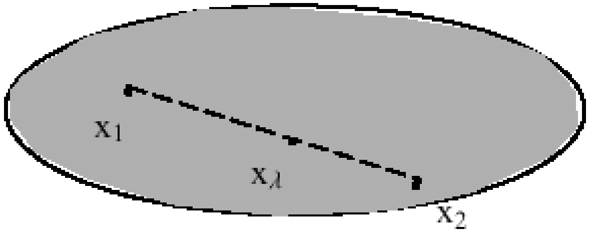
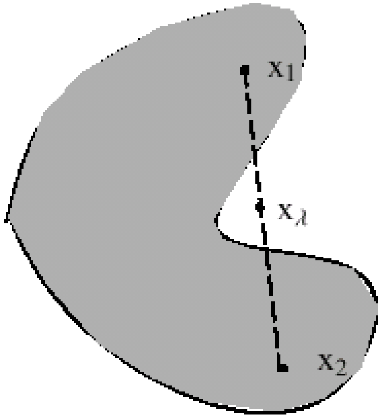
Chap. 16 Convex set and optimization

A is a convex set $\Leftrightarrow$

$$\forall\lambda(0<\lambda<1)\quad \forall \mathbf{x}_1,\mathbf{x}_2\in A\quad \lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2\in A$$

convex



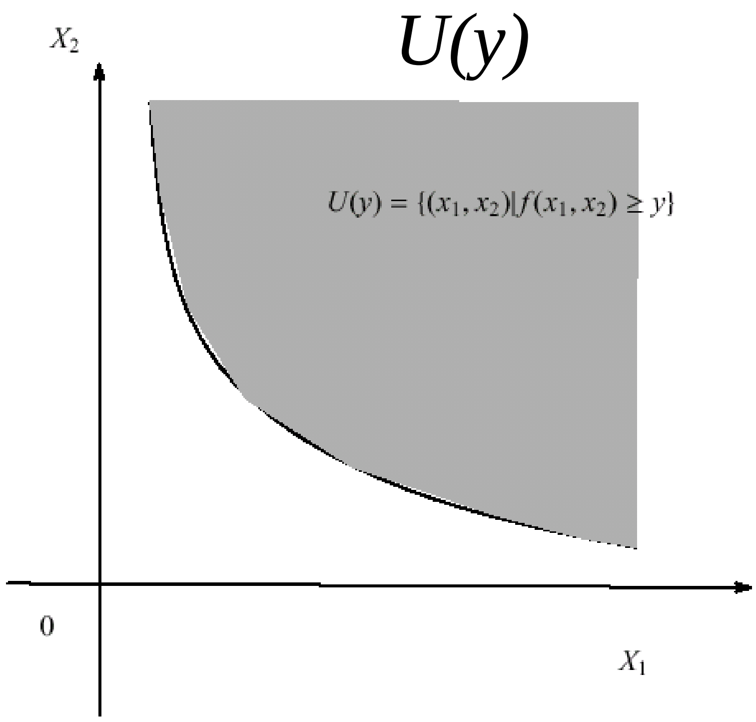


non-convex

$$\boldsymbol{x}_\lambda \equiv \lambda\boldsymbol{x}_1+(1-\lambda)\boldsymbol{x}_2$$

