

Relaxation in an L^∞ -optimization problem

Hitoshi Ishii

Department of Mathematics, School of Education, Waseda University, 1-6-1 Nishi-Waseda, Shinjuku-ku, Tokyo 169-8050, Japan
(ishii@edu.waseda.ac.jp)

Paola Loreti

Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università degli Studi di Roma “La Sapienza”, Via Scarpa n. 16, 00161 Roma, Italy
(loreti@dmmm.uniroma1.it)

Let Ω be an open bounded subset of \mathbf{R}^n and f a continuous function on $\overline{\Omega}$ satisfying $f(x) > 0$ for all $x \in \overline{\Omega}$. We consider the maximization problem for the integral $\int_{\Omega} f(x)u(x)dx$ over all Lipschitz continuous functions u subject to the Dirichlet boundary condition $u = 0$ on $\partial\Omega$ and to the gradient constraint of the form $H(Du(x)) \leq 1$, and prove that the supremum is “achieved” by the viscosity solution of $\widehat{H}(Du(x)) = 1$ in Ω and $u = 0$ on $\partial\Omega$, where \widehat{H} denotes the convex envelope of H . This result is applied to an asymptotic problem, as $p \rightarrow \infty$, for quasi-minimizers of the integral $\int_{\Omega} [(1/p)H(Du(x))^p - f(x)u(x)]dx$. An asymptotic problem as $k \rightarrow \infty$ for $\inf \int_{\Omega} [k \operatorname{dist}(Du(x), K) - f(x)u(x)]dx$ is also considered, where the infimum is taken all over $u \in W_0^{1,1}(\Omega)$ and the set K is given by $\{\xi \mid H(\xi) \leq 1\}$.

1. Introduction

In this paper we investigate the maximization problem of the functional

$$\int_{\Omega} f(x)u(x)dx \tag{1.1}$$

in the space $W_0^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ under gradient constraint

$$H(Du(x)) \leq 1 \quad \text{almost everywhere in } \Omega, \tag{1.2}$$

where Ω is a given bounded open set in \mathbf{R}^n , f is a continuous function on $\overline{\Omega}$ satisfying $f(x) > 0$ for all $x \in \overline{\Omega}$, and H is a continuous function on \mathbf{R}^n such that $H(\lambda\xi) = \lambda H(\xi)$ for all $(\lambda, \xi) \in [0, \infty) \times \mathbf{R}^n$ and $H(\xi) > 0$ for all $\xi \neq 0$.

One of our results (Theorem 2.2 below) asserts that, if \widehat{H} denotes the convex envelope of H , then the supremum of (1.1) in the above maximization problem is “achieved” by the unique viscosity solution of

$$\widehat{H}(Du(x)) = 1 \quad \text{in } \Omega \quad \text{and} \quad u(x) = 0 \quad \text{for } x \in \partial\Omega. \tag{1.3}$$

Roughly speaking, this says that, in the process of maximization (1.1)–(1.2), the inequality $H \leq 1$ is replaced by the inequality $\hat{H} \leq 1$. We may classify this phenomenon as one of relaxations in optimization. This phenomenon is precisely stated in Theorem 2.2 that the viscosity solution of (1.3) is the pointwise supremum of solutions of (1.2) in the almost everywhere sense. This observation is quite well-known for a long time in the case when H is convex and therefore (1.2) is equivalent to the inequality $H(Du(x)) \leq 1$ in the viscosity sense, but it establishes a new connection in our generality between solutions in the viscosity sense and in the almost everywhere sense. We refer the reader to [7] and to [9] for general overviews, respectively, on viscosity solutions and on solutions in the almost everywhere sense. We refer to [18] for observations related to relaxation in eikonal equations.

Our motivation to studying the maximization problem (1.1)–(1.2) is in its application to an asymptotic problem for the minimizers in $W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ of the functionals

$$\int_{\Omega} [\frac{1}{p} H(Du(x))^p - f(x)u(x)] dx \quad (1.4)$$

as $p \rightarrow \infty$. The minimization problems for functionals (1.4) appear in the study of torsional creep in elasticity. The case when $H(\xi)$ is the Euclidean norm of ξ , i.e., $H(\xi) = |\xi|$, has been studied by several authors (see, for instance, [2, 16] and references therein). Recently Ishibashi–Koike [12] have studied the case when H is the general l_p -norm ($1 \leq p \leq \infty$).

Theorem 3.1 below says that if u_p is a minimizer for (1.4) and u is the viscosity solution of (1.3), then $u_p(x) \rightarrow u(x)$ uniformly in $\overline{\Omega}$ as $p \rightarrow \infty$. The maximization problem (1.1)–(1.2) formally corresponds to solving (1.4) in the case $p = \infty$ and, in this spirit, we may regard (1.1)–(1.2) as an L^∞ -optimization problem. See [1] for an overview on the subjects of L^∞ -optimization. See also [5] for related subjects.

We also study an asymptotic problem similar to (1.4), but with slightly different characters. That is, the asymptotic problem for

$$\inf \left\{ \int_{\Omega} [k \operatorname{dist}(Du(x), K) - f(x)u(x)] dx \mid u \in W_0^{1,1}(\Omega) \right\} \quad (1.5)$$

as $k \rightarrow \infty$, where $K := \{\xi \in \mathbf{R}^n \mid H(\xi) \leq 1\}$. One of basic difference between (1.4) and (1.5) is in the growth order of the integrands in Du variable: (1.4) has a superlinear growth while (1.5) has the linear growth. The maximization problem (1.1)–(1.2) is equivalent to the problem

$$\inf \left\{ \int_{\Omega} [\delta_K(Du(x)) - f(x)u(x)] dx \mid u \in W_0^{1,1}(\Omega) \right\},$$

where δ_K denotes the indicator function of K , i.e., $\delta_K(\xi) = 0$ if $\xi \in K$ and $\delta_K(\xi) = \infty$ otherwise, and a natural way of approximating the indicator function is given by a sequence of Lipschitz continuous functions

$$k \operatorname{dist}(\xi, K), \quad k = 1, 2, \dots$$

These considerations lead us to study the asymptotic problem for (1.5) as $k \rightarrow \infty$.

