developed by Barles and Souganidis [BS]. Fathi's approach has been developed by Roquejoffre [R] and Davini-Siconolfi [DS].

Motivated by these developments the authors [FIL1] have recently investigated the asymptotic problem for viscous Hamilton-Jacobi equations in  $\mathbf{R}^n$  with Ornstein-Uhlenbeck operator and have established a convergence result similar to the one stated above. The equations treated in [FIL1] have the form

$$(1.3) \quad u_t - \Delta u + \alpha x \cdot Du + H(Du) = f(x).$$

In this paper we study the Cauchy problem

$$(1.4) \quad u_t + \alpha x \cdot Du + H(Du) = f(x) \quad \text{in } \mathbf{R}^n \times (0, \infty),$$

and

$$(1.5) \quad u|_{t=0} = \phi.$$

To be precise, here  $u$  represents the real-valued unknown function on  $\mathbf{R}^n \times [0, \infty)$ ,  $\alpha$  is a given positive constant,  $H, f, \phi$  are given real-valued functions on  $\mathbf{R}^n$ ,  $u_t$  and  $Du$  denote the  $t$ -derivative and  $x$ -gradient of  $u$ , respectively, and  $x \cdot y$  denotes the Euclidean inner product of  $x, y \in \mathbf{R}^n$ . The result in this paper extends, in regard to the generality of  $H, f$ , and  $\phi$ , our previous work [FIL2] on (1.4) and (1.5), where we assumed that  $H$  has the form  $\beta|p|^2 + b \cdot p$ , with  $\beta > 0$  and  $b \in \mathbf{R}^n$ .

We assume the following conditions on  $H, f, \phi$  throughout this paper:

$$(A1) \quad H, f, \phi \in C(\mathbf{R}^n).$$

$$(A2) \quad H \text{ is convex on } \mathbf{R}^n.$$

$$(A3) \quad \lim_{|p| \rightarrow \infty} \frac{H(p)}{|p|} = \infty.$$

PDE (1.4) can be seen as the dynamic programming equation of the control system in which the state equation is given by

$$\dot{X}(t) + \alpha X(t) = \xi(t) \quad \text{for } t \in (0, T), \quad X(0) = x,$$

where  $0 < T < \infty$ ,  $x \in \mathbf{R}^n$ , and  $\xi \in L^1(0, T)$  is a control, and in which the value function  $u$  is given by

$$(1.6) \quad u(x, T) = \inf_{\xi \in L^1(0, T)} \left\{ \int_0^T [f(X(t)) + L(-\xi(t))] dt + \phi(X(T)) \right\},$$

where  $L$  denotes the convex conjugate  $H^*$  of  $H$ , i.e.,

$$L(\xi) := H^*(\xi) \equiv \sup\{\xi \cdot p - H(p) \mid p \in \mathbf{R}^n\} \quad \text{for } \xi \in \mathbf{R}^n.$$

As is well-known, the function  $L$  is continuous on  $\mathbf{R}^n$  and satisfies

$$\lim_{|\xi| \rightarrow \infty} \frac{L(\xi)}{|\xi|} = \infty.$$

We assume furthermore that there is a convex function  $l : \mathbf{R}^n \rightarrow \mathbf{R}$  having the properties:

$$(A4) \quad \lim_{|x| \rightarrow \infty} (L(x) - l(x)) = \infty.$$

$$(A5) \quad \inf\{f(x) + l(-\alpha x) \mid x \in \mathbf{R}^n\} > -\infty.$$

$$(A6) \quad \inf\{\phi(x) + \frac{1}{\alpha}l(-\alpha x) \mid x \in \mathbf{R}^n\} > -\infty.$$

In view of (A4) and (A5), we see that the function  $x \mapsto f(x) + L(-\alpha x)$  attains a minimum over  $\mathbf{R}^n$ , and we set

$$(1.7) \quad c = \min\{f(x) + L(-\alpha x) \mid x \in \mathbf{R}^n\} \quad \text{and} \quad f_c(x) = f(x) - c \quad \text{for } x \in \mathbf{R}^n.$$

We observe as well that

$$(1.8) \quad Z := \{x \in \mathbf{R}^n \mid f(x) + L(-\alpha x) = c\}$$

is a compact subset of  $\mathbf{R}^n$ .

This set  $Z$  corresponds to the projected Aubry set although we do not need to introduce the projected Aubry set for (1.4) in our approach. We shall return this point in a future publication.

Our approach in this paper is based on the representation formula (1.6) of the viscosity solution of (1.4) and (1.5) and a convex analysis lemma (see Lemma 2.2 below) which takes advantage of the special form of (1.4).

A typical case where (A1)–(A6) are satisfied is: let  $H$ ,  $f$ , and  $\phi$  satisfy (A1)–(A3). Assume furthermore that there is a constant  $C_0 > 0$  such that

$$f(x) \geq -C_0(|x| + 1), \quad \phi(x) \geq -C_0(|x| + 1) \quad \text{for } x \in \mathbf{R}^n.$$

In this situation, if we take  $l$  to be the function given by  $l(x) = (\alpha + 1)C_0(|x| + 1)$ , then conditions (A4)–(A6) hold.

For  $(x, y, T) \in \mathbf{R}^n \times \mathbf{R}^n \times (0, \infty)$  let  $\mathcal{C}(x, T)$  and  $\mathcal{C}(x, y, T)$  denote the spaces of absolutely continuous functions  $X : [0, T] \rightarrow \mathbf{R}^n$  satisfying, respectively,  $X(0) = x$

and  $(X(0), X(T)) = (x, y)$ . Define the functions  $d : \mathbf{R}^n \times \mathbf{R}^n \rightarrow \mathbf{R} \cup \{-\infty\}$  and  $\psi : \mathbf{R}^n \rightarrow \mathbf{R} \cup \{-\infty\}$  by

$$(1.9) \quad d(x, y) = \inf \left\{ \int_0^T [f_c(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt \mid T > 0, X \in \mathcal{C}(x, y, T) \right\},$$

and

$$(1.10) \quad \psi(x) = \inf \left\{ \int_0^T [f_c(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)) \mid T > 0, X \in \mathcal{C}(x, T) \right\},$$

respectively.

Define the function  $v : \mathbf{R}^n \rightarrow \mathbf{R} \cup \{-\infty\}$  by

$$(1.11) \quad v(x) = \inf_{y \in Z} (d(x, y) + \psi(y)).$$

**Proposition 1.1.** *The functions  $d$ ,  $\psi$ , and  $v$  are real-valued continuous functions on  $\mathbf{R}^n \times \mathbf{R}^n$ ,  $\mathbf{R}^n$ , and  $\mathbf{R}^n$ , respectively.*

Henceforth  $B(x, R)$  denotes the closed ball of  $\mathbf{R}^n$  with center at  $x$  and radius  $R \geq 0$ .

**Theorem 1.2.** *There is a unique viscosity solution  $u \in C(\mathbf{R}^n \times [0, \infty))$  of (1.4) and (1.5) which satisfies for any  $0 < T < \infty$ ,*

$$(1.12) \quad \lim_{r \rightarrow \infty} \inf \{ u(x, t) + \frac{1}{\alpha} L(-\alpha x) \mid (x, t) \in (\mathbf{R}^n \setminus B(0, r)) \times [0, T] \} = \infty.$$

The main result in this paper is the following.

**Theorem 1.3.** *Let  $u \in C(\mathbf{R}^n \times [0, \infty))$  be the unique viscosity solution of (1.4) and (1.5) satisfying (1.12). Then*

$$(1.13) \quad \lim_{t \rightarrow \infty} \max_{x \in B(0, R)} |u(x, t) - (ct + v(x))| = 0 \quad \text{for } R > 0.$$

We remark that formula (1.11) for the asymptotic solution  $v$  has been shown in [DS] for a fairly general Hamilton-Jacobi equation in the periodic setting. The function  $v$  is a viscosity solution of

$$(1.14) \quad c + \alpha x \cdot Dv + H(Dv) = f(x) \quad \text{in } \mathbf{R}^n.$$

For instance, this follows from Theorem 1.3 and the stability of viscosity solutions of (1.4) under locally uniform convergence.

The paper is organized as follows: in Section 2 we establish the convex analysis lemma mentioned above and some of basic properties of the functions  $d$ ,  $\psi$ , and  $u$ . Section 3 is devoted to the proof of Theorem 1.3. In Section 4 we establish the existence part of Theorem 1.2 together with some estimates on the solution of (1.4) and (1.5). In Section 5 we establish a comparison theorem for viscosity solutions of (1.4).