Upper contour set
lower contour set

$$y = f(x_1, x_2, .., x_n)$$

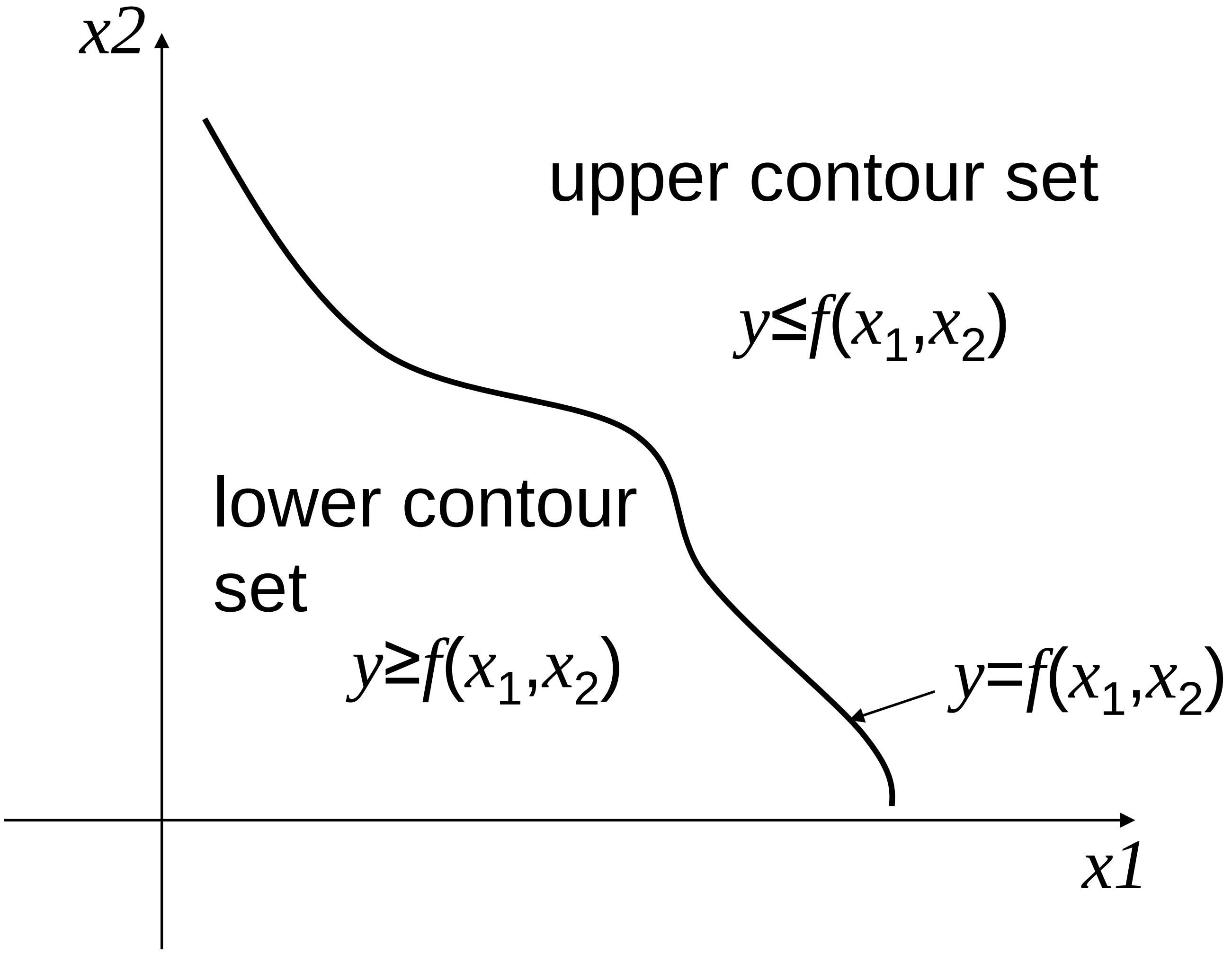


Upper contour set

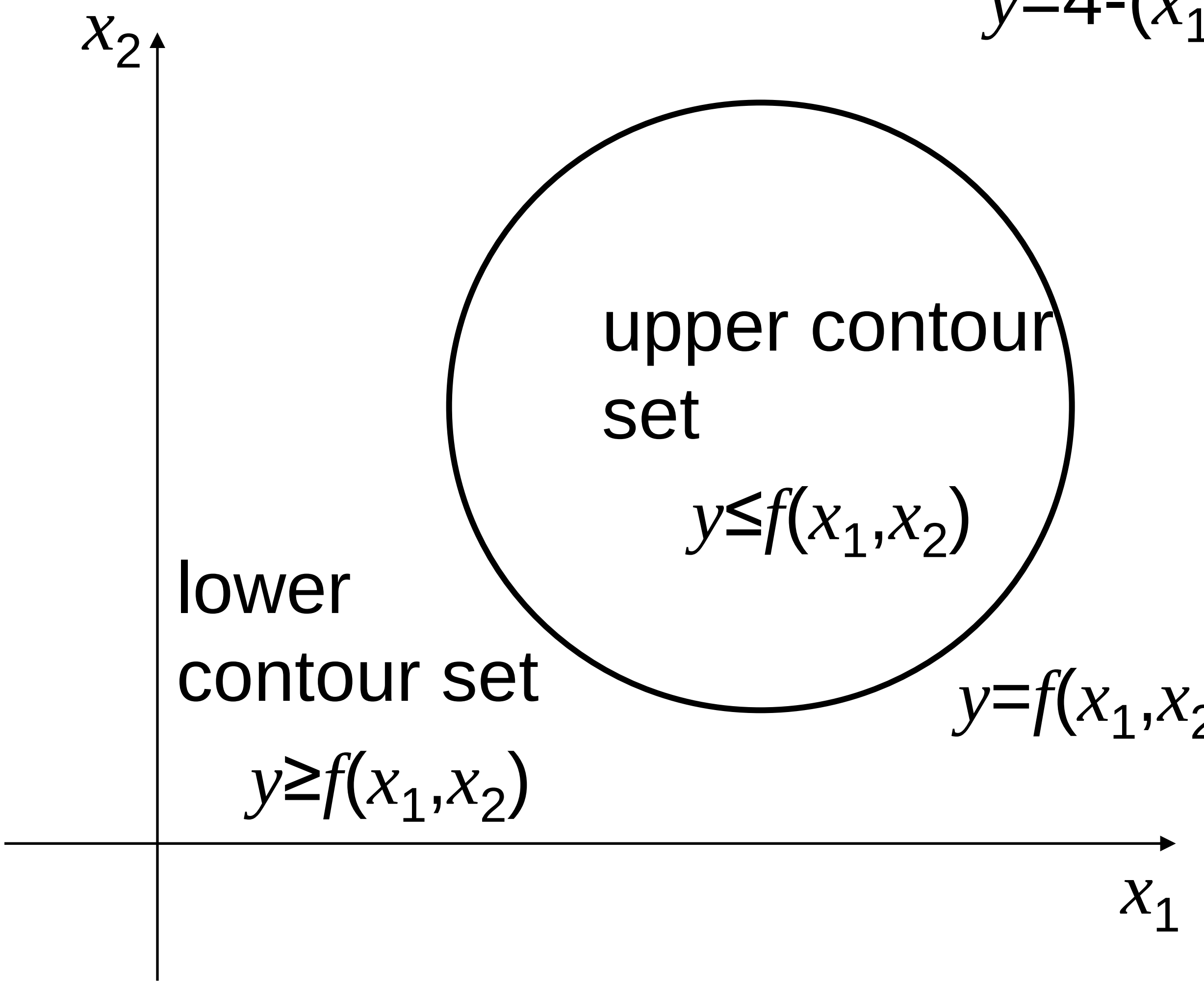
$$U(y) = \{(x_1, x_2, ..., x_n) \in \Re^n \mid f(x_1, x_2, ..., x_n) \geq y\}$$