There has been much attention on problems (1.4) where f vanishes in a region of Ω . We refer the reader to [15, 2, 4, 6] and references therein for this topics.

This paper is organized as follows. Sections 2 and 3 are devoted to the optimization problem (1.1)–(1.2) and to the asymptotic problem mentioned above for quasi-minimizers for (1.4), respectively. In Section 4 we briefly explain another approach to the asymptotic problem above, which relies on relaxation for (1.4) in $W^{1,p}(\Omega)$, but not on relaxation for (1.1)–(1.2). In Section 5 we study the asymptotic problem for (1.5) as $k \rightarrow \infty$. Appendix provides a proof of a standard fact on eikonal equations.

Finally, we have restricted our considerations on the zero Dirichlet data. However, all our results in this paper are valid for general Dirichlet data $g \in C(\partial\Omega)$ provided that there exists a function $\varphi \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ such that $H(D\varphi(x)) \leq 1$ almost everywhere in Ω and $\varphi(x) = g(x)$ for $x \in \partial\Omega$.

2. Relaxation in eikonal equations with non-convex Hamiltonian

Let $H : \mathbf{R}^n \rightarrow \mathbf{R}$ be a continuous function which is positively homogeneous of degree one and satisfies $H(\xi) > 0$ for all $\xi \neq 0$. Here we do not assume that H is convex, and \widehat{H} denotes the convex envelope of H , i.e.,

$$\begin{aligned} \widehat{H}(\xi) &= \sup\{a \cdot \xi + b \mid a \in \mathbf{R}^n, b \in \mathbf{R}, a \cdot \xi + b \leq H(\xi) \quad \forall \xi \in \mathbf{R}^n\} \\ &= \sup\{a \cdot \xi \mid a \in \mathbf{R}^n, a \cdot \xi \leq H(\xi) \quad \forall \xi \in \mathbf{R}^n\}. \end{aligned}$$

Let Ω be a bounded open subset of \mathbf{R}^n . Set $X = W_0^{1,\infty}(\Omega) \cap C(\overline{\Omega})$. We consider the maximization problem

$$j_\infty := \sup\left\{\int_\Omega f(x)u(x)dx \mid u \in X, H(Du(x)) \leq 1 \quad \text{a.e. } x \in \Omega\right\}. \quad (2.1)$$

The following lemma collects standard facts on which our arguments will rely.

LEMMA 2.1. (a) *The problem*

$$\widehat{H}(Du(x)) = 1 \quad \text{in } \Omega, \quad (2.2)$$

$$u(x) = 0 \quad \text{for } x \in \partial\Omega \quad (2.3)$$

has a unique viscosity solution $u \in C(\overline{\Omega})$. (b) Any solution $v \in C(\overline{\Omega})$ of (2.2) is locally Lipschitz continuous and satisfies

$$\widehat{H}(Dv(x)) = 1 \quad \text{a.e. } x \in \Omega. \quad (2.4)$$

(c) If $u, v \in C(\overline{\Omega})$ are viscosity sub- and supersolutions of (2.2), respectively, and $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω . (d) The (unique) viscosity solution $u \in C(\overline{\Omega})$ of (2.2)–(2.3) can be represented as

$$u(x) = \inf\{L(x - y) \mid y \in \partial\Omega\}, \quad (2.5)$$

where $L(z) := \sup\{z \cdot p \mid \widehat{H}(p) \leq 1\} = \sup\{z \cdot p \mid H(p) \leq 1\}$.

Proof. We refer for a proof of (a) and (c) to [17, 13] and do not give it here. Assertion (b) is now a well-known fact (see [8]). A proof of assertion (d), which is a well-known fact, can be found in the appendix (Lemma A). \square

We remark that (a), (c) of Lemma 2.1 are valid with H in place of \widehat{H} .

Here let us recall the following well-known fact that, since \widehat{H} is convex in \mathbf{R}^n , any locally Lipschitz function satisfying (2.4) is a viscosity subsolution of (2.2), which can be checked by mollifying the given locally Lipschitz function, observing with help of the Jensen inequality that the resulting function is a classical subsolution of (2.4) with a small error, and using the stability of viscosity solutions. This is the converse to a half (subsolution part) of assertion (b) above.

The main result in this section is:

THEOREM 2.2. *Let $d \in C(\overline{\Omega})$ be the (unique) viscosity solution of (2.2)–(2.3). Then*

$$d(x) = \sup\{u(x) \mid u \in X, H(Du(y)) \leq 1 \text{ a.e. } y \in \Omega\} \quad \text{for all } x \in \overline{\Omega}, \quad (2.6)$$

and

$$j_\infty = \int_{\Omega} f(x)d(x) dx.$$

By the previous lemma, we know that the function d above is given by $d(x) = \inf_{y \in \partial\Omega} L(x - y)$, where $L(x) = \sup\{x \cdot \xi \mid \widehat{H}(\xi) \leq 1\} = \sup\{x \cdot \xi \mid H(\xi) \leq 1\}$.

Formula (2.6) is the principal assertion of this paper, which can be phrased as follows: the pointwise supremum of solutions of

$$H(Du(x)) \leq 1 \text{ a.e. in } \Omega, \quad u|_{\partial\Omega} = 0$$

is exactly the viscosity solution of

$$\widehat{H}(Du(x)) = 1 \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

The following is an immediate consequence of the above theorem.

COROLLARY 2.3. *Let $d \in C(\overline{\Omega})$ be the viscosity solution of (2.2)–(2.3). Then problem (2.1) has a maximizer if and only if $H(Dd(x)) \leq 1$ a.e. $x \in \Omega$. Moreover, if (2.1) has a maximizer, then the maximizer is unique and given by d .*

Proof of Theorem 2.2. We write $U(x)$ for the right hand side of (2.6).