## 2. Preliminaries

We prepare in this section to prove the main theorem.

We normalize as follows by replacing  $f$  by  $f_c$ :

$$(2.1) \quad c \equiv \min\{f(x) + L(-\alpha x) \mid x \in \mathbf{R}^n\} = 0.$$

We may as well assume, by replacing  $l$  by the convex envelope  $g^{**}$  of the function  $g$  given by  $g(x) = \min\{L(x), l(x) + A(|x| + 1)\}$ , where  $A$  is a constant chosen sufficiently large, that

$$(2.2) \quad L(x) \geq l(x) \quad \text{for } x \in \mathbf{R}^n,$$

$$(2.3) \quad f(x) + l(-\alpha x) \geq 0 \quad \text{for } x \in \mathbf{R}^n,$$

$$(2.4) \quad \phi(x) + \frac{1}{\alpha}l(-\alpha x) \geq -C \quad \text{for } x \in \mathbf{R}^n,$$

where  $C$  is a positive constant.

We use the following two technical lemmas for convex functions.

**Lemma 2.1.** *Let  $T > 0$ . For any convex function  $g \in C(\mathbf{R}^n)$  and absolutely continuous curve  $X : [0, T] \rightarrow \mathbf{R}^n$  we have*

$$\frac{d}{dt}g(X(t)) = \xi \cdot \dot{X}(t) \quad \text{for all } \xi \in D^-g(X(t))$$

for almost all  $t \in (0, T)$ .

See [Br, Lemme 3.3] for a proposition similar to the above. The following proof is an adaptation of that of [Br, Lemme 3.3].

**Proof.** Since  $g$  is locally Lipschitz continuous in  $\mathbf{R}^n$ , the function  $t \mapsto g(X(t))$  is absolutely continuous on  $[0, T]$ . Therefore, there is a null set  $N$  of  $(0, T)$  such that for all  $s \in (0, T) \setminus N$  the functions  $X$  and  $t \mapsto g(X(t))$  are differentiable at  $s$ . Let  $s \in (0, T) \setminus N$  and  $\xi \in D^-g(X(s))$ . For  $h > 0$  sufficiently small, we have

$$g(X(s+h)) - g(X(s)) \geq \xi \cdot (X(s+h) - X(s))$$

and

$$g(X(s-h)) - g(X(s)) \geq \xi \cdot (X(s-h) - X(s)).$$

Dividing these by  $h$  and sending  $h \rightarrow 0$ , we find that

$$\left. \frac{d}{dt} g(X(t)) \right|_{t=s} = \xi \cdot \dot{X}(s),$$

which completes the proof.  $\square$

**Lemma 2.2.** *Under the same hypotheses as in Lemma 2.1, we have*

$$\int_0^T g(-\alpha X(t) - \dot{X}(t)) dt \geq \int_0^T g(-\alpha X(t)) dt + \frac{1}{\alpha} [g(-\alpha X(T)) - g(-\alpha X(0))]$$

**Proof.** We observe by Lemma 2.1 that for almost all  $t \in (0, T)$  and for all  $\xi \in D^-g(-\alpha X(t))$ ,

$$\frac{d}{dt} g(-\alpha X(t)) = -\alpha \xi \cdot \dot{X}(t),$$

and therefore

$$\begin{aligned} g(-\alpha X(t) - \dot{X}(t)) &\geq g(-\alpha X(t)) - \xi \cdot \dot{X}(t) \\ &= g(-\alpha X(t)) + \frac{1}{\alpha} \frac{d}{dt} g(-\alpha X(t)). \end{aligned}$$

Next, integrating the above over  $[0, T]$ , we get

$$\begin{aligned} \int_0^T g(-\alpha X(t) - \dot{X}(t)) dt &\geq \int_0^T \left[ g(-\alpha X(t)) + \frac{1}{\alpha} \frac{d}{dt} g(-\alpha X(t)) \right] dt \\ &= \int_0^T g(-\alpha X(t)) dt + \frac{1}{\alpha} (g(-\alpha X(T)) - g(-\alpha X(0))), \end{aligned}$$

which was to be shown.  $\square$

**Lemma 2.3.** *The function  $d$  is a continuous function on  $\mathbf{R}^n \times \mathbf{R}^n$  and has the properties:*

$$(2.5) \quad d(x, y) \leq d(x, z) + d(z, y) \quad \text{and} \quad d(x, x) = 0 \quad \text{for all } x, y, z \in \mathbf{R}^n.$$

**Proof.** 1. Fix any  $x, y \in \mathbf{R}^n$  and  $\varepsilon > 0$ . Choose  $T > 0$  and  $X \in \mathcal{C}(x, y, T)$  so that

$$d(x, y) + \varepsilon > \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt.$$

By virtue of Lemma 2.2 we get

$$d(x, y) + \varepsilon > \frac{1}{\alpha} (L(-\alpha y) - L(-\alpha x)),$$

which readily yields

$$(2.6) \quad d(x, y) \geq \frac{1}{\alpha}(L(-\alpha y) - L(-\alpha x)) \quad \text{for } x, y \in \mathbf{R}^n.$$

In particular we find that  $d(x, y) \in \mathbf{R}$  for all  $x, y \in \mathbf{R}^n$ . We have as well

$$(2.7) \quad d(x, y) \geq \frac{1}{\alpha}(l(-\alpha y) - l(-\alpha x)) \quad \text{for } x, y \in \mathbf{R}^n.$$

2. We fix  $R > 0$  and  $x, y \in B(0, R)$ . We set

$$C_R = \|f\|_{L^\infty(B(0, R))} + \|L\|_{L^\infty(B(0, \alpha R + 1))}.$$

We assume that  $x \neq y$ . Define the curve  $X \in \mathcal{C}(x, y, |x - y|)$  by

$$X(t) = x + \frac{t}{|x - y|}(y - x) \quad \text{for } 0 \leq t \leq |x - y|,$$

and observe that

$$(2.8) \quad d(x, y) \leq \int_0^{|x-y|} [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt \leq C_R |x - y|.$$

Next we consider the case when  $x = y$ . Fix any  $T > 0$  and set  $X(t) = x$  for  $t \in [0, T]$ . Then we have

$$d(x, y) \leq \int_0^T [f(x) + L(-\alpha x)] dt \leq C_R T,$$

and hence, we find that (2.8) holds also in the case when  $x = y$ . From (2.6) and (2.8), we see that  $d(x, x) = 0$  for all  $x \in \mathbf{R}^n$ .

3. Let  $x, y, z \in \mathbf{R}^n$ . Let  $T > 0$ ,  $S > 0$ ,  $X \in \mathcal{C}(x, z, T)$ , and  $Y \in \mathcal{C}(z, y, S)$ . Define  $Z \in \mathcal{C}(x, y, T + S)$  by

$$Z(t) = \begin{cases} X(t) & \text{for } 0 \leq t \leq T, \\ Y(t - T) & \text{for } T \leq t \leq T + S. \end{cases}$$

Then we calculate that

$$\begin{aligned} & \int_0^{T+S} [f(Z(t)) + L(-\alpha Z(t) - \dot{Z}(t))] dt \\ &= \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \int_0^S [f(Y(t)) + L(-\alpha Y(t) - \dot{Y}(t))] dt \end{aligned}$$

and observe that

$$\begin{aligned} d(x, y) &\leq \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt \\ &\quad + \int_0^S [f(Y(t)) + L(-\alpha Y(t) - \dot{Y}(t))] dt, \end{aligned}$$

and conclude that

$$(2.9) \quad d(x, y) \leq d(x, z) + d(z, y) \quad \text{for } x, y, z \in \mathbf{R}^n.$$

4. Let  $R > 0$  and  $x, y, \xi, \eta \in B(0, R)$ . We observe that

$$\begin{aligned} d(x, y) - d(\xi, \eta) &\leq d(x, \xi) + d(\xi, \eta) + d(\eta, y) - d(\xi, \eta) \\ &= d(x, \xi) + d(\eta, y) \leq C_R(|x - \xi| + |\eta - y|). \end{aligned}$$

By symmetry we have

$$|d(x, y) - d(\xi, \eta)| \leq C_R(|x - \xi| + |y - \eta|).$$

Thus we see that  $d$  is locally Lipschitz continuous on  $\mathbf{R}^n \times \mathbf{R}^n$ .  $\square$

**Lemma 2.4.**  $\psi \in C(\mathbf{R}^n)$ .

**Proof.** 1. We show first that

$$(2.10) \quad \psi(x) \geq -\frac{1}{\alpha}l(-\alpha x) - C \quad \text{for } x \in \mathbf{R}^n,$$

which, in particular, implies that  $\psi(x) \in \mathbf{R}$  for all  $x \in \mathbf{R}^n$ . Let  $x \in \mathbf{R}^n$  and  $\varepsilon > 0$ . Choose  $T > 0$  and  $X \in \mathcal{C}(x, T)$  so that

$$\psi(x) + \varepsilon > \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)).$$

Hence, by Lemma 2.2, we get

$$\begin{aligned} \psi(x) + \varepsilon &> \int_0^T [f(X(t)) + l(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)) \\ &\geq \frac{1}{\alpha}(l(-\alpha X(T)) - l(-\alpha x)) + \phi(X(T)) \geq -\frac{1}{\alpha}l(-\alpha x) - C, \end{aligned}$$

proving (2.10).