Lower contour set

$$D(y) = \{(x_1, x_2, ..., x_n) \in \Re^n \mid f(x_1, x_2, ..., x_n) \leq y\}$$

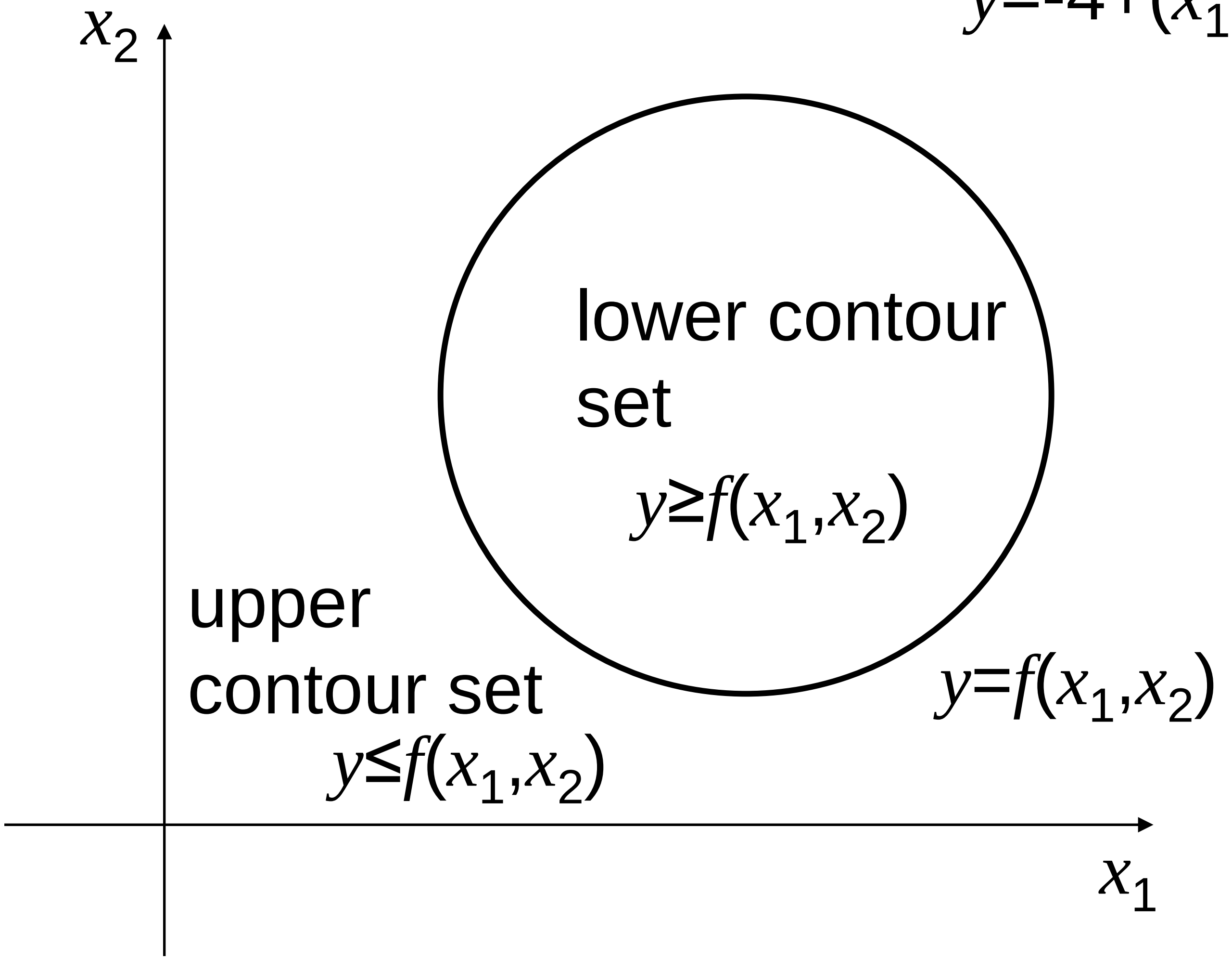

x2
upper contour set
y≤f(x1,x2)
lower contour set
y≥f(x1,x2)
y=f(x1,x2)
x1

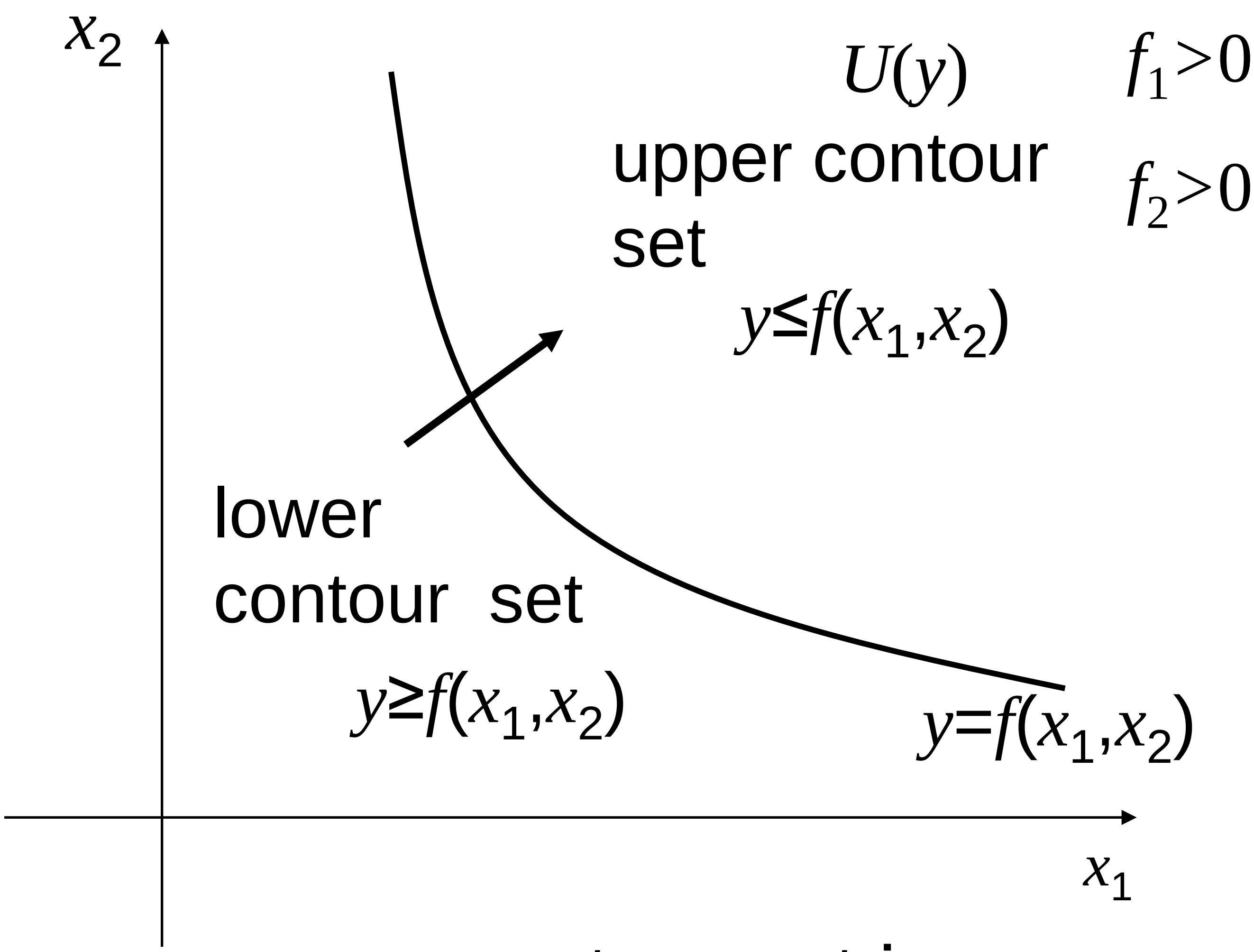
$$y=4-(x_1-4)^2-(x_2-3)^2$$