First we observe that if $u \in X$ satisfies

$$H(Du(x)) \leq 1 \quad \text{a.e. } x \in \Omega, \quad (2.7)$$

then

$$\widehat{H}(Du(x)) \leq 1 \quad \text{a.e. } x \in \Omega,$$

since $\widehat{H} \leq H$, and hence u is a viscosity subsolution of (2.2).

Let d be the unique viscosity solution of (2.2)–(2.3). Then, what we have seen just above and (c) of Lemma 2.1 yield that if $u \in X$ satisfies (2.7), then

$$d(x) \geq u(x) \quad \text{for } x \in \Omega,$$

and therefore

$$d(x) \geq U(x) \quad \text{for } x \in \Omega \quad \text{and} \quad \int_{\Omega} f(x)d(x) \, dx \geq j_{\infty}. \quad (2.8)$$

We want to prove the opposite inequalities of these. We know from (d) of Lemma 2.1 that

$$d(x) = \inf\{L(x - y) \mid y \in \partial\Omega\},$$

where $L(z) = \sup\{z \cdot \xi \mid \xi \in \mathbf{R}^n, \widehat{H}(\xi) \leq 1\}$.

The first step is to prove that at every point $x \in \mathbf{R}^n$ of differentiability of L ,

$$H(DL(x)) = 1. \quad (2.9)$$

Set

$$K = \{\xi \in \mathbf{R}^n \mid H(\xi) \leq 1\} \quad \text{and} \quad S = \text{co } K,$$

where $\text{co } K$ denotes the convex hull of the set K , and note that

$$S = \{\xi \in \mathbf{R}^n \mid \widehat{H}(\xi) \leq 1\}.$$

Of course, we have

$$L(x) = \max\{\xi \cdot x \mid \xi \in S\} = \max\{\xi \cdot x \mid \xi \in K\}.$$

Note that L is the convex conjugate function (Legendre transform) of the indicator function

$$\delta_S(\xi) = \begin{cases} 0 & \text{if } \xi \in S \\ +\infty & \text{if } \xi \notin S. \end{cases}$$

Observe as well that L is convex and positively homogeneous of degree one and satisfies $L(x) > 0$ for $x \neq 0$ and that the function L is Lipschitz continuous in \mathbf{R}^n .

In order to show (2.9), we use a standard fact from convex analysis that if L is differentiable at $x \in \mathbf{R}^n$, then the maximization problem

$$L(x) = \sup\{\xi \cdot x \mid \xi \in S\} \quad (2.10)$$

has a unique maximizer. A way to see this is as follows. Let $y \in \mathbf{R}^n$ and $q \in D^+L(y)$, and choose a C^1 function φ such that $L - \varphi$ has a maximum at y , so that $D\varphi(y) = q$. Next choose $r \in S$ so that $L(y) = r \cdot y$. Then, using that the function $x \mapsto x \cdot r - \varphi(x)$ on \mathbf{R}^n attains a maximum at y , we observe that $D\varphi(y) = r$ and hence $r = q$. This shows that q is the unique maximizer of $\xi \mapsto \xi \cdot y$ over S .

By the same reasoning, we have that if L is differentiable at $x \in \mathbf{R}^n$, then the maximization problem

$$L(x) = \sup\{\xi \cdot x \mid \xi \in K\} \quad (2.11)$$

has a unique maximizer.

Let $x \in \mathbf{R}^n$ be a point of differentiability of L , and let $\xi \in S$ and $\eta \in K$ be the unique maximizers, respectively, of (2.10) and of (2.11). Since $K \subset S$ and ξ is the unique maximizer of (2.10), we see that $\xi = \eta \in K$. Since $\xi \in K$ is the unique maximizer of problem (2.11), we see that $\xi \in \partial K$, i.e., $H(\xi) = 1$, which proves (2.9).

The next step is to introduce a sequence of functions which approximates the function d in a nice way. Choose a sequence $\{z_j\}_{j \in \mathbf{N}} \subset \partial\Omega$ which is dense in $\partial\Omega$, and set

$$d_k(x) = \min_{j \leq k} L(x - z_j).$$

In view of Dini's lemma, it is obvious that

$$d_k(x) \rightarrow d(x) \quad \text{uniformly on } \overline{\Omega} \quad \text{as } k \rightarrow \infty.$$

We show that for all $k \in \mathbf{N}$,

$$H(Dd_k(x)) = 1 \quad \text{a.e. } x \in \Omega. \quad (2.12)$$

Let $Z \subset \Omega$ be the set of point $x \in \Omega$ at which the function L is not differentiable and $Z_k \subset \Omega$, for $k \in \mathbf{N}$, the set of points at which d_k is not differentiable. Since L and d_k are

Lipschitz continuous and therefore almost everywhere differentiable in Ω , the Lebesgue measure of each of these sets Z and Z_k is zero. Set

$$Z_0 = \bigcup_{j \in \mathbf{N}} (z_j + Z) \cup \bigcup_{k \in \mathbf{N}} Z_k.$$

Then the set Z_0 is also a set of measure zero, and we have

$$H(DL(x - z_j)) = 1 \quad \text{for } x \in \Omega \setminus Z_0 \text{ and } j \in \mathbf{N}.$$

Fix $y \in \Omega \setminus Z_0$ and $k \in \mathbf{N}$. The function d_k is differentiable at y , and there is a $j \leq k$ such that $L(y - z_j) = d_k(y)$. Since $d_k(x) \leq L(x - z_j)$ for all $x \in \Omega$, we see that $q := Dd_k(y) \in D^-L(y - z_j)$. Furthermore, since $x \mapsto L(x - z_j)$ is differentiable at y , we see that $q = DL(y - z_j)$ and therefore $H(q) = 1$, and conclude the proof of (2.12).