2. Let  $\varepsilon > 0$  and  $x, y \in B(0, R)$ . Set as before

$$C_R = \|f\|_{L^\infty(B(0, R))} + \|L\|_{L^\infty(B(0, \alpha R + 1))}.$$

We show that

$$(2.11) \quad |\psi(x) - \psi(y)| \leq C_R |x - y|.$$

We may assume that  $x \neq y$ . Choose  $T > 0$  and  $X \in \mathcal{C}(x, T)$  so that

$$\psi(x) + \varepsilon > \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)).$$

Define  $Y \in \mathcal{C}(y, T + |x - y|)$  by

$$Y(t) = \begin{cases} y + \frac{t}{|x - y|}(x - y) & \text{for } 0 \leq t \leq |x - y|, \\ X(t - |x - y|) & \text{for } |x - y| \leq t \leq T + |x - y|. \end{cases}$$

Then we have

$$\begin{aligned} \psi(y) &\leq \int_0^{T+|x-y|} [f(Y(t)) + L(-\alpha Y(t) - \dot{Y}(t))] dt + \phi(Y(T + |x - y|)) \\ &= \int_0^{|x-y|} [f(Y(t)) + L(-\alpha Y(t) - \dot{Y}(t))] dt \\ &\quad + \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)) \\ &< C_R |x - y| + \varepsilon + \psi(x). \end{aligned}$$

From this we obtain (2.11) and conclude the proof.  $\square$

We are now in a position to prove Proposition 1.1.

**Proof of Proposition 1.1.** In view of Lemmas 2.3 and 2.4, we only need to show that  $v$  is continuous on  $\mathbf{R}^n$ . For each  $R > 0$  the collection of the functions

$$x \mapsto d(x, y) + \psi(y)$$

on  $B(0, R)$ , with  $y \in Z$ , is uniformly bounded and equi-continuous because of Lemmas 2.3 and 2.4 and the compactness of  $Z$ . From this observation, it is easy to see that  $v \in C(B(0, R))$  for all  $R > 0$ , which shows that  $v \in C(\mathbf{R}^n)$ .  $\square$

**Lemma 2.5.** *Let  $R > 0$  and  $\varepsilon > 0$ . Then there is a constant  $T > 0$  such that for each  $x, y \in B(0, R)$  there are  $S \in (0, T]$  and  $X \in \mathcal{C}(x, y, S)$ ,*

$$d(x, y) + \varepsilon > \int_0^S [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt.$$

**Proof.** Let  $R > 0$  and  $\varepsilon \in (0, 1)$ . Fix  $(\bar{x}, \bar{y}) \in B(0, R) \times B(0, R)$ . Choose  $\bar{T} > 0$  and  $Y \in \mathcal{C}(\bar{x}, \bar{y}, \bar{T})$  so that

$$d(\bar{x}, \bar{y}) + \frac{\varepsilon}{4} > \int_0^{\bar{T}} [f(Y(t)) + L(-\alpha Y(t) - \dot{Y}(t))] dt.$$

Let  $C_R$  be a positive constant such that

$$\|f\|_{L^\infty(B(0, R+1))} + \|L\|_{L^\infty(B(0, \alpha(R+1)+1))} \leq C_R.$$

Fix  $\delta \in (0, 1)$  so that

$$2C_R\delta \leq \frac{\varepsilon}{4},$$

$$|d(x, y) - d(\bar{x}, \bar{y})| < \frac{\varepsilon}{2} \quad \text{for } x \in B(\bar{x}, \delta), y \in B(\bar{y}, \delta).$$

Let  $x \in B(\bar{x}, \delta)$  and  $y \in B(\bar{y}, \delta)$ . Define  $\xi \in \mathcal{C}(x, \bar{x}, \delta)$  and  $\eta \in \mathcal{C}(\bar{y}, y, \delta)$ , respectively, by

$$\xi(t) = x + \frac{t}{\delta}(\bar{x} - x) \quad \text{for } 0 \leq t \leq \delta,$$

$$\eta(t) = \bar{y} + \frac{t}{\delta}(y - \bar{y}) \quad \text{for } 0 \leq t \leq \delta.$$

Noting that  $\dot{\xi}(t), \dot{\eta}(t) \in B(0, 1)$  for all  $t \in [0, \delta]$ , we see that

$$\int_0^\delta [f(\xi(t)) + L(-\alpha\xi(t) - \dot{\xi}(t))] dt \leq C_R\delta,$$

$$\int_0^\delta [f(\eta(t)) + L(-\alpha\eta(t) - \dot{\eta}(t))] dt \leq C_R\delta.$$

Next define the function  $X \in \mathcal{C}(x, y, \bar{T} + 2\delta)$  by

$$X(t) = \begin{cases} \xi(t) & \text{for } t \in [0, \delta], \\ Y(t - \delta) & \text{for } t \in [\delta, \bar{T} + \delta], \\ \eta(t - \bar{T} - \delta) & \text{for } t \in [\bar{T} + \delta, \bar{T} + 2\delta]. \end{cases}$$

Then we have

$$\begin{aligned} & \int_0^{\bar{T}+2\delta} [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt \\ &= \int_0^\delta [f(\xi(t)) + L(-\alpha\xi(t) - \dot{\xi}(t))] dt \\ &+ \int_0^{\bar{T}} [f(Y(t)) + L(-\alpha Y(t) - \dot{Y}(t))] dt \\ &+ \int_0^\delta [f(\eta(t)) + L(-\alpha\eta(t) - \dot{\eta}(t))] dt \\ &\leq 2C_R\delta + d(\bar{x}, \bar{y}) + \frac{\varepsilon}{4} \leq d(\bar{x}, \bar{y}) + \frac{\varepsilon}{2} \\ &< d(x, y) + \varepsilon. \end{aligned}$$

Thus we conclude that for each  $(\bar{x}, \bar{y}) \in B(0, R) \times B(0, R)$  there are constants  $\bar{S} > 0$  and  $\delta > 0$  such that for any  $x \in B(\bar{x}, \delta)$  and  $y \in B(\bar{y}, \delta)$  we have

$$d(x, y) + \varepsilon > \int_0^{\bar{S}} [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt$$

for some  $X \in \mathcal{C}(x, y, \bar{S})$ .

By the compactness of  $B(0, R) \times B(0, R)$ , there is a finite collection of  $(x_k, y_k) \in B(0, R) \times B(0, R)$ ,  $S_k > 0$ , and  $\delta_k > 0$ , where  $k = 1, 2, \dots, N$ , such that

$$B(0, R) \times B(0, R) \subset \bigcup_{k=1}^N B(x_k, \delta_k) \times B(y_k, \delta_k),$$

and such that for any  $k \in \{1, 2, \dots, N\}$ ,  $x \in B(x_k, \delta_k)$ , and  $y \in B(y_k, \delta_k)$ ,

$$d(x, y) + \varepsilon > \int_0^{S_k} [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt$$

for some  $X \in \mathcal{C}(x, y, S_k)$ . Setting  $T = \max_{1 \leq k \leq N} S_k$ , we observe that for any  $(x, y) \in B(0, R) \times B(0, R)$ ,

$$d(x, y) + \varepsilon > \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt$$

for some  $S \in (0, T]$  and  $X \in \mathcal{C}(x, y, S)$ . The proof is now complete.  $\square$

**Lemma 2.6.** *For each  $R > 0$  and  $\varepsilon > 0$  there is a constant  $T > 0$  such that for any  $x \in B(0, R)$ ,*

$$\psi(x) + \varepsilon > \int_0^S [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(S))$$

for some  $S \in (0, T]$  and  $X \in \mathcal{C}(x, S)$ .