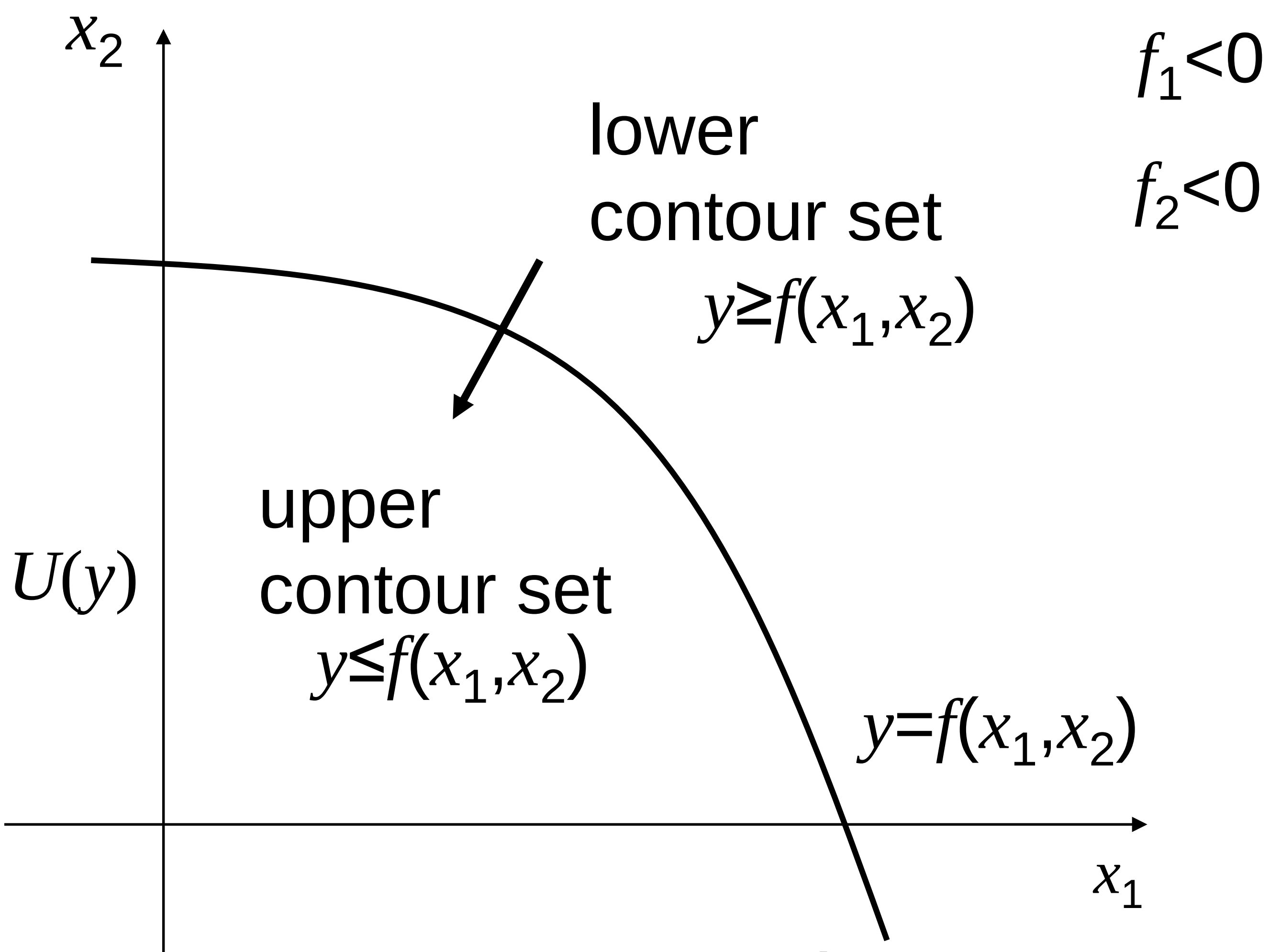
x_2
upper contour set
$y \leq f(x_1,x_2)$
lower contour set
$y \geq f(x_1,x_2)$
$y=f(x_1,x_2)$
x_1

$y=-4+(x_1-4)^2+(x_2-3)^2$


x_2
lower contour set
$y \geq f(x_1,x_2)$
upper contour set
$y \leq f(x_1,x_2)$
$y=f(x_1,x_2)$
x_1