Now, we set

$$u_k(x) = \left(d_k(x) - \max_{\partial\Omega} d_k \right)_+ \quad \text{for } x \in \overline{\Omega}$$

and for all $k \in \mathbf{N}$. Then observe that $u_k \in X$ for all $k \in \mathbf{N}$, that for all $k \in \mathbf{N}$,

$$H(Du_k(x)) \leq 1 \quad \text{a.e. } x \in \Omega,$$

and that

$$u_k(x) \rightarrow d(x) \quad \text{uniformly on } \overline{\Omega} \text{ as } k \rightarrow \infty.$$

Here we have used the fact that $Du_k(x) = Dd_k(x)$ or $Du_k(x) = 0$ a.e. $x \in \Omega$ for all $k \in \mathbf{N}$, an observation due to Stampacchia [11, Theorem 7.8]. Now, it is immediate to see that

$$d(x) \leq U(x) \quad \text{for } x \in \overline{\Omega} \quad \text{and} \quad \int_{\Omega} f(x) d(x) dx \leq j_{\infty}.$$

This together with (2.8) completes the proof. \square

The following example shows that in general there is no maximizer for problem (2.1).

Consider the case when $n = 2$. Define the function $H : \mathbf{R}^2 \rightarrow \mathbf{R}$ by the two conditions that

$$\begin{aligned} K &\equiv \{\xi \in \mathbf{R}^2 \mid H(\xi) \leq 1\} = \{\xi \in \mathbf{R}^2 \mid |\xi_1| + |\xi_2| \leq 1\} \cap \{\xi \in \mathbf{R}^2 \mid \xi_2 \leq (1 - \xi_1)^2\}; \\ H(\lambda\xi) &= \lambda H(\xi) \quad \text{for all } \lambda \geq 0, \xi \in \mathbf{R}^2. \end{aligned}$$

Then the convex envelope \widehat{H} is just the function $\widehat{H}(\xi) = |\xi_1| + |\xi_2|$, and the convex hull of K is given by $S = \{\xi \in \mathbf{R}^2 \mid |\xi_1| + |\xi_2| \leq 1\}$. The function $L(x) = \sup\{x \cdot \xi \mid \xi \in S\}$ associated with H is given explicitly by $L(x) = \max\{|x_1|, |x_2|\}$.

Now let Ω be the set $\{x \in \mathbf{R}^2 \mid x_1 + x_2 > 0, x_1^2 + x_2^2 < 1\}$ and $d : \overline{\Omega} \rightarrow \mathbf{R}$ the function given by

$$d(x) = \inf_{y \in \partial\Omega} L(x - y).$$

In a small neighborhood of the origin relative to Ω , we have

$$d(x) = \frac{x_1 + x_2}{2},$$

and so,

$$Dd(x) = (\frac{1}{2}, \frac{1}{2}) \notin K.$$

Thus, in this example, we have

$$H(Dd(x)) > 1 \quad \text{in a set } \subset \Omega \text{ of positive measure,}$$

and, in view of Corollary 2.3, there is no maximizer of (2.1).

3. An asymptotic problem for variational problems

As in the previous section we assume throughout this section that Ω is a bounded open subset of \mathbf{R}^n and $f \in C(\overline{\Omega})$ satisfies $f(x) > 0$ for all $x \in \overline{\Omega}$ and that $H \in C(\mathbf{R}^n)$ is positively homogeneous function of degree one and satisfies $H(\xi) > 0$ for all $\xi \neq 0$.

In this section we consider the minimization problem, for $p > 1$, of the functional

$$I_p(u) = \int_{\Omega} [\frac{1}{p} H(Du(x))^p - f(x)u(x)] dx \quad (3.1)$$

over all functions u in the space $X_p := W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, where $W_0^{1,p}(\Omega)$ denotes the standard Sobolev space (see e.g. [11]), and study the asymptotic behavior of quasi-minimizer u_p of I_p as $p \rightarrow \infty$.

For $p > 1$ we define

$$i_p = \inf\{I_p(u) \mid u \in X_p\}. \quad (3.2)$$

Fix a net $\{\varepsilon_p\}_{p \in (1, \infty)} \subset (0, \infty)$ such that $\varepsilon_p \rightarrow 0$ as $p \rightarrow \infty$ and such that $\sup_{p > 1} \varepsilon_p < \infty$. Select $u_p \in W_0^{1,p}(\Omega)$ so that

$$I_p(u_p) < i_p + \varepsilon_p.$$

THEOREM 3.1. (a) For any $q > 1$,

$$u_p(x) \rightarrow d(x) := \inf_{y \in \partial\Omega} L(x - y) \quad \text{weakly in } W_0^{1,q}(\Omega)$$

as $p \rightarrow \infty$, where L is the function on \mathbf{R}^n defined by

$$L(x) := \sup\{x \cdot \xi \mid \xi \in \mathbf{R}^n, \widehat{H}(\xi) \leq 1\}.$$

$$(b) \quad \lim_{p \rightarrow \infty} i_p = - \int_{\Omega} f(x) d(x) dx.$$

We remark that, since the injection of $W_0^{1,q}(\Omega)$ into $C(\overline{\Omega})$ is compact for any $q > n$ by the Rellich theorem (see e.g. [11]), the above theorem yields the convergence of $\{u_p\}$ to d in $C(\overline{\Omega})$.

PROPOSITION 3.2. *For any $q > 1$ the net $\{u_p\}_{p \geq q}$ is bounded in $W_0^{1,q}(\Omega)$.*

Proof. Let $p > 1$. Since $i_p \leq I_p(0) = 0$, we have

$$\begin{aligned} \int_{\Omega} \frac{1}{p} H(Du_p(x))^p dx &\leq \int_{\Omega} f(x) u_p(x) dx + \varepsilon_p \leq C \left(\int_{\Omega} H(Du_p(x)) dx + 1 \right) \\ &\leq C \left(\int_{\Omega} H(Du_p(x))^p dx \right)^{\frac{1}{p}} |\Omega|^{1-\frac{1}{p}} + C \\ &\leq \frac{1}{2p} \int_{\Omega} H(Du_p(x))^p dx + \left(1 - \frac{1}{p}\right) \left(2^{\frac{1}{p}} C |\Omega|^{1-\frac{1}{p}}\right)^{(1-\frac{1}{p})^{-1}} + C, \end{aligned}$$

where C denotes a positive constant independent of p . Here we have used the Poincaré inequality

$$\|u\|_{L^1(\Omega)} \leq C_1 \|H(Du)\|_{L^1(\Omega)} \quad \forall u \in W_0^{1,1}(\Omega),$$

where $C_1 > 0$ is a constant depending only on n , Ω , and H . Hence we have

$$\frac{1}{2p} \int_{\Omega} H(Du_p(x))^p dx \leq C + 2^{\frac{1}{p-1}} C^{\frac{p}{p-1}} |\Omega|,$$

and therefore,

$$\|H(Du_p)\|_{L^p(\Omega)} \leq \left(2p \left(C + 2^{\frac{1}{p-1}} C^{\frac{p}{p-1}} |\Omega|\right)\right)^{\frac{1}{p}}. \quad (3.3)$$