The proof of this lemma is similar to that of Lemma 2.5 and we omit presenting it here.  $\square$

We shall give a proof of Theorem 1.2 in Section 4 and we concede its validity in the following arguments in this and the next sections. In what follows we let  $u \in C(\mathbf{R}^n \times [0, \infty))$  denote the unique viscosity solution of (1.4) and (1.5).

**Lemma 2.7.** *For  $(x, T) \in \mathbf{R}^n \times (0, \infty)$ ,*

$$u(x, T) = \inf \left\{ \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)) \mid X \in \mathcal{C}(x, T) \right\}.$$

**Lemma 2.8.** *For each  $R > 0$  there is a constant  $C_1(R) > 0$  such that*

$$|u(x, T)| \leq C_1(R) \quad \text{for } (x, T) \in B(0, R) \times [0, \infty).$$

**Lemma 2.9.** *For each  $R > 0$  there is a constant  $C_2(R) > 0$  such that for  $x \in B(0, R)$ ,  $T > 0$ , and  $X \in \mathcal{C}(x, T)$ , if*

$$u(x, T) + 1 > \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)),$$

*then*

$$|X(t)| \leq C_2(R) \quad \text{for } t \in (0, T).$$

The proof of the above three lemmas will be presented in Section 4.

### 3. Proof of Theorem 1.3

This section will be devoted to the proof of Theorem 1.3.

1. Fix  $R > 0$  and we show that

$$\lim_{t \rightarrow \infty} \max_{B(0, R)} |u(x, t) - v(x)| = 0.$$

Note in view of Proposition 1.1 and the compactness of  $Z$  that

$$v(x) = \min\{d(x, y) + \psi(y) \mid y \in Z\}.$$

2. Choose  $\rho > 0$  so that  $Z \subset B(0, \rho)$ . Fix any  $\varepsilon > 0$  and choose a constant  $T > 0$  so that for any  $(x, y) \in B(0, R) \times B(0, \rho)$ ,

$$(3.1) \quad d(x, y) + \varepsilon > \int_0^A [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt,$$

$$(3.2) \quad \psi(y) + \varepsilon > \int_0^B [f(Y(t)) + L(-\alpha Y(t) - \dot{Y}(t))] dt + \phi(Y(B))$$

for some  $A, B \in (0, T]$ ,  $X \in \mathcal{C}(x, y, A)$ , and  $Y \in \mathcal{C}(y, B)$ .

3. We show that for all  $t \geq 2T$  and  $x \in B(0, R)$ ,

$$u(x, t) \leq v(x) + 2\varepsilon.$$

Fix  $x \in B(0, R)$  and choose  $y \in Z$  so that

$$v(x) = d(x, y) + \psi(y).$$

We choose  $A, B \in (0, T]$ ,  $X \in \mathcal{C}(x, y, A)$ , and  $Y \in \mathcal{C}(y, B)$  so that (3.1) and (3.2) hold.

For any  $t \geq 2T$  we define  $\xi \in \mathcal{C}(x, t)$  by

$$\xi(s) = \begin{cases} X(s) & \text{for } s \in [0, A], \\ y & \text{for } s \in [A, t - B], \\ Y(s - (t - B)) & \text{for } s \in [t - B, t]. \end{cases}$$

Using the fact that  $f(y) + L(-\alpha y) = 0$  because  $y \in Z$ , we observe that

$$\begin{aligned} & \int_0^t [f(\xi(s)) + L(-\alpha \xi(s) - \dot{\xi}(s))] ds + \phi(\xi(t)) \\ &= \int_0^A [f(X(s)) + L(-\alpha X(s) - \dot{X}(s))] ds + \int_A^{t-B} [f(y) + L(-\alpha y)] ds \\ & \quad + \int_0^B [f(Y(s)) + L(-\alpha Y(s) - \dot{Y}(s))] ds + \phi(Y(B)) \\ &< d(x, y) + \varepsilon + 0 + \psi(y) + \varepsilon = v(x) + 2\varepsilon. \end{aligned}$$

From this and Lemma 2.7, we find that for all  $t \geq 2T$ ,

$$u(x, t) \leq v(x) + 2\varepsilon,$$

which shows that

$$\lim_{t \rightarrow \infty} \max_{B(0, R)} \max\{u(x, t) - v(x), 0\} = 0.$$

4. Fix  $\varepsilon \in (0, 1)$ . Let  $C_1(R) > 0$  and  $C_2(R) > 0$  be the constants, respectively, from Lemmas 2.8 and 2.9. We may assume that

$$Z \subset B(0, C_2(R)).$$

We may assume that

$$\begin{aligned} & \|f\|_{L^\infty(B(0, C_2(R)))} + \|L\|_{L^\infty(B(0, \alpha C_2(R)+1))} \leq C_1(R), \\ & \frac{1}{\alpha} \|L\|_{L^\infty(B(0, R))} \leq C_1(R). \end{aligned}$$

Choose a constant  $\delta \in (0, 1)$  so that

$$2C_2(R)\delta \leq \varepsilon.$$

We set

$$Z_\delta = \{x \in \mathbf{R}^n \mid \text{dist}(x, Z) < \delta\}.$$

Since  $f(x) + L(-\alpha x) \geq L(-\alpha x) - l(-\alpha x)$  for all  $x \in \mathbf{R}^n$  by (2.3), recalling (A4), we see that the function  $f(x) + L(-\alpha x)$  attains a minimum over the closed set  $\mathbf{R}^n \setminus Z_\delta$ . Let  $\gamma$  be the minimum value of this function over  $\mathbf{R}^n \setminus Z_\delta$ . Note that  $\gamma > 0$ . We set  $T = \gamma^{-1}(C + 2C_1(R) + 1)$ .

5. Let  $x \in B(0, R)$  and  $t \geq T$ . We select  $X \in \mathcal{C}(x, t)$  in view of Lemma 2.7 so that

$$u(x, t) + \varepsilon > \int_0^t [f(X(s)) + L(-\alpha X(s) - \dot{X}(s))] ds + \phi(X(t)).$$

From this, calculating as in the proof of Lemma 2.2 and using (2.2) and (2.4), we find that

$$u(x, t) + 1 > \int_0^t [f(X(s)) + L(-\alpha X(s))] ds - \frac{1}{\alpha} L(-\alpha x) - C.$$

Hence we have

$$(3.3) \quad \int_0^t [f(X(s)) + L(-\alpha X(s))] ds < C + 2C_1(R) + 1.$$

We show that

$$(3.4) \quad X(s) \in Z_\delta \quad \text{for some } s \in [0, t].$$

Indeed, if this is not the case, we get

$$f(X(s)) + L(-\alpha X(s)) \geq \gamma \quad \text{for all } s \in [0, t].$$

Consequently we get

$$\int_0^t [f(X(s)) + L(-\alpha X(s))] ds \geq \gamma t \geq \gamma T = C + 2C_1(R) + 1,$$

which contradicts (3.3).

By (3.4), we can choose  $\tau \in (0, t)$  and  $z \in Z$  such that  $|X(\tau) - z| < \delta$ . By Lemma 2.9, we get

$$|X(s)| \leq C_2(R) \quad \text{for } s \in [0, t].$$

Define  $\xi \in \mathcal{C}(X(\tau), z, \delta)$  and  $\eta \in \mathcal{C}(z, X(\tau), \delta)$ , respectively, by

$$\begin{aligned} \xi(s) &= X(\tau) + \frac{s}{\delta}(z - X(\tau)) \quad \text{for } s \in [0, \delta], \\ \eta(s) &= z + \frac{s}{\delta}(X(\tau) - z) \quad \text{for } s \in [0, \delta]. \end{aligned}$$

Noting that

$$\begin{aligned} \xi(s), \eta(s) &\in B(0, C_2(R)) \quad \text{for } s \in [0, \delta], \\ \dot{\xi}(s), \dot{\eta}(s) &\in B(0, 1) \quad \text{for } s \in [0, \delta], \end{aligned}$$

we see that

$$\begin{aligned} \int_0^\delta [f(\xi(s)) + L(-\alpha\xi(s) - \dot{\xi}(s))] ds &\leq C_1(R)\delta, \\ \int_0^\delta [f(\eta(s)) + L(-\alpha\eta(s) - \dot{\eta}(s))] ds &\leq C_1(R)\delta. \end{aligned}$$

We define the function  $Y \in \mathcal{C}(x, t + 2\delta)$  by

$$Y(s) = \begin{cases} X(s) & \text{for } s \in [0, \tau], \\ \xi(s - \tau) & \text{for } s \in [\tau, \tau + \delta], \\ \eta(s - \tau - \delta) & \text{for } s \in [\tau + \delta, \tau + 2\delta], \\ X(s - 2\delta) & \text{for } s \in [\tau + 2\delta, t + 2\delta]. \end{cases}$$