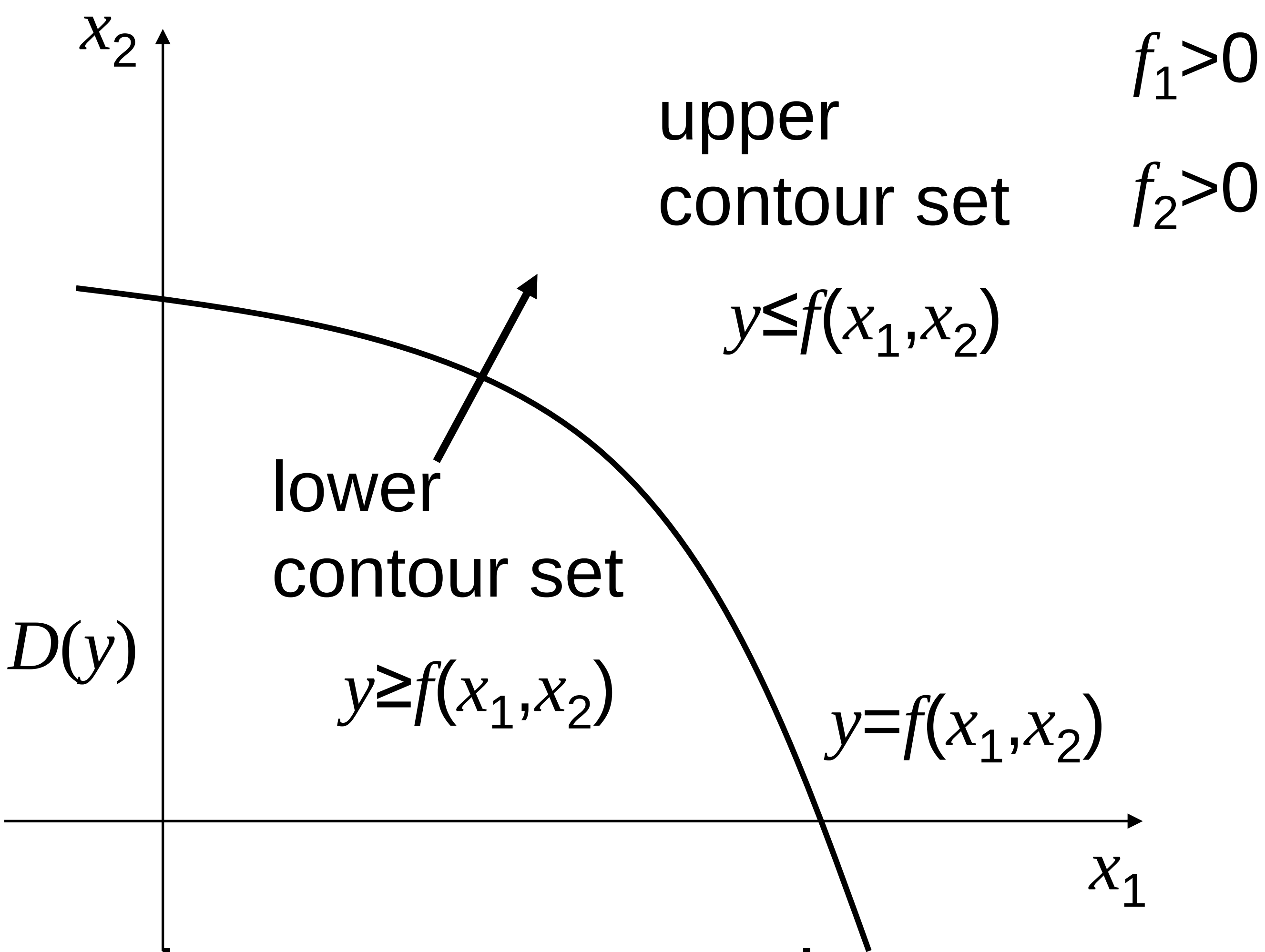


upper contour set is convex
=quasi-concave function

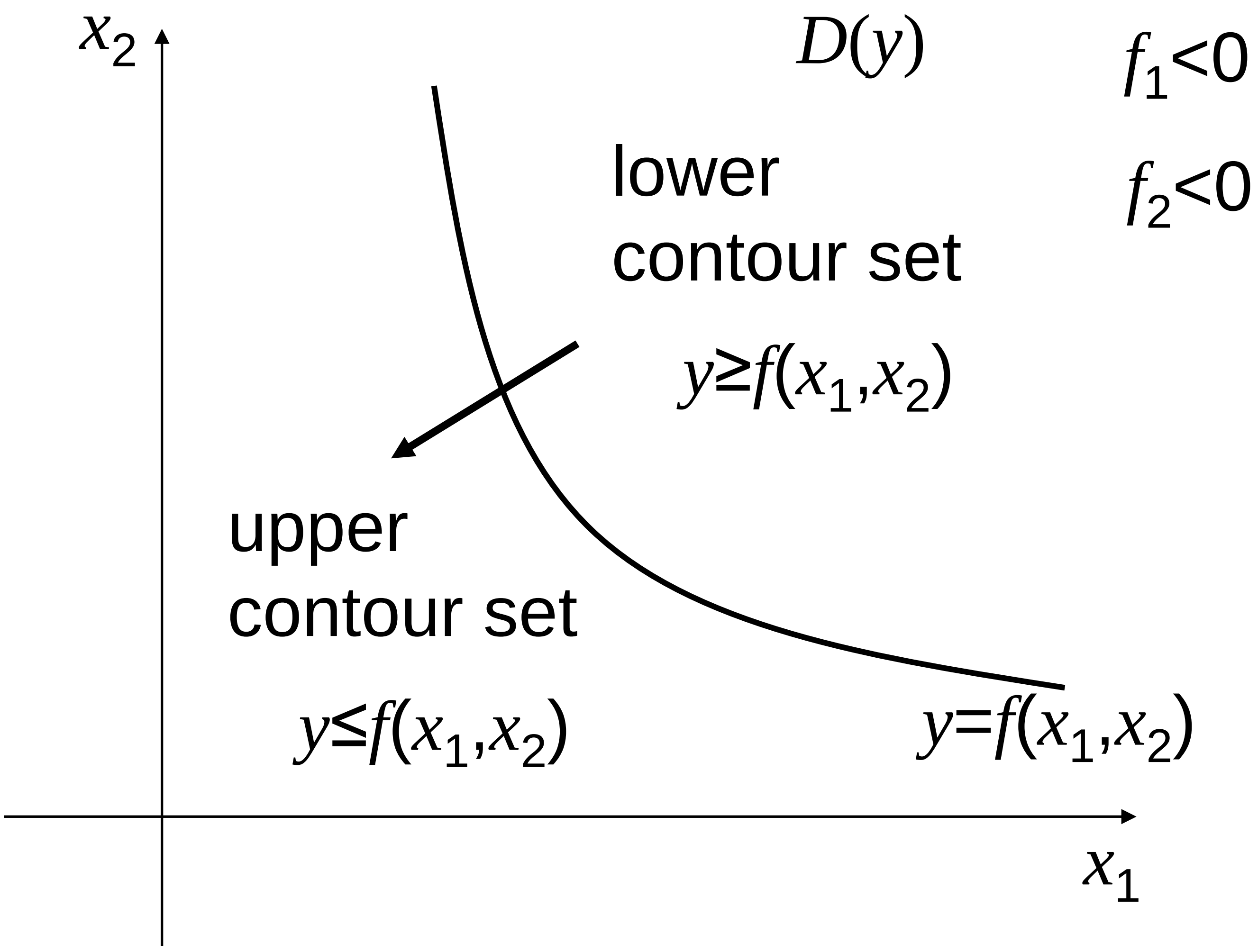


upper contour set is convex

=quasi-concave function

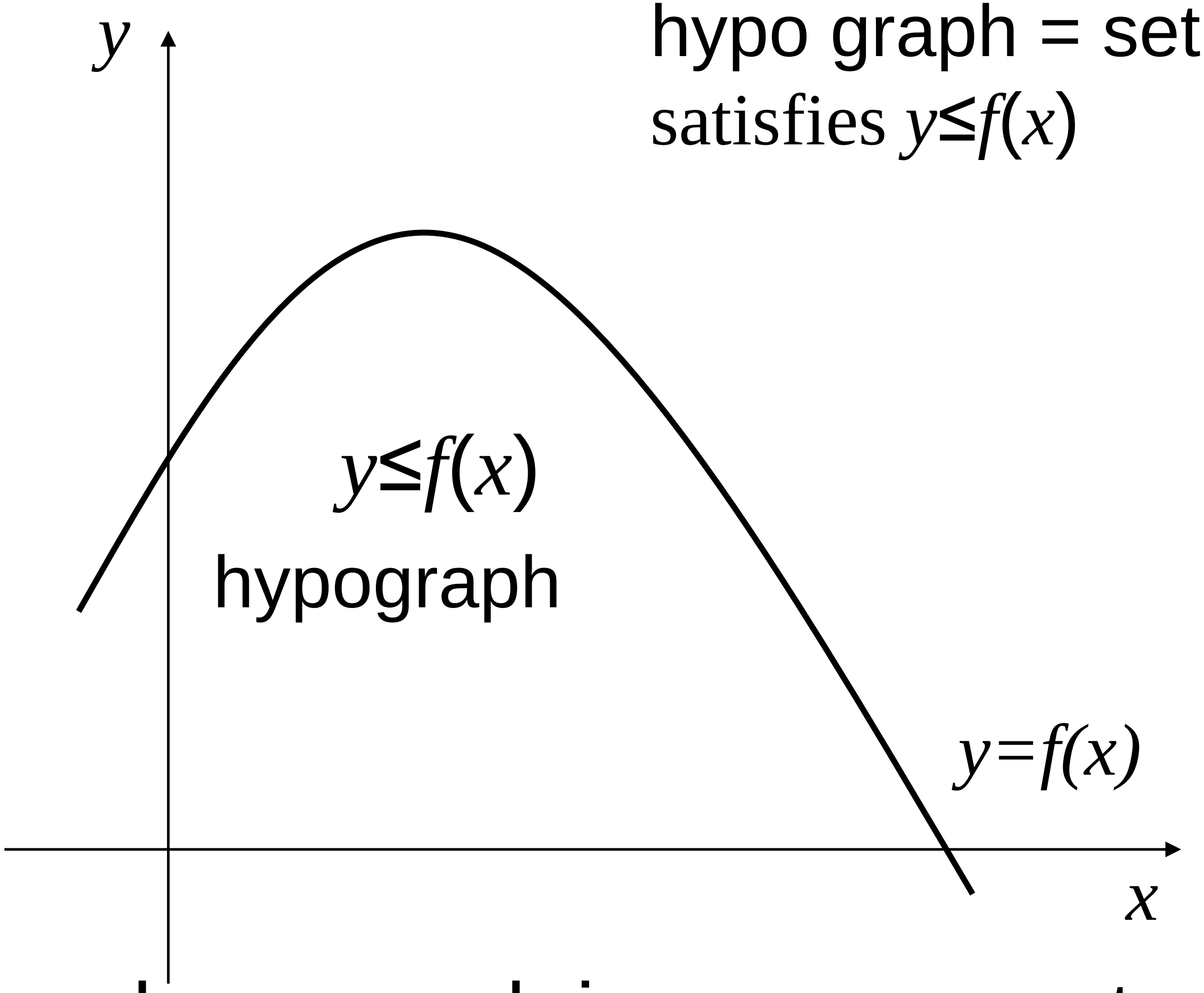


lower contour set is convex
=quasi-convex function



lower contour set is convex
=quasi-convex function

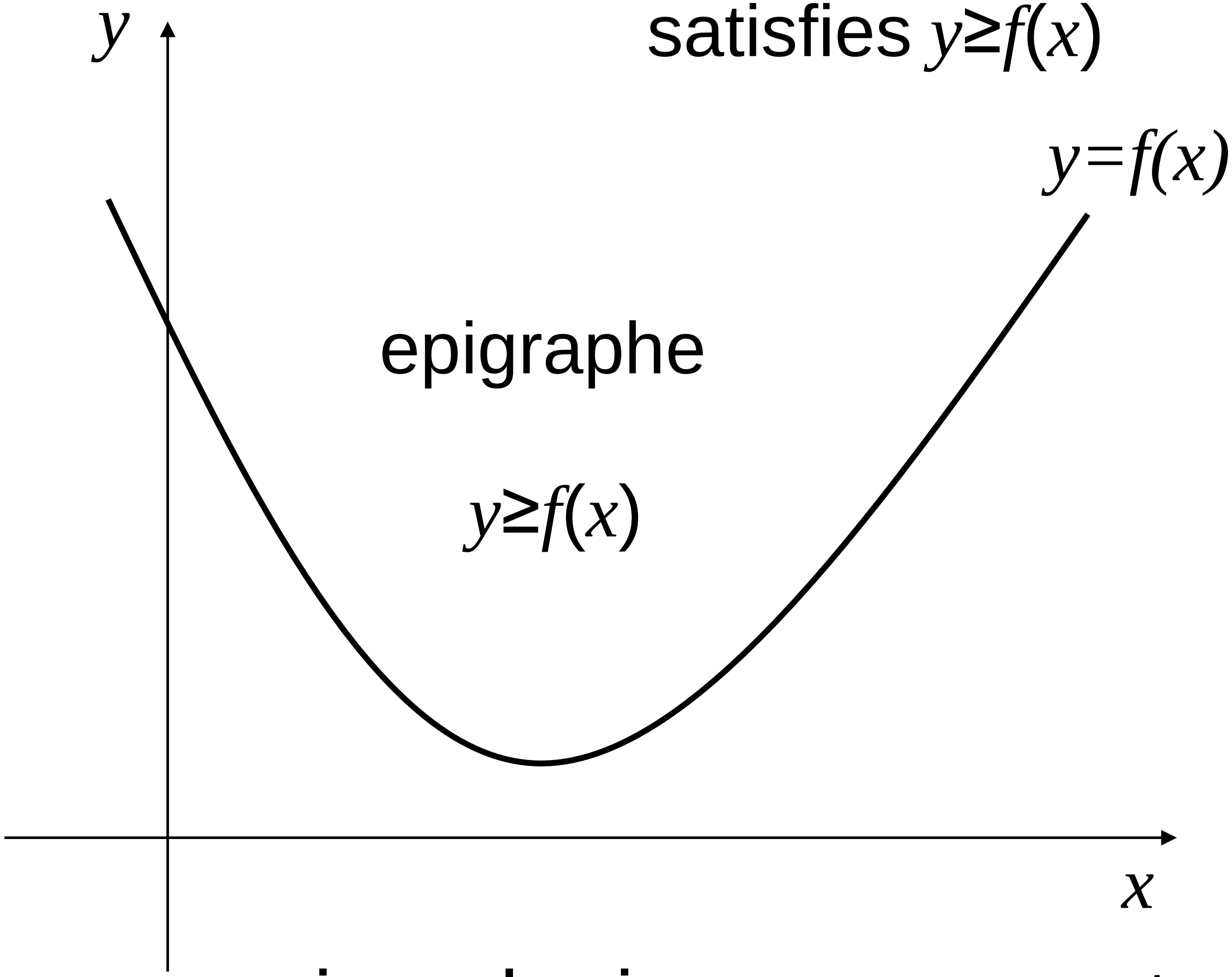
hypo graph = set of (x,y) which satisfies $y \leq f(x)$



hypo graph is a convex set

=a concave function

epigraphe = set of (x,y) which satisfies $y \geq f(x)$



epigraphe is a convex set
=a convex function

The semi- concavity function, the semi- convexity function

multivariable function $y=f(x)$ $\quad (\boldsymbol{x} \in \Re^n)$

$$y = f(\mathbf{x}) \text{ is quasi-concave}$$

$$\Leftrightarrow U(y) \text{ is a convex set for any } y$$

$$y = f(\mathbf{x}) \text{ is quasi-convex}$$

$$\Leftrightarrow D(y) \text{ is a convex set for any } y$$

The necessary and sufficient condition of quasi-concavity

$$y = f(\mathbf{x}) \text{ is quasi-concave} \Leftrightarrow$$

$$f(\mathbf{x}_1) \geq f(\mathbf{x}_2) \Rightarrow f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \geq f(\mathbf{x}_2)$$

$$\forall \lambda (0 < \lambda < 1)$$

$$\mathbf{x}_1, \mathbf{x}_2 \in U(y) \Rightarrow f(\mathbf{x}_1) \geq y,\ \ f(\mathbf{x}_2) \geq y$$

$$\text{Suppose } f(\mathbf{x}_1) \geq f(\mathbf{x}_2) \geq y$$

$$f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \geq f(\mathbf{x}_2) \geq y \ \Rightarrow \mathbf{x}_\lambda \in U(y)$$