Now let $q > 1$ and $p > q$. Using the Hölder inequality, we compute that

$$\|H(Du_p)\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p}} \|H(Du_p)\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{q}} \left(2p \left(C |\Omega|^{-1} + 2^{\frac{1}{p-1}} C^{\frac{p}{p-1}}\right)\right)^{\frac{1}{p}}.$$

This shows that $\{u_p\}_{p > q}$ is bounded in $W_0^{1,q}(\Omega)$. \square

Let $\{p_j\}_{j \in \mathbb{N}}$ be a sequence of $p_j > 1$ such that $p_j \rightarrow \infty$ as $j \rightarrow \infty$ and such that for any $q > 1$,

$$u_{p_j} \rightarrow u_{\infty} \quad \text{weakly in } W_0^{1,q}(\Omega)$$

for some function $u_{\infty} \in \bigcap_{p > 1} W_0^{1,p}(\Omega)$. Notice that, according to Proposition 3.2, one can extract such a subsequence from any sequence of numbers $p > 1$ which goes to ∞ .

PROPOSITION 3.3. *The function u_{∞} satisfies*

$$\hat{H}(Du_{\infty}(x)) \leq 1 \quad \text{a.e. } x \in \Omega,$$

where \widehat{H} denotes the convex envelope of H as before.

We divide the proof of this proposition into two lemmas.

LEMMA 3.4. *If $v \in \bigcap_{p>1} L^p(\Omega)$ satisfies*

$$\liminf_{p \rightarrow \infty} \|v\|_{L^p(\Omega)} \leq 1,$$

then $|v(x)| \leq 1$ a.e. in Ω .

Proof. For $n \in \mathbf{N}$ define

$$\Omega_n = \{x \in \Omega \mid |v(x)| \geq 1 + \frac{1}{n}\}.$$

Then we have

$$\|v\|_{L^p(\Omega)} \geq |\Omega_n|^{\frac{1}{p}} \left(1 + \frac{1}{n}\right).$$

This together with the assumption of the lemma yields that if $|\Omega_n| > 0$ then

$$1 + \frac{1}{n} \leq \liminf_{p \rightarrow \infty} \|v\|_{L^p(\Omega)} \leq 1,$$

which is impossible. Thus we conclude that $|\Omega_n| = 0$ for all $n \in \mathbf{N}$ and that $|v(x)| \leq 1$ a.e. in Ω . \square

LEMMA 3.5. *Let $\{w_j\}_{j \in \mathbf{N}}$ be a sequence of functions on Ω such that $w_j \in L^{p_j}(\Omega)^n$ for all $j \in \mathbf{N}$, where $1 < p_j \rightarrow \infty$ as $j \rightarrow \infty$. Assume that $w_j \rightarrow w$ weakly in $L^r(\Omega)^n$ as $j \rightarrow \infty$ for some $r > 1$ and for some $w \in L^r(\Omega)^n$. Assume further that*

$$\limsup_{j \rightarrow \infty} \|H(w_j)\|_{L^{p_j}(\Omega)} \leq 1. \tag{3.4}$$

Then $\widehat{H}(w(x)) \leq 1$ a.e. in Ω .

Proof. First we note that if $p_j > q > 1$, then

$$\|H(w_j)\|_{L^q(\Omega)} \leq |\Omega|^{\frac{1}{q} - \frac{1}{p_j}} \|H(w_j)\|_{L^{p_j}(\Omega)}. \tag{3.5}$$

From this we see that for any $q > 1$ the sequence $\{w_j\}$ is bounded in $L^q(\Omega)^n$, and so we may assume that, as $j \rightarrow \infty$, $w_j \rightarrow w$ weakly in $L^q(\Omega)^n$ for all $q > 1$.

For $q > 1$ and $u \in L^q(\Omega)^n$ we write

$$\Phi_q(u) = \|\widehat{H}(u)\|_{L^q(\Omega)}.$$

Fix $q > 1$. By a well-known theorem due to Mazur (see e.g. [19, Theorem 2, p. 120]), for each $k \in \mathbf{N}$ there is a finite sequence $\lambda_k^k, \lambda_{k+1}^k, \lambda_{k+2}^k, \dots \in [0, 1]$ such that

$$\sum_{j \geq k} \lambda_j^k = 1,$$

$$\sum_{j \geq k} \lambda_j^k w_j \rightarrow w \quad \text{strongly in } L^q(\Omega)^n \text{ as } k \rightarrow \infty.$$

By using the convexity of Φ_q , the inequality $\hat{H} \leq H$, and (3.5), we compute that

$$\begin{aligned} \Phi_q(w) &= \lim_{k \rightarrow \infty} \Phi_q\left(\sum_{j \geq k} \lambda_j^k w_j\right) \leq \liminf_{k \rightarrow \infty} \sum_{j \geq k} \lambda_j^k \Phi_q(w_j) \\ &\leq \limsup_{j \rightarrow \infty} \Phi_q(w_j) \leq \limsup_{j \rightarrow \infty} |\Omega|^{\frac{1}{q} - \frac{1}{pj}} \|H(w_j)\|_{L^{pj}(\Omega)} \leq |\Omega|^{\frac{1}{q}}. \end{aligned}$$

Now Lemma 3.4 guarantees that $|\hat{H}(w(x))| \leq 1$ a.e. $x \in \Omega$. \square

Proof of Proposition 3.3. From (3.3), we get

$$\limsup_{p \rightarrow \infty} \|H(Du_p)\|_{L^p(\Omega)} \leq 1.$$

Then we conclude by Lemma 3.5 that $\hat{H}(Du_\infty(x)) \leq 1$ a.e. in Ω . \square

We consider the minimization problem for $p = \infty$:

$$i_\infty = \inf \left\{ - \int_{\Omega} f(x)u(x)dx \mid u \in X, H(Du(x)) \leq 1 \text{ a.e. } x \in \Omega \right\},$$

where $X = C(\overline{\Omega}) \cap W_0^{1,\infty}(\Omega)$.