We calculate that

$$\begin{aligned} &\int_0^{t+2\delta} [f(Y(s)) + L(-\alpha Y(s) - \dot{Y}(s))] ds + \phi(Y(t + 2\delta)) \\ &= \int_0^t [f(X(s)) + L(-\alpha X(s) - \dot{X}(s))] ds + \phi(X(t)) \\ &\quad + \int_0^\delta [f(\xi(s)) + L(-\alpha\xi(s) - \dot{\xi}(s))] ds \\ &\quad + \int_0^\delta [f(\eta(s)) + L(-\alpha\eta(s) - \dot{\eta}(s))] ds \\ &< u(x, t) + \varepsilon + 2C_1(R)\delta \leq u(x, t) + 2\varepsilon. \end{aligned}$$

On the other hand, we have

$$d(x, z) \leq \int_0^{\tau+\delta} [f(Y(s)) + L(-\alpha Y(s) - \dot{Y}(s))] ds,$$

and, since  $Y(\cdot + \tau + \delta)|_{[0, t-\tau+\delta]} \in \mathcal{C}(z, t - \tau + \delta)$ ,

$$\psi(z) \leq \int_{\tau+\delta}^{t+2\delta} [f(Y(s)) + L(-\alpha Y(s) - \dot{Y}(s))] ds + \phi(Y(t + 2\delta)).$$

These together yield

$$d(x, z) + \psi(z) \leq u(x, t) + 2\varepsilon,$$

Since  $z \in Z$ , we get

$$v(x) \leq u(x, t) + 2\varepsilon.$$

This shows that

$$\lim_{t \rightarrow \infty} \max_{x \in B(0, R)} \max\{v(x) - u(x, t), 0\} = 0.$$

□

#### 4. Existence of a solution of the Cauchy problem

In this section we establish the existence part of Theorem 1.2 together with some estimates on the solution of (1.4) and (1.5). Our strategy here for proving existence of a viscosity solution of (1.4) and (1.5) is to prove (i) the continuity of the function  $u$  on  $\mathbf{R}^n \times [0, \infty)$  given by

$$u(x, T) = \inf \left\{ \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)) \mid X \in \mathcal{C}(x, T) \right\}$$

and then (ii) to show that the function  $u$  is a viscosity solution of (1.4) and (1.5) by using the dynamic programming principle.

We assume as in the previous sections that (A1)–(A4) and (2.1)–(2.4) hold.

**Lemma 4.1.** *We have*

$$u(x, T) \geq -\frac{1}{\alpha} l(-\alpha x) - C \quad \text{for } (x, T) \in \mathbf{R}^n \times [0, \infty),$$

where  $C$  is the constant from (2.4).

**Proof.** For each  $\varepsilon \in (0, 1)$  and  $(x, T) \in \mathbf{R}^n \times (0, \infty)$  there is a curve  $X \in \mathcal{C}(x, T)$  such that

$$u(x, T) + \varepsilon > \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)).$$

We then compute with use of Lemma 2.2 that

$$\begin{aligned} u(x, T) + \varepsilon &> \int_0^T [f(X(t)) + l(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)) \\ &\geq \frac{1}{\alpha} [l(-\alpha X(T)) - l(-\alpha x)] + \phi(X(T)) \\ &\geq -\frac{1}{\alpha} l(-\alpha x) - C. \end{aligned} \quad \square$$

**Lemma 4.2.** *For all  $(x, T) \in \mathbf{R}^n \times (0, \infty)$ , we have*

$$u(x, T) \leq \phi(x) + (f(x) + L(-\alpha x))T.$$

**Proof.** For any  $(x, T) \in \mathbf{R}^n \times (0, \infty)$ , by choosing the curve  $X(t) \equiv x$ , we find that

$$\begin{aligned} u(x, T) &= \inf \left\{ \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt \right. \\ &\quad \left. + \phi(X(T)) \mid X \in \mathcal{C}(x, T) \right\} \\ &\leq \int_0^T [f(x) + L(-\alpha x)] dt + \phi(x) = \phi(x) + [f(x) + L(-\alpha x)]T. \end{aligned} \quad \square$$

**Lemma 4.3.** *Let  $x_0 \in Z$ . For  $(x, T) \in \mathbf{R}^n \times [1, \infty)$  we have*

$$u(x, T) \leq \phi(x_0) + \|f\|_{L^\infty(B(0, \rho_0(x)))} + \|L\|_{L^\infty(B(0, \rho_1(x)))},$$

where  $\rho_0(x) = \max\{|x|, |x_0|\}$  and  $\rho_1(x) := \alpha\rho_0(x) + |x - x_0|$ .

Notice at this point that Lemmas 4.1, 4.2, and 4.3 yield Lemma 2.8.

**Proof.** Define the curve  $X \in \mathcal{C}(x, T)$  by

$$X(t) = \begin{cases} x + t(x_0 - x) & \text{for } 0 \leq t \leq 1, \\ x_0 & \text{for } 1 \leq t. \end{cases}$$

Observe that

$$\begin{aligned} |X(t)| &\leq \max\{|x|, |x_0|\} = \rho_0(x), \\ |\dot{X}(t)| &\leq |x - x_0|, \\ |\alpha X(t) + \dot{X}(t)| &\leq \alpha\rho_0(x) + |x - x_0| = \rho_1(x). \end{aligned}$$

Therefore we get

$$\begin{aligned} u(x, T) &\leq \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)) \\ &= \int_0^1 [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(x_0) \\ &\leq \int_0^1 (\|f\|_{L^\infty(B(0, \rho_0(x)))} + \|L\|_{L^\infty(B(0, \rho_1(x)))}) dt + \phi(x_0) \\ &= \|f\|_{L^\infty(B(0, \rho_0(x)))} + \|L\|_{L^\infty(B(0, \rho_1(x)))} + \phi(x_0). \quad \square \end{aligned}$$

We have to pay attention on the dependence of our estimates on  $\phi$  in the following arguments. To describe the necessary dependence on  $\phi$ , we introduce collections  $\{M_R\}_{R>0}$  of positive constants and  $\{\omega_R\}_{R>0}$  of moduli such that for all  $R > 0$ ,

$$(4.1) \quad \|\phi\|_{L^\infty(B(0, R))} + |\phi(x_0)| \leq M_R,$$

$$(4.2) \quad |\phi(x) - \phi(y)| \leq \omega_R(|x - y|) \quad \text{for } x, y \in B(0, R).$$

Here and henceforth a modulus means a real-valued function  $\omega \in C([0, \infty))$  such that  $\omega$  is non-decreasing on  $[0, \infty)$  and  $\omega(0) = 0$ . In what follows we fix such a pair of collections  $\{M_R\}_{R>0}$  and  $\{\omega_R\}_{R>0}$  and set  $\Lambda = (\{M_R\}_{R>0}, \{\omega_R\}_{R>0})$  for notational simplicity.

**Lemma 4.4.** *Let  $x_0 \in Z$ . For each  $R > 0$  there is a constant  $C(R) > 0$ , depending on  $\phi$  only through  $M_R$  and  $\phi(x_0)$ , such that for  $(x, T) \in B(0, R) \times (0, \infty)$  and  $X \in \mathcal{C}(x, T)$ , if*

$$u(x, T) + 1 > \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)),$$

*then*

$$|X(t)| \leq C(R) \quad \text{for } t \in [0, T].$$

Note that the above lemma is equivalent to Lemma 2.9 except the description of the dependence of the constant  $C(R)$  on  $\phi$ .

**Proof.** Due to Lemmas 4.2 and 4.3, there is a constant  $C_1(R) > 0$ , independent of  $T \in [0, \infty)$ , such that

$$u(x, T) \leq C_1(R) \quad \text{for } x \in B(0, R).$$

Note that  $C_1(R)$  may be chosen so as to depend on  $\phi$  only through  $M_R$  and  $\phi(x_0)$ .