Strictly quasi-concave functions, strictly quasi-convex functions

$$y = f(\mathbf{x}) \text{ is strictly quasi-concave} \Leftrightarrow$$

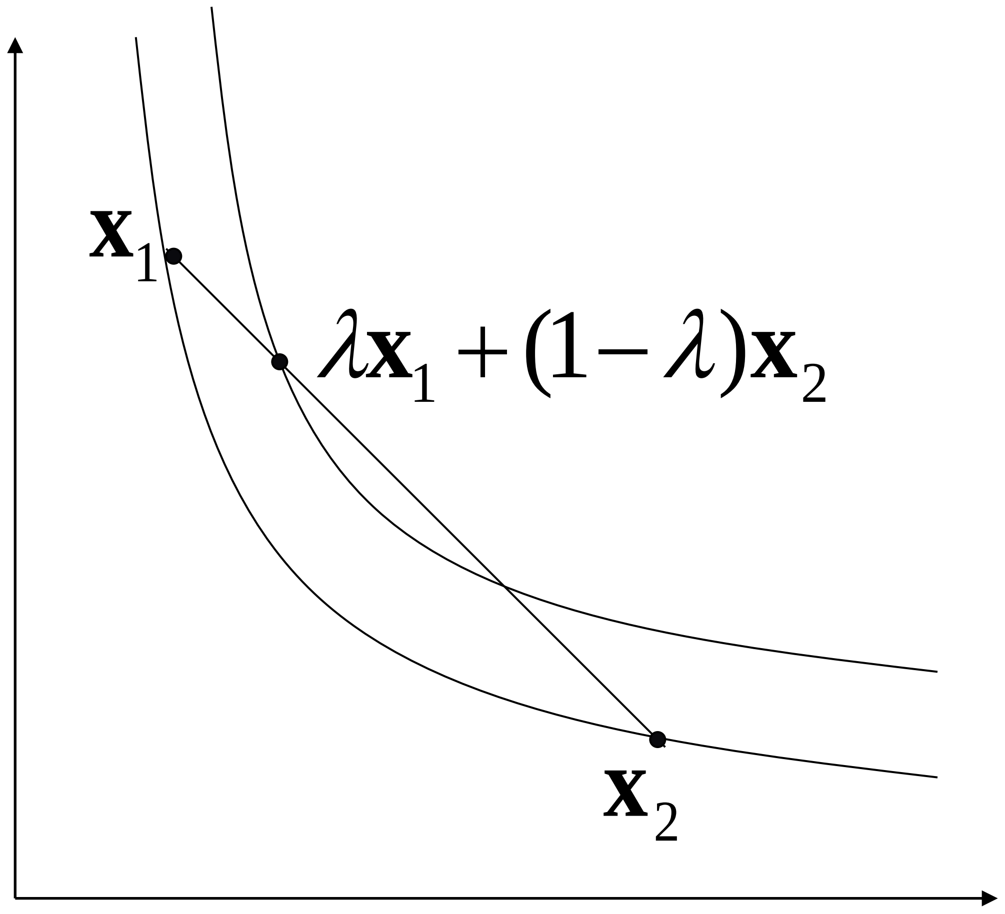
$$f(\mathbf{x}_1) \geq f(\mathbf{x}_2) \Rightarrow f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) > f(\mathbf{x}_2)$$

$$\forall \lambda (0 < \lambda < 1)$$

$$y = f(\mathbf{x}) \text{ is (strictly) quasi-convex} \Leftrightarrow$$

$$f(\mathbf{x}_1) \leq f(\mathbf{x}_2) \Rightarrow f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \underset{[<]}{\leq} f(\mathbf{x}_2)$$

$$\forall \lambda (0 < \lambda < 1)$$


$\mathbf{x}_1$
$\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$
$\mathbf{x}_2$

Strictly quasi-concave functions and marginal rate of substitution

$y = f(x_1, x_2)$ is strictly quasi-concave

$f_2 > 0$

On the border line of $U(y)$ the absolute of

the slope $-\dfrac{dx_2}{dx_1}$ is decreasing

By the implicit function theorem $-\dfrac{dx_2}{dx_1} = \dfrac{f_1}{f_2}$

Strictly quasi-concave functions and marginal rate of substitution

$$\frac{d}{dx_1}\left(-\frac{dx_2}{dx_1}\right)=\frac{d}{dx_1}\left(\frac{f_1}{f_2}\right)=\frac{\partial}{\partial x_1}\left(\frac{f_1}{f_2}\right)+\frac{\partial}{\partial x_2}\left(\frac{f_1}{f_2}\right)\frac{dx_2}{dx_1}$$

$$=\frac{f_{11}f_2-f_1f_{21}}{(f_2)^2}+\frac{f_{12}f_2-f_1f_{22}}{(f_2)^2}\left(-\frac{f_1}{f_2}\right)$$

$$=\frac{f_{11}f_2^{\,2}-f_1f_{21}f_2-f_{12}f_2f_1+f_1^{\,2}f_{22}}{(f_2)^3}=\frac{f_{11}f_2^{\,2}+f_{22}f_1^{\,2}-2f_{12}f_1f_2}{(f_2)^3}<0$$

Necessary and sufficient conditions for strictly quasi-concave functions

If $f_2 > 0$, f is strictly quasi-concave

$$\Leftrightarrow 2 f_{21} f_1 f_2 - f_{11} f_2^{\ 2} - f_{22} f_1^{\ 2} > 0$$

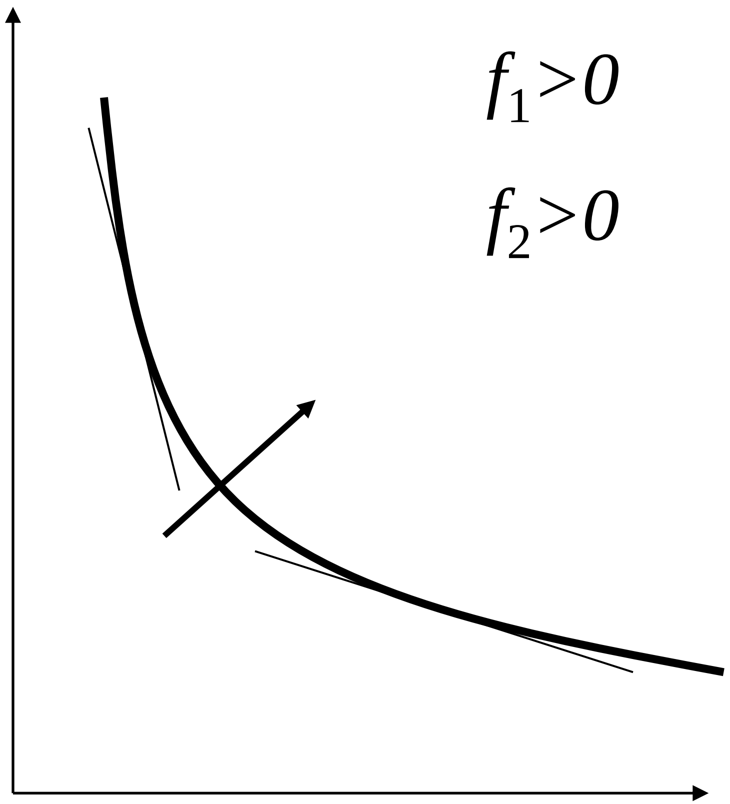
If $f_2 < 0$, with $\dfrac{d}{dx_1}\left(-\dfrac{dx_2}{dx_1}\right) > 0$,