By Lemma 2.1 and Theorem 2.2, we know that d is the unique viscosity solution of

$$\left. \begin{aligned} \hat{H}(Du(x)) &= 1 && \text{in } \Omega, \\ u(x) &= 0 && \text{on } \partial\Omega \end{aligned} \right\} \quad (3.6)$$

and that

$$d(x) = \sup\{u(x) \mid u \in X, H(Du(y)) \leq 1 \text{ a.e. } y \in \Omega\} \quad (3.7)$$

and

$$i_\infty = - \int_{\Omega} f(x)d(x) dx. \quad (3.8)$$

Proof of Theorem 3.1. First of all we show that

$$\limsup_{p \rightarrow \infty} i_p \leq i_\infty. \quad (3.9)$$

Fix any $\gamma > 0$ and choose a function $u \in X$ so that

$$H(Du(x)) \leq 1 \quad \text{a.e. } x \in \Omega \quad \text{and} \quad i_\infty + \gamma > - \int_{\Omega} f(x)u(x)dx.$$

Then compute that

$$\begin{aligned} i_p &\leq I_p(u) = \int_{\Omega} \left(\frac{1}{p} H(Du(x))^p - f(x)u(x) \right) dx \\ &\leq \int_{\Omega} \left(\frac{1}{p} - f(x)u(x) \right) dx < \frac{1}{p} |\Omega| + i_\infty + \gamma. \end{aligned}$$

Sending $p \rightarrow \infty$, we see that (3.9) holds.

Next, we fix a sequence $\{p_k\}_{k \in \mathbf{N}} \subset (1, \infty)$ such that $p_k \rightarrow \infty$ as $k \rightarrow \infty$. By taking a subsequence if necessary, we may assume moreover that for any $q > 1$, as $k \rightarrow \infty$,

$$u_{p_k} \rightarrow u_\infty \quad \text{weakly in } W_0^{1,q}(\Omega)$$

for some function $u_\infty \in \cap_{p>1} W_0^{1,p}(\Omega)$.

By Proposition 3.3, we see that

$$\widehat{H}(Du_\infty(x)) \leq 1 \quad \text{a.e. } x \in \Omega.$$

Since \widehat{H} is convex, it follows that u_∞ is a viscosity subsolution of (3.6). By comparison (see (c) of Lemma 2.1), we have

$$u_\infty(x) \leq d(x) \quad \text{for all } x \in \overline{\Omega}. \quad (3.10)$$

Now, observe that as $k \rightarrow \infty$,

$$i_{p_k} \geq - \int_{\Omega} f(x)u_{p_k}(x)dx \rightarrow - \int_{\Omega} f(x)u_\infty(x)dx \geq - \int_{\Omega} f(x)d(x)dx = i_\infty.$$

Hence, using (3.9), we obtain

$$\liminf_{k \rightarrow \infty} i_{p_k} \geq - \int_{\Omega} f(x)u_\infty(x)dx \geq - \int_{\Omega} f(x)d(x)dx = i_\infty \geq \limsup_{k \rightarrow \infty} i_{p_k},$$

which implies

$$\lim_{k \rightarrow \infty} i_{p_k} = i_\infty = \int_{\Omega} f(x)u_\infty(x)dx = \int_{\Omega} f(x)d(x)dx.$$

The last equality together with (3.10) yields that $u_\infty = d$ in $\overline{\Omega}$.

Thus we have shown that if $p_k \in (1, \infty)$ and $\lim_{k \rightarrow \infty} p_k = \infty$, then there exists a subsequence, which we denote by the same symbol, such that as $k \rightarrow \infty$,

$$u_{p_k} \rightarrow d \quad \text{weakly in } W_0^{1,q}(\Omega) \quad \text{for all } q > 1 \quad \text{and} \quad i_{p_k} \rightarrow i_\infty.$$

Therefore, using a simple argument by contradiction, we see that as $p \rightarrow \infty$,

$$u_p \rightarrow d \quad \text{weakly in } W_0^{1,q}(\Omega) \quad \text{for all } q > 1 \quad \text{and} \quad i_p \rightarrow i_\infty,$$

which completes the proof. \square

4. Another approach to the asymptotic problem

In this section we briefly explain a variant of the previous proof of Theorem 3.1. The new proof is based on the relaxation in the space $W^{1,p}(\Omega)$ for the functionals I_p defined by (3.1) and does not rely on Theorem 2.2 for non-convex H .

We use the same notation and assumptions as in the previous section. We assume in addition that Ω is Lipschitz (see, e.g., [10, Definition X. 2. 2]).

We define the functionals \widehat{I}_p on X_p , with $p > 1$, by

$$\widehat{I}_p(u) = \int_{\Omega} \left[\frac{1}{p} \widehat{H}(Du(x))^p - f(x)u(x) \right] dx \quad \text{for } u \in X_p. \quad (4.1)$$

The relaxation in $W^{1,p}(\Omega)$ for the functionals I_p is stated as follows:

LEMMA 4.1. $i_p = \inf\{\widehat{I}_p(u) \mid u \in X_p\}$ for $p \in (1, \infty)$.

We do not give the proof of Lemma 4.1 and instead refer the reader to [11, Proposition X. 3. 6, Theorem X. 3. 3] together with their proofs for the proof and to [11] for an overview on relaxation in calculus of variations.