Let  $\tau \in [0, T]$ . Using Lemmas 2.2 and 4.1, we compute that

$$\begin{aligned} C_1(R) + 1 &\geq u(x, T) + 1 > \int_0^\tau [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt \\ &\quad + \int_\tau^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)) \\ &\geq \int_0^\tau [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + u(X(\tau), T - \tau) \\ &\geq \int_0^\tau [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt - \frac{1}{\alpha} l(-\alpha X(\tau)) - C \\ &\geq \frac{1}{\alpha} [L(-\alpha X(\tau)) - L(-\alpha x) - l(-\alpha X(\tau))] - C, \end{aligned}$$

which yields

$$L(-\alpha X(\tau)) - l(-\alpha X(\tau)) \leq \alpha(C + C_1(R) + 1 + C_2(R)),$$

where  $C_2(R) > 0$  is a constant such that

$$\frac{1}{\alpha} L(-\alpha x) \leq C_2(R) \quad \text{for } x \in B(0, R).$$

Thus, thanks to (A4), we find a constant  $C(R) > 0$ , depending on  $\phi$  only through  $M_R$  and  $\phi(x_0)$ , such that

$$|X(t)| \leq C(R) \quad \text{for } t \in [0, T].$$

□

**Lemma 4.5.** *For each  $R > 0$  there is a modulus  $\sigma_R$ , depending on  $\phi$  only through  $M_R$  and  $\phi(x_0)$ , such that for  $(x, T) \in B(0, R) \times (0, 1]$  and  $X \in \mathcal{C}(x, T)$ , if*

$$u(x, T) + 1 > \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)),$$

then

$$|X(t) - x| \leq \sigma_R(t) \quad \text{for } t \in [0, T].$$

**Proof.** According to Lemmas 4.3 and 4.4, there is a constant  $C(R) > 0$ , depending on  $\phi$  only through  $M_R$  and  $\phi(x_0)$ , such that

$$\begin{aligned} |X(t)| &\leq C(R) \quad \text{for } t \geq 0, \\ u(x, T) &\leq C(R) \quad \text{for } (x, T) \in B(0, R) \times [0, \infty). \end{aligned}$$

We choose a constant  $C_1(R) > 0$  so that

$$\begin{aligned} \|f\|_{L^\infty(B(0, C(R)))} &\leq C_1(R), \\ \|u\|_{L^\infty(B(0, C(R)) \times [0, \infty))} &\leq C_1(R). \end{aligned}$$

In view of Lemmas 4.1, 4.2, 4.3, and 4.4, the constant  $C_1(R)$  can be chosen so as to depend on  $\phi$  only through  $M_R$  and  $\phi(x_0)$ .

For any  $A > 0$  there is a constant  $C_A > 0$  such that

$$L(x) \geq A|x| - C_A \quad \text{for } x \in \mathbf{R}^n.$$

Fix  $A > 0$ , and we calculate that

$$\begin{aligned} C(R) + 1 &> \int_0^\tau [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + u(X(\tau), T - \tau) \\ &\geq \int_0^\tau [-C_1(R) + L(-\alpha X(t) - \dot{X}(t))] dt - C_1(R) \\ &\geq \int_0^\tau [-C_1(R) + A|\dot{X}(t) + \alpha X(t)| - C_A] dt - C_1(R) \\ &\geq \int_0^\tau [-C_1(R) - A\alpha C(R) + A|\dot{X}(t)| - C_A] dt - C_1(R). \end{aligned}$$

Hence we get

$$\int_0^\tau |\dot{X}(t)| dt \leq A^{-1}(C(R) + 1 + C_1(R)) + (\alpha C(R) + A^{-1}C_A + A^{-1}C_1(R))\tau.$$

There is a modulus  $\sigma_R$ , depending on  $\phi$  only through  $M_R$  and  $\phi(x_0)$ , such that

$$\inf_{A>0} [A^{-1}(C(R) + C_1(R) + 1) + (\alpha C(R) + A^{-1}C_A + A^{-1}C_1(R))t] \leq \sigma_R(t).$$

Fix such a modulus  $\sigma_R$ , and we have

$$\int_0^\tau |\dot{X}(t)| dt \leq \sigma_R(\tau),$$

which implies that

$$|X(\tau) - x| \leq \sigma_R(\tau). \quad \square$$

**Lemma 4.6.** *For each  $R > 0$  there is a modulus  $\nu_R$ , depending on  $\phi$  only through  $\Lambda$ , such that for  $(x, t) \in B(0, R) \times [0, 1]$ ,*

$$(4.3) \quad |u(x, t) - \phi(x)| \leq \nu_R(t).$$

**Proof.** For any  $\varepsilon \in (0, 1)$ ,  $R > 0$ , and  $(x, t) \in B(0, R) \times (0, 1]$  there is a curve  $X \in \mathcal{C}(x, T)$  such that

$$u(x, t) + \varepsilon > \int_0^t [f(X(s)) + L(-\alpha X(s) - \dot{X}(s))] ds + \phi(X(t)).$$

According to Lemma 4.5, there is a modulus  $\sigma_R$ , depending on  $\phi$  only through  $\Lambda$ , such that

$$|X(t) - x| \leq \sigma_R(t) \quad \text{for } t \in [0, 1].$$

Recall that

$$\int_0^t [f(X(s)) + L(-\alpha X(s) - \dot{X}(s))] ds \geq \frac{1}{\alpha} [L(-\alpha X(t)) - L(-\alpha x)].$$

Then we have

$$\begin{aligned} u(x, t) + \varepsilon &> \phi(x) + \frac{1}{\alpha} [L(-\alpha X(t)) - L(-\alpha x)] + [\phi(X(t)) - \phi(x)] \\ &\geq \phi(x) + \frac{1}{\alpha} \mu_{L,R}(\alpha \sigma_R(t)) + \mu_{\phi,R}(\sigma_R(t)), \end{aligned}$$

where  $\mu_{L,R}$  and  $\mu_{\phi,R}$  are the moduli of continuity of  $L(x)$  and of  $\phi$ , respectively, on the set  $B(0, R_\alpha)$ , with  $R_\alpha = (R + \sigma_R(1)) \max\{1, \alpha\}$ . Since  $\varepsilon \in (0, 1)$  is arbitrary, the above inequality guarantees together with Lemma 4.2 existence of a modulus  $\nu_R$ , which depends on  $\phi$  only through  $\Lambda$ , such that (4.3) holds.  $\square$

**Lemma 4.7.** *For each  $R > 0$  there is a modulus  $\gamma_R$ , depending on  $\phi$  only through  $\Lambda$ , such that for  $x, y \in B(0, R)$  and  $T > 0$ ,*

$$(4.4) \quad |u(x, T) - u(y, T)| \leq \gamma_R(|x - y|).$$

**Proof.** Let  $R > 0$ ,  $x, y \in B(0, R)$ , and  $T > 0$ . We may assume that  $|x - y| \leq 1$ .

We consider the case when  $|x - y| \geq T$ . By Lemma 4.6, there is a modulus  $\nu_R$ , depending on  $\phi$  only through  $\Lambda$ , such that

$$\begin{aligned} |u(x, T) - \phi(x)| &\leq \nu_R(T), \\ |u(y, T) - \phi(y)| &\leq \nu_R(T). \end{aligned}$$

We may assume that

$$|\phi(x) - \phi(y)| \leq \nu_R(|x - y|).$$

Consequently we have

$$\begin{aligned} |u(x, T) - u(y, T)| &\leq |\phi(x) - \phi(y)| + |u(x, T) - \phi(x)| + |u(y, T) - \phi(y)| \\ &\leq \nu_R(|x - y|) + 2\nu_R(T) \leq 3\nu_R(|x - y|). \end{aligned}$$

Next we consider the case when  $|x - y| \leq T$ . Fix  $\varepsilon \in (0, 1)$ . We select  $X \in \mathcal{C}(x, T)$  so that

$$u(x, T) + \varepsilon > \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)).$$

We know that there is a constant  $C(R) \geq R$ , which depends on  $\phi$  only through  $\Lambda$ , such that

$$|X(t)| \leq C(R) \quad \text{for } t \in [0, T].$$

Define  $Y \in \mathcal{C}(y, T)$  by

$$Y(t) = \begin{cases} y + \frac{t}{|x - y|}(x - y) & \text{for } 0 \leq t \leq |x - y|, \\ X(t - |x - y|) & \text{for } |x - y| \leq t \leq T. \end{cases}$$

We may assume by replacing  $\nu_R$  by a larger modulus if necessary that

$$|u(\xi, |x - y|) - \phi(\xi)| \leq \nu_R(|x - y|) \quad \text{for } \xi \in B(0, C(R)).$$

Then we have

$$\begin{aligned} &\int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T)) \\ &\geq \int_0^{T-|x-y|} [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + u(X(T - |x - y|), |x - y|) \\ &\geq \int_0^{T-|x-y|} [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T - |x - y|)) - \nu_R(|x - y|). \end{aligned}$$

There is a constant  $C_1(R) > 0$  which depends on  $\phi$  only through  $\Lambda$  such that

$$|f(\xi)| + |L(\alpha\xi + \eta)| \leq C_1(R) \quad \text{for } (\xi, \eta) \in B(0, C(R)) \times B(0, 1).$$