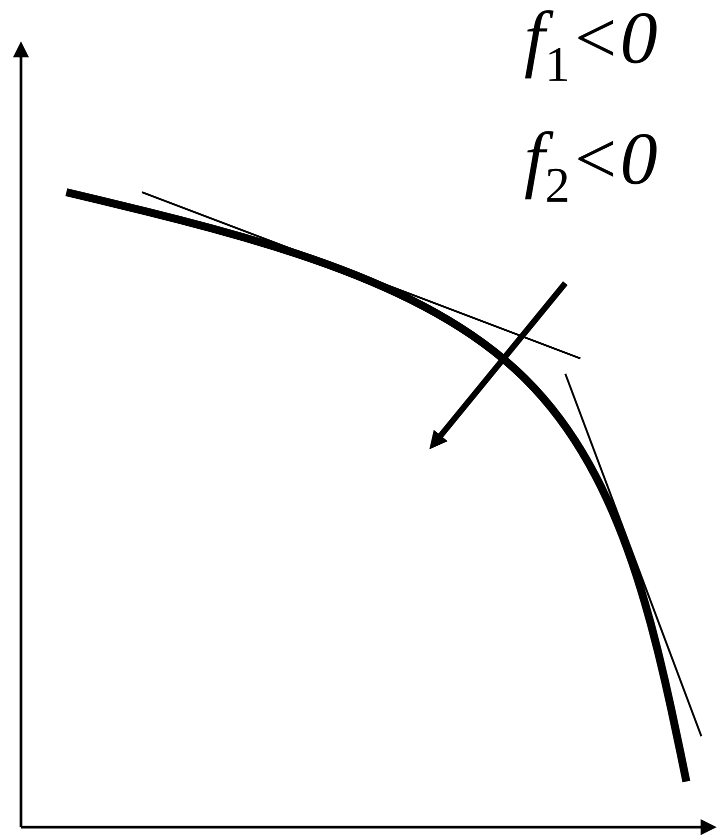
$$\frac{f_{11} f_2^{\ 2} + f_{22} f_1^{\ 2} - 2 f_{12} f_1 f_2}{(f_2)^3} > 0$$

f is stricly quasi-concave

$$\Leftrightarrow 2 f_{21} f_1 f_2 - f_{11} f_2^{\ 2} - f_{22} f_1^{\ 2} > 0$$

Strictly quasi-concave functions





The slope approaches 0 as x_1 goes up.

The slope increases as x_1 goes up.

Necessary and sufficient conditions for strictly quasi- concave, quasi-convex functions

f is strictly quasi-concave

$$\Leftrightarrow 2 f_{21} f_1 f_2 - f_{11} f_2^{\,2} - f_{22} f_1^{\,2} > 0$$

f is quasi-concave

$$\Leftrightarrow 2 f_{21} f_1 f_2 - f_{11} f_2^{\,2} - f_{22} f_1^{\,2} \geq 0$$

f is (strictly) quasi-convex

$$\Leftrightarrow 2 f_{21} f_1 f_2 - f_{11} f_2^{\,2} - f_{22} f_1^{\,2} \leq 0 \quad [<]$$

Quasi- concavity of concave functions

$$y = f(\mathbf{x}) \text{ is concave} \Rightarrow \text{quasi-concave}$$

$$f(\mathbf{x}_\lambda) = f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \geq \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2)$$

$$\forall \mathbf{x}_1, \mathbf{x}_2, \forall \lambda (0 < \lambda < 1)$$

$$\mathbf{x}_1, \mathbf{x}_2 \in U(y) \Rightarrow f(\mathbf{x}_1) \geq y \quad f(\mathbf{x}_2) \geq y$$

$$f(\mathbf{x}_\lambda) \geq \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2) \geq \lambda y + (1-\lambda) y = y$$

$$\Rightarrow \mathbf{x}_\lambda \in U(y)$$

$$y = f(\mathbf{x}) \text{ is convex} \Rightarrow \text{quasi-convex}$$

Quasi-concavity of Cobb-Douglas function

$$F(K,L) = AK^{\alpha}L^{\beta} \qquad A, \alpha, \beta > 0$$

The iso-quant curve

$$Q = F(K,L) = AK^{\alpha}L^{\beta}$$

$$L = h(K,Q) = K^{-\frac{\alpha}{\beta}}\left(\frac{Q}{A}\right)^{\frac{1}{\beta}}$$

$$-RTS_{12} = \frac{\partial h(K,Q)}{\partial K} = -\frac{\alpha}{\beta}K^{-\frac{\alpha}{\beta}-1}\left(\frac{Q}{A}\right)^{\frac{1}{\beta}}$$

Quasi-concavity of Cobb-Douglas function

$$-RTS_{12} = \frac{\partial h(K,Q)}{\partial K} = -\frac{\alpha}{\beta} K^{-\frac{\alpha}{\beta}-1} \left(\frac{Q}{A}\right)^{\frac{1}{\beta}}$$

$$\frac{\partial^2 h(K,Q)}{\partial K^2} = \frac{\alpha}{\beta}\left(\frac{\alpha}{\beta}+1\right) K^{-\frac{\alpha}{\beta}-2} \left(\frac{Q}{A}\right)^{\frac{1}{\beta}} > 0$$

always quasi-concave

$$F_{KK}F_{LL} - F_{KL}{}^2 = \{\alpha(\alpha-1)\beta(\beta-1) - \alpha^2\beta^2\} A^2 K^{2\alpha-2} L^{2\beta-2}$$

$$= \alpha\beta(1-\alpha-\beta) A^2 K^{2\alpha-2} L^{2\beta-2} > 0$$

$$1 > \alpha + \beta \Leftrightarrow \text{convex}$$

It mediates between Cobb-Douglas function.

$$F(K,L) = AK^{\alpha}L^{\beta} \qquad A, \alpha, \beta > 0$$

always quasi-concave

$$\alpha + \beta < 1 \Leftrightarrow \text{decreasing return to scale, concave}$$

$$\alpha + \beta = 1 \Leftrightarrow \text{constant return to scale}$$

$$\alpha + \beta > 1 \Leftrightarrow \text{increasing return to scale}$$