Sketch of another proof of Theorem 3.1. Let $\{\varepsilon_p\}_{p>1}$ and $\{u_p\}_{p>1}$ be chosen as in Theorem 3.1.

By Lemma 4.1, since $\widehat{I}_p(u) \leq I_p(u)$ for $u \in X_p$ by definition, we have

$$\widehat{I}_p(u_p) < i_p + \varepsilon_p = \inf\{\widehat{I}_p(u) \mid u \in X_p\} + \varepsilon_p \quad \text{for all } p > 1.$$

This shows that the net $\{u_p\}_{p>1}$ satisfies the same relation with \widehat{I}_p as that with I_p .

We repeat the same arguments in Section 3, but with \widehat{H} in place of H , to conclude the proof. The main difference is that in the new proof we use Theorem 2.2 with \widehat{H} in place of H , the proof of which is much easier than the original Theorem 2.2. \square

5. Another asymptotic problem

In this section we consider an asymptotic problem similar to that in Sections 3 and 4, but with slightly different characters. We use the same notation and assumptions as in the previous section.

The maximization problem (2.1) is equivalent to the minimization problem

$$\inf\left\{\int_{\Omega}[\delta_K(Du(x)) - f(x)u(x)]dx \mid u \in W_0^{1,1}(\Omega)\right\},$$

where $K := \{\xi \in \mathbf{R}^n \mid H(\xi) \leq 1\}$ and δ_K denotes the indicator function of K . One of natural approximations of the indicator function $\delta_K(\xi)$ is given by a sequence of functions

$$k \operatorname{dist}(\xi, K), \quad \text{with } k \in \mathbf{N}.$$

Thus we are led to the asymptotic problem for

$$\inf\left\{\int_{\Omega}[k \operatorname{dist}(Du(x), K) - f(x)u(x)]dx \mid u \in W_0^{1,1}(\Omega)\right\},$$

as $k \rightarrow \infty$.

Observe from standard relaxation theory (see [11, Theorem X. 3. 3] together with its proof) that for all $k \in \mathbf{N}$,

$$\begin{aligned} & \inf\left\{\int_{\Omega}[k \operatorname{dist}(Du(x), K) - f(x)u(x)]dx \mid u \in W_0^{1,1}(\Omega)\right\}, \\ &= \inf\left\{\int_{\Omega}[k \operatorname{dist}(Du(x), \operatorname{co} K) - f(x)u(x)]dx \mid u \in W_0^{1,1}(\Omega)\right\}. \end{aligned}$$

Concerning this asymptotic problem we have the following result.

THEOREM 5.1. *As $k \rightarrow \infty$,*

$$\inf\left\{\int_{\Omega}[k \operatorname{dist}(Du(x), K) - f(x)u(x)]dx \mid u \in W_0^{1,1}(\Omega)\right\} \rightarrow i_{\infty}.$$

Proof. Set

$$\begin{aligned} \rho(\xi) &= \operatorname{dist}(\xi, \operatorname{co} K), \\ J_k(u) &= \int_{\Omega}[k\rho(Du(x)) - f(x)u(x)]dx, \\ l_k &= \inf\{J_k(u) \mid u \in W_0^{1,1}(\Omega)\}. \end{aligned}$$

We need to show that $l_k \rightarrow i_{\infty}$ as $k \rightarrow \infty$. Since

$$l_k \leq J_k(d) = - \int_{\Omega} f(x)d(x)dx = i_{\infty} \quad \text{for all } k \in \mathbf{N},$$

we only need to show that

$$l_\infty := \lim_{k \rightarrow \infty} l_k \geq i_\infty,$$

for which we argue in a way similar to the proof of Theorem 3.1. Notice here that the sequence $\{l_k\}$ is non-decreasing.

Fix a non-increasing sequence $\{\varepsilon_k\}_{k \in \mathbf{N}}$ of positive numbers such that $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Choose a sequence $\{u_k\}_{k \in \mathbf{N}}$ of functions in $W_0^{1,1}(\Omega)$ so that

$$l_k + \varepsilon_k > J_k(u_k) \quad \text{for } k \in \mathbf{N}.$$

In view of Poincaré's inequality, there is a constant $C > 0$ such that

$$\|u_k\|_{L^1(\Omega)} \leq C \|Du_k\|_{L^1(\Omega)} \quad \text{for } k \in \mathbf{N}. \quad (5.1)$$

We may assume that C is large enough so that $\text{co } K$ is contained in the ball with radius C centered at the origin. Then

$$\rho(\xi) \geq |\xi| - C \quad \text{for } \xi \in \mathbf{R}^n.$$

Observe that

$$\begin{aligned} l_k + \varepsilon_k &> J_k(u_k) \geq k \int_{\Omega} \rho(Du_k(x)) dx - C \|f\|_{L^\infty(\Omega)} \|Du_k\|_{L^1(\Omega)} \\ &\geq k \int_{\Omega} \rho(Du_k(x)) dx - C \|f\|_{L^\infty(\Omega)} \int_{\Omega} (C + \rho(Du_k(x))) dx, \end{aligned}$$

which yields that

$$\int_{\Omega} \rho(Du_k(x)) dx = O(k^{-1}) \quad \text{as } k \rightarrow \infty.$$

We now claim that there is a sequence $\{v_k\}_{k \in \mathbf{N}} \subset W_0^{1,1}(\Omega)$ converging to v in $W^{1,1}(\Omega)$ such that $J_k(v_k) \leq l_\infty + \varepsilon_k$ for all $k \in \mathbf{N}$.