Also we have

$$\begin{aligned} u(y, T) &\leq \int_0^T [f(Y(t)) + L(-\alpha Y(t) - \dot{Y}(t))] dt + \phi(Y(T)) \\ &\leq \int_0^{|x-y|} [f(Y(t)) + L(-\alpha Y(t) - \dot{Y}(t))] dt \\ &\quad + \int_0^{T-|x-y|} [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(T - |x - y|)) \\ &\leq C_1(R)|x - y| + \int_0^{T-|x-y|} [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt \\ &\quad + \phi(X(T - |x - y|)). \end{aligned}$$

Therefore we get

$$u(y, T) - u(x, T) < \varepsilon + \nu_R(|x - y|) + C_1(R)|x - y|,$$

and moreover

$$u(y, T) - u(x, T) \leq \nu_R(|x - y|) + C_1(R)|x - y|,$$

which completes the proof of (4.4).  $\square$

**Lemma 4.8 (Dynamic Programming Principle).** *For  $S > 0$ ,  $T > 0$ , and  $x \in \mathbf{R}^n$ , we have*

$$(4.5) \quad u(x, S + T) = \inf \left\{ \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + u(X(T), S) \mid X \in \mathcal{C}(x, T) \right\}.$$

We omit giving the proof of this lemma and we refer to [L] for a proof in a standard case.

We extend the domain of definition of  $u$  to  $\mathbf{R}^n \times [0, \infty)$  by setting

$$(4.6) \quad u(x, 0) = \phi(x) \quad \text{for } x \in \mathbf{R}^n.$$

**Lemma 4.9.**  $u \in C(\mathbf{R}^n \times [0, \infty))$ .

**Proof.** In view of Lemma 4.7, there is a pair of collections  $\{N_R\}_{R>0}$  and  $\{\gamma_R\}_{R>0}$  such that for any  $R > 0$ ,

$$(4.7) \quad \|u(\cdot, S)\|_{L^\infty(B(0, R))} \leq N_R \quad \text{for } S \in [0, \infty),$$

$$(4.8) \quad |u(x, S) - u(y, S)| \leq \gamma_R(|x - y|) \quad \text{for } x, y \in B(0, R), S \in [0, \infty).$$

Fix any  $S \geq 0$  and note by Lemma 4.8 that

$$u(x, S + T) = \inf \left\{ \int_0^T [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + u(X(T), S) \mid X \in \mathcal{C}(x, T) \right\}.$$

Then apply Lemma 4.6, with  $\phi = u(\cdot, S)$  and  $\Lambda = (\{N_R\}_{R>0}, \{\gamma_R\}_{R>0})$ , to conclude that for each  $R > 0$  there is a modulus  $\nu_R$  such that

$$|u(x, S + T) - u(x, S)| \leq \nu_R(T) \quad \text{for } (x, S) \in B(0, R) \times [0, \infty) \text{ and } T > 0.$$

That is, we have

$$|u(x, t) - u(x, s)| \leq \nu_R(|t - s|) \quad \text{for } x \in B(0, R) \text{ and } t, s \in [0, \infty).$$

This and Lemma 4.7 ensure that  $u$  is uniformly continuous on  $B(0, R) \times [0, \infty)$  for any  $R > 0$ . In particular, we see that  $u \in C(\mathbf{R}^n \times [0, \infty))$ .  $\square$

**Theorem 4.10.** *The function  $u$  is a viscosity solution of (1.4) and (1.5).*

Now, Lemma 4.1 and Theorem 4.10 above guarantee that the existence part of Theorem 1.2 is valid.

**Proof.** Let  $\varphi \in C^1(\mathbf{R}^n \times (0, \infty))$  and  $(\hat{x}, \hat{t}) \in \mathbf{R}^n \times (0, \infty)$ .

We first assume that  $u - \varphi$  attains a maximum at  $(\hat{x}, \hat{t})$ . We may assume without loss of generality that  $(u - \varphi)(\hat{x}, \hat{t}) = 0$ , so that  $u \leq \varphi$  in  $\mathbf{R}^n \times (0, \infty)$ .

Let  $\varepsilon \in (0, \hat{t})$  and  $z \in \mathbf{R}^n$ . Setting  $X(t) := \hat{x} + tz$ , by Lemma 4.8, we get

$$\begin{aligned} \varphi(\hat{x}, \hat{t}) = u(\hat{x}, \hat{t}) &\leq \int_0^\varepsilon [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + u(X(\varepsilon), \hat{t} - \varepsilon) \\ &\leq \int_0^\varepsilon [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \varphi(X(\varepsilon), \hat{t} - \varepsilon) \end{aligned}$$

and hence

$$0 \leq \int_0^\varepsilon [f(X(t)) + L(-\alpha X(t) - z) - \varphi_t(X(t), \hat{t} - t) + D\varphi(X(t), \hat{t} - t) \cdot z] dt.$$

Dividing this by  $\varepsilon$  and sending  $\varepsilon \rightarrow 0$  yield

$$0 \leq f(\hat{x}) + L(-\alpha \hat{x} - z) - \varphi_t(\hat{x}, \hat{t}) + D\varphi(\hat{x}, \hat{t}) \cdot z \quad \text{for } z \in \mathbf{R}^n,$$

which implies that for  $\xi \in \mathbf{R}^n$ ,

$$\varphi_t(\hat{x}, \hat{t}) + \alpha \hat{x} \cdot D\varphi(\hat{x}, \hat{t}) + \xi \cdot D\varphi(\hat{x}, \hat{t}) - L(\xi) \leq f(\hat{x}).$$

Thus we get

$$\varphi_t(\hat{x}, \hat{t}) + \alpha x \cdot D\varphi(\hat{x}, \hat{t}) + H(D\varphi(\hat{x}, \hat{t})) \leq f(\hat{x}),$$

which was to be shown.

We next assume that  $u - \varphi$  attains a minimum at  $(\hat{x}, \hat{t})$ . We may assume that  $u(\hat{x}, \hat{t}) = \varphi(\hat{x}, \hat{t})$  and  $u \geq \varphi$  in  $\mathbf{R}^n \times (0, \infty)$ .

Let  $\varepsilon \in (0, 1)$  and choose, in view of Lemma 4.8,  $X \in \mathcal{C}(\hat{x}, \varepsilon)$  so that

$$\begin{aligned} \varphi(\hat{x}, \hat{t}) + \varepsilon^2 &= u(\hat{x}, \hat{t}) + \varepsilon^2 \\ &> \int_0^\varepsilon [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + u(X(\varepsilon), \hat{t} - \varepsilon) \\ &\geq \int_0^\varepsilon [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \varphi(X(\varepsilon), \hat{t} - \varepsilon), \end{aligned}$$

which yields

$$\begin{aligned} \varepsilon^2 &> \int_0^\varepsilon [f(X(t)) + L(-\alpha X(t) - \dot{X}(t)) \\ &\quad - \varphi_t(X(t), \hat{t} - t) + D\varphi(X(t), \hat{t} - t) \cdot \dot{X}(t)] dt. \end{aligned}$$

Theorefore, noting that for  $x = X(t)$ ,  $z = \dot{X}(t)$ ,

$$\begin{aligned} &L(-\alpha x - z) + D\varphi(x, \hat{t} - t) \cdot z \\ &= L(-\alpha x - z) - D\varphi(x, \hat{t} - t) \cdot (-\alpha x - z) - \alpha x \cdot D\varphi(x, \hat{t} - t) \\ &\geq -\alpha x \cdot D\varphi(x, \hat{t} - t) - H(D\varphi(x, \hat{t} - t)), \end{aligned}$$

we get

$$(4.9) \quad \int_0^\varepsilon [f(X(t)) - \varphi_t(X(t), \hat{t} - t) - \alpha X(t) \cdot D\varphi(X(t), \hat{t} - t) - H(D\varphi(X(t), \hat{t} - t))] dt < \varepsilon^2.$$

We can extend the domain of definition of  $X$  to  $[0, \hat{t}]$  so that  $X \in \mathcal{C}(\hat{x}, \hat{t})$  and

$$\varepsilon^2 > \int_0^{\hat{t}} [f(X(t)) + L(-\alpha X(t) - \dot{X}(t))] dt + \phi(X(\hat{t})).$$

Taking into account that  $X$  depends on  $\varepsilon$ , we write  $X_\varepsilon$  for  $X$ . By Lemma 4.5, there is a modulus  $\omega$  such that  $|X_\varepsilon(t) - \hat{x}| \leq \omega(t)$  for all  $t \in [0, \hat{t}]$  and  $\varepsilon \in (0, 1)$ . Thus dividing (4.9) by  $\varepsilon$  and sending  $\varepsilon \rightarrow 0$  yield

$$f(\hat{x}) - \varphi_t(\hat{x}, \hat{t}) - \alpha \hat{x} \cdot D\varphi(\hat{x}, \hat{t}) - H(D\varphi(\hat{x}, \hat{t})) \leq 0,$$

which completes the proof.  $\square$

## 5. A comparison theorem for solutions of (1.4)

In this section we establish the following comparison theorem. Let  $0 < T < \infty$ .

