On the criticality of viscous Hamilton-Jacobi equations

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In this talk we discuss a nonlinear additive eigenvalue problem (ergodic problem) for the following viscous Hamilton-Jacobi equation:

$$\lambda - \frac{1}{2} \Delta \phi + \frac{c(x)}{2} |D\phi|^2 + \beta V(x) = 0 \quad \text{in} \quad \mathbb{R}^N, \quad \phi(0) = 0,$$

(EP)

where $\beta$ is a real parameter, and functions $c(x)$, $V(x)$ satisfy the following conditions:

(H1) $c \in C^2_b(\mathbb{R}^N)$ and $\kappa_1 \leq c(x) \leq \kappa_2$ in $\mathbb{R}^N$ for some $\kappa_1, \kappa_2 > 0$.

(H2) $V \in C^2_b(\mathbb{R}^N)$, $V \geq 0$, and $|x|^2 V(x) \to 0$ as $|x| \to \infty$.

We seek for a pair $(\lambda, \phi) \in \mathbb{R} \times C^2(\mathbb{R}^N)$ satisfying (EP) in the classical sense. Note that the constraint $\phi(0) = 0$ is imposed to avoid the ambiguity of additive constants with respect to $\phi$. Our objective is to investigate qualitative properties of the 'principal eigenvalue' and the associated 'ground state' for (EP).

**Theorem 1.** Let (H1) and (H2) hold.

(i) For each $\beta \in \mathbb{R}$, there exists a real constant $\lambda^* = \lambda^*(\beta)$ such that (EP) has a solution $\phi \in C^2(\mathbb{R}^N)$ if and only if $\lambda \leq \lambda^*$.

(ii) The mapping $\beta \mapsto \lambda^*(\beta)$ is non-positive, non-increasing, and concave.

(iii) There exists a $\beta_c \geq 0$ such that $\lambda^*(\beta) = 0$ for $\beta \leq \beta_c$ and $\lambda^*(\beta) < 0$ for $\beta > \beta_c$.

(iv) Let $\beta_c$ be the constant in (iii). Then, $\beta_c = 0$ for $N \leq 2$ and $\beta_c > 0$ for $N \geq 3$.

**Theorem 2.** Let (H1) and (H2) hold. Let $\beta_c$ be the constant in Theorem 1.

(i) For any $\beta \geq \beta_c$, there exists at most one solution $\phi$ of (EP) with $\lambda = \lambda^*(\beta)$.

(ii) Suppose that $\beta > \beta_c$. Then, the solution $\phi$ satisfies

$$C^{-1} |x| - C \leq \phi(x) \leq C(1 + |x|), \quad x \in \mathbb{R}^N$$

for some $C > 0$.

(ii) Suppose that $\beta = \beta_c$. Then, the solution $\phi$ satisfies

$$C^{-1} \log(1 + |x|) - C \leq \phi(x) \leq C \log(1 + |x|) + C, \quad x \in \mathbb{R}^N$$

for some $C > 0$.

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Equation (EP) is closely related to the following minimizing problems:

Minimize \( J_\beta(\xi) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T \left\{ \frac{|\xi_t|^2}{2c(X_t^\xi)} - \beta V(X_t^\xi) \right\} dt \right] \),

subject to \( X_t^\xi = x - \int_0^t \xi_s \, ds + W_t, \quad t \geq 0 \),

where \( W = (W_t) \) is an \( N \)-dimensional standard Brownian motion defined on some probability space, and \( \xi = (\xi_t) \) stands for an \( \mathbb{R}^N \)-valued control process belonging to a suitable admissible class. It turns out that the optimal value \( \Lambda(\beta) := \inf_\xi J_\beta(\xi) \) coincides with \( \lambda^*(\beta) \) given in Theorem 1. Furthermore, the solution \( \phi \) of (EP) with \( \lambda = \lambda^*(\beta) \) plays a crucial role in constructing an optimal control. More precisely, let \( X = (X_t) \) be the feedback diffusion governed by the stochastic differential equation

\[
dX_t = -c(X_t)D\phi(X_t) \, dt + dW_t, \quad X_0 = x, \tag{1}
\]

where \( \phi = \phi(x) \) is a solution of (EP) with \( \lambda = \lambda^*(\beta) \). Then, the following theorem holds.

**Theorem 3.** Let (H1) and (H2) hold. Let \( \beta_c \) be the constant in Theorem 1.

(i) Suppose that \( \beta < \beta_c \). Then \( X \) is transient.

(ii) Suppose that \( \beta > \beta_c \). Then \( X \) is positive recurrent.

(iii) Suppose that \( \beta = \beta_c \). Then \( X \) is recurrent.

Let \( X = (X_t) \) be the diffusion governed by (1), and set \( \xi^*_t := c(X_t)D\phi(X_t) \).

**Theorem 4.** Suppose that (H1) and (H2) hold. Let \( \lambda^*(\beta) \) be the constant given in Theorem 1. Then, \( \Lambda(\beta) = \lambda^*(\beta) = J_\beta(\xi^*) \) for \( \beta \geq \beta_c \) and \( \Lambda(\beta) = \lambda^*(\beta) = J_\beta(0) \) for \( \beta \leq \beta_c \).

**References**