To see this, define functions $g_k, b_k \in L^1(\Omega)^n$ for $k \in \mathbf{N}$ by

$$\begin{aligned} g_k(x) &= \begin{cases} Du_k(x) & \text{if } \rho(Du_k(x)) \leq 1 \\ 0 & \text{otherwise,} \end{cases} \\ b_k(x) &= Du_k(x) - g_k(x). \end{aligned}$$

Note that $|g_k(x)| \leq C + 1$ for $x \in \Omega$ and $k \in \mathbf{N}$ and that $Du_k = g_k + b_k$ for $k \in \mathbf{N}$. Note as well that if $\xi \in \mathbf{R}^n$ and $\rho(\xi) > 1$ then $|\xi| \leq C + \rho(\xi) \leq (C + 1)\rho(\xi)$ and hence

$$\|b_k\|_{L^1(\Omega)} \leq (C + 1) \int_{\Omega} \rho(b_k(x)) dx \leq (C + 1) \int_{\Omega} \rho(Du_k(x)) dx = O(k^{-1}) \quad \text{as } k \rightarrow \infty.$$

In view of Sobolev's embedding theorem and (5.1), we see that the sequence $\{u_k\}$ is bounded in $L^{n/(n-1)}(\Omega)$. Noting that the sequence $\{g_k\}$ is bounded in $L^\infty(\Omega)^n$, we see that, along a common subsequence, the sequences $\{u_k\}$ and $\{g_k\}$ converge to some functions $u \in L^{n/(n-1)}(\Omega)$ and $g \in L^\infty(\Omega)^n$, respectively, in $L^{n/(n-1)}(\Omega)$ and in $L^{n/(n-1)}(\Omega)^n$. Invoking a theorem due to Mazur, for each $j \in \mathbf{N}$ we may choose a finite sequence $\lambda_j^j, \lambda_{j+1}^j, \dots \in [0, 1]$ such that

$$\begin{aligned} \sum_{i \geq j} \lambda_i^j &= 1, \\ \sum_{i \geq j} \lambda_i^j u_i &\rightarrow u \quad \text{strongly in } L^{n/(n-1)}(\Omega) \quad \text{as } j \rightarrow \infty, \\ \sum_{i \geq j} \lambda_i^j g_i &\rightarrow g \quad \text{strongly in } L^{n/(n-1)}(\Omega)^n \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Set

$$v_k = \sum_{i \geq k} \lambda_i^k u_i \in W_0^{1,1}(\Omega) \quad \text{for } k \in \mathbf{N}.$$

Since

$$Dv_k = \sum_{i \geq k} \lambda_i^k g_i + \sum_{i \geq k} \lambda_i^k b_i \rightarrow g \quad \text{strongly in } L^1(\Omega)^n \quad \text{as } k \rightarrow \infty,$$

we see that $v_k \rightarrow u$ in $W^{1,1}(\Omega)$ as $k \rightarrow \infty$. Using the convexity of the functionals J_k , we observe that for $k \in \mathbf{N}$,

$$J_k(v_k) \leq \sum_{i \geq k} \lambda_i^k J_k(u_i) \leq \sum_{i \geq k} \lambda_i^k J_i(u_i) \leq \sum_{i \geq k} \lambda_i^k (l_i + \varepsilon_i) \leq l_\infty + \varepsilon_k.$$

proving our claim.

Fix such a sequence $\{v_k\}$. By the same reason as for u_k , we have

$$\lim_{k \rightarrow \infty} \int_{\Omega} \rho(Dv_k(x)) dx = 0,$$

which implies

$$\int_{\Omega} \rho(Du(x)) dx = 0$$

and furthermore

$$Du(x) \in \text{co } K \quad \text{or, equivalently,} \quad \widehat{H}(Du(x)) \leq 1 \quad \text{a.e. } x \in \Omega.$$

Thus we may assume that $u \in W_0^{1,\infty}(\Omega) \cap C(\overline{\Omega})$, and then we have $d(x) \geq u(x)$ for all $x \in \overline{\Omega}$.

Since

$$l_\infty + \varepsilon_k > J_k(v_k) \geq - \int_{\Omega} f(x)v_k(x)dx \quad \text{for all } k \in \mathbf{N},$$

we get in the limit as $k \rightarrow \infty$,

$$l_\infty \geq - \int_{\Omega} f(x)u(x)dx \geq - \int_{\Omega} f(x)d(x)dx = i_\infty,$$

which completes the proof. \square

Appendix.

Let $H : \mathbf{R}^n \rightarrow \mathbf{R}$ be a convex function satisfying the conditions:

- (a) H is positively homogeneous of degree one,
- (b) $H(\xi) > 0$ for $\xi \neq 0$.

As before we define $L : \mathbf{R}^n \rightarrow \mathbf{R}$ by

$$L(x) = \sup\{x \cdot \xi \mid \xi \in \mathbf{R}^n, H(\xi) \leq 1\}.$$

Let $B \subset \mathbf{R}^n$ be a non-empty closed set and define $d : \mathbf{R}^n \rightarrow \mathbf{R}$ by

$$d(x) = \inf\{L(x - y) \mid y \in B\}.$$

LEMMA A. *The function d is a viscosity solution of*

$$H(Dd(x)) = 1 \quad \text{in } \mathbf{R}^n \setminus B. \tag{A.1}$$

Proof. We set $\Omega = \mathbf{R}^n \setminus B$. Let $y \in \Omega$ and $q \in D^-d(y)$. We show that $H(q) = 1$. To this end, choose a point $z \in B$ so that $d(y) = L(y - z)$.

Since $d(x) \leq L(x - z)$ for all $x \in \Omega$, we see that $q \in D^-L(y - z)$. Since L is a convex function, by the convex duality we have

$$y - z \in D^- \delta_K(q),$$

where $K = \{\xi \in \mathbf{R}^n \mid H(\xi) \leq 1\}$ and δ_K denotes the indicator function of K as before. Since $y - z \neq 0$ and

$$D^- \delta_K(q) = \begin{cases} \{0\} & \text{if } q \in \text{Int } K, \\ \emptyset & \text{if } q \notin K, \end{cases}$$

we infer that $q \in \partial K$, and hence $H(q) = 1$. To see that d is a viscosity subsolution of (A.1), we invoke a result due to Barron-Jensen (see, e.g., [3, 14]), and then conclude that d is also a viscosity subsolution of (A.1). \square

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