**Theorem 5.1.** *Let  $u \in \text{USC}(\mathbf{R}^n \times [0, T])$  and  $v \in \text{LSC}(\mathbf{R}^n \times [0, T])$  be, respectively, a viscosity subsolution and a viscosity supersolution of*

$$(5.1) \quad u_t + \alpha x \cdot Du + H(Du) = f(x) \quad \text{in } \mathbf{R}^n \times (0, T).$$

Assume that

$$\lim_{r \rightarrow \infty} \inf \{v(x, t) + \frac{1}{\alpha} L(-\alpha x) \mid (x, t) \in (\mathbf{R}^n \setminus B(0, r)) \times [0, T]\} = \infty,$$

and that  $u(x, 0) \leq v(x, 0)$  for all  $x \in \mathbf{R}^n$ . Then  $u \leq v$  in  $\mathbf{R}^n \times [0, T]$ .

We remark that the uniqueness part of Theorem 1.2 follows from the above theorem. Thus we have completed the proof of Theorem 1.2, and now Theorems 1.2 and 4.10 together yield Lemma 2.7.

The proof below follows the outline of that of [I, Theorem 4].

**Proof.** Let  $L^\alpha$  denote the function  $L^\alpha \in C(\mathbf{R}^n)$  given by

$$L^\alpha(x) = \frac{1}{\alpha} L(-\alpha x).$$

It is enough to prove that for any  $R > 0$  and for all  $(x, t) \in B(0, R) \times [0, T]$ ,

$$(5.2) \quad \min\{u(x, t) + \alpha t + L^\alpha(x), R\} \leq v(x, t) + \alpha t + L^\alpha(x).$$

To show (5.2), we fix  $R > 0$ . Recalling that

$$\lim_{r \rightarrow \infty} \inf \{v(x, t) + L^\alpha(x) \mid (x, t) \in (\mathbf{R}^n \setminus B(0, r)) \times [0, T]\} = \infty,$$

we choose a constant  $r \geq R$  so that

$$(5.3) \quad v(x, t) + L^\alpha(x) \geq R + 1 \quad \text{for } (x, t) \in \partial B(0, r) \times [0, T].$$

For convex function  $F \in C(\mathbf{R}^n)$  and  $\varepsilon > 0$ , we denote the  $\varepsilon$  inf-convolution by  $F_\varepsilon$ , i.e.,

$$F_\varepsilon(x) = \inf \{F(y) + \frac{1}{2\varepsilon} |x - y|^2 \mid y \in \mathbf{R}^n\}.$$

Recall that  $F_\varepsilon \in C^1(\mathbf{R}^n)$ ,  $F_\varepsilon$  is convex in  $\mathbf{R}^n$ ,  $F_\varepsilon \leq F$  in  $\mathbf{R}^n$ , and  $F_\varepsilon(\xi) \rightarrow F(\xi)$  uniformly on compact subsets of  $\mathbf{R}^n$  as  $\varepsilon \rightarrow 0$ . We set  $L_\varepsilon^\alpha := (L^\alpha)_\varepsilon$ . Note that for  $x \in \mathbf{R}^n$ ,

$$(5.4) \quad \begin{aligned} L_\varepsilon^\alpha(x) &= \inf_{y \in \mathbf{R}^n} \left( \frac{1}{\alpha} L(-\alpha y) + \frac{1}{2\varepsilon} |x - y|^2 \right) \\ &= \frac{1}{\alpha} \inf_{z \in \mathbf{R}^n} \left( L(z) + \frac{1}{2\alpha\varepsilon} |\alpha x + z|^2 \right) = \frac{1}{\alpha} L_{\alpha\varepsilon}(-\alpha x). \end{aligned}$$

We fix  $\delta > 0$  so that for any  $\varepsilon \in (0, \delta)$ ,

$$L_\varepsilon^\alpha(x) + 1 \geq L^\alpha(x) \quad \text{for } x \in B(0, r),$$

which guarantees that

$$\begin{aligned} v(x, t) + \alpha t + L_\varepsilon^\alpha(x) &\geq R \quad \text{for } (x, t) \in \partial B(0, r) \times [0, T], \\ L_{\alpha\varepsilon}(-\alpha x) + \alpha &\geq L(-\alpha x) \quad \text{for } x \in B(0, r). \end{aligned}$$

Fix any  $\varepsilon \in (0, \alpha)$ , and set

$$\begin{aligned} u_\varepsilon(x, t) &= u(x, t) + \alpha t + L_\varepsilon^\alpha(x) \quad \text{for } (x, t) \in \mathbf{R}^n \times [0, T], \\ v_\varepsilon(x, t) &= v(x, t) + \alpha t + L_\varepsilon^\alpha(x) \quad \text{for } (x, t) \in \mathbf{R}^n \times [0, T]. \end{aligned}$$

Observe that  $u_\varepsilon$  and  $v_\varepsilon$  are, respectively, a viscosity subsolution and supersolution of

$$(5.5) \quad w_t + \alpha x \cdot Dw + \sup_{z \in \mathbf{R}^n} (z \cdot Dw - f_\varepsilon(x, z)) = 0 \quad \text{in } \mathbf{R}^n \times (0, T),$$

where  $f_\varepsilon \in C(\mathbf{R}^n \times \mathbf{R}^n)$  is given by

$$f_\varepsilon(x, z) = L(z) + \alpha + f(x) + (\alpha x + z) \cdot DL_\varepsilon^\alpha(x).$$

Observe furthermore that

$$f_\varepsilon(x, z) \geq 0 \quad \text{for } (x, z) \in B(0, r) \times \mathbf{R}^n.$$

Indeed, by the convexity of  $L_{\alpha\varepsilon}$ , we have

$$L_{\alpha\varepsilon}(z) \geq L_{\alpha\varepsilon}(-\alpha x) + DL_{\alpha\varepsilon}(-\alpha x) \cdot (z + \alpha x) \quad \text{for } z, x \in \mathbf{R}^n,$$

and therefore, noting in view of (5.4) that

$$DL_{\alpha\varepsilon}(-\alpha x) = -DL_\varepsilon^\alpha(x) \quad \text{for } x \in \mathbf{R}^n,$$

we get

$$f_\varepsilon(x, z) \geq L_{\alpha\varepsilon}(-\alpha x) + \alpha + f(x) \geq L(-\alpha x) + f(x) \geq 0 \quad \text{for } (x, z) \in B(0, r) \times [0, T].$$

As observed in [A, I], for any non-decreasing function  $\theta \in C(\mathbf{R})$  such that  $\theta'(t) \leq 1$  a.e., the function  $\theta \circ u_\varepsilon$  is a viscosity subsolution of (5.5) in  $\text{int } B(0, r) \times (0, T)$ . In particular, the function:  $(x, t) \mapsto \min\{u_\varepsilon(x, t), R\}$  on  $B(0, r) \times [0, T]$  is a viscosity

subsolution of (5.5) in  $\text{int } B(0, r) \times (0, T)$ . For  $(x, t) \in \partial B(0, r) \times [0, T)$ , we have  $v_\varepsilon(x, t) \geq R$  and hence,

$$\min\{u_\varepsilon(x, t), R\} \leq R \leq v_\varepsilon(x, t).$$

Applying a standard comparison theorem (see [CIL, Ba, BC]) to viscosity sub- and supersolutions of (5.5) in  $B(0, r) \times [0, T)$ , we conclude that for  $(x, t) \in B(0, r) \times [0, T)$ ,

$$\min\{u_\varepsilon(x, t), R\} \leq v_\varepsilon(x, t),$$

and moreover that  $(x, t) \in B(0, r) \times [0, T)$ ,

$$\min\{u(x, t) + \alpha t + L^\alpha(x), R\} \leq v(x, t) + \alpha t + L^\alpha(x),$$

since  $\varepsilon \in (0, \delta)$  is arbitrary.  $\square$

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