Note: throughout this course \( C^\times = \mathbb{C} \setminus \{0\} \) and \( C_\infty = \mathbb{C} \cup \{\infty\} \). For simplicity we often denote partial derivatives by subscripts, e.g., \( f_z = \partial f / \partial z \).

Recall from topology that, by definition, a continuous map \( \varphi : M \to N \) between topological spaces pulls open sets back to open sets (equally, pulls closed sets back to closed sets) and maps compact sets to compact sets. Recall also the Heine-Borel theorem: a closed bounded subset of Euclidean space is compact.

(1) Let \( S^2 = \{(a, b, c) \in \mathbb{R}^3 : a^2 + b^2 + c^2 = 1\} \).

Let \( e_3 = (0, 0, 1) \) be the “north pole” and define \( V_1 = S^2 \setminus \{e_3\} \), \( V_2 = S^2 \setminus \{-e_3\} \). Stereographic projection from, respectively, \( e_3 \) and \(-e_3\) to the plane \( c = 0 \), which we identify with \( \mathbb{C} \), gives two maps

\[
\begin{align*}
  f_1 : V_1 &\to \mathbb{C}; & f_1(a, b, c) &= (a + ib) / (1 - c), \\
  f_2 : V_2 &\to \mathbb{C}; & f_2(a, b, c) &= (a - ib) / (1 + c).
\end{align*}
\]

Show that \( \{(V_j, f_j) : j = 1, 2\} \) is a complex atlas for \( S^2 \). Deduce that this makes \( S^2 \) a compact Riemann surface.

(2) Complex projective space. Let \( \mathbb{CP}^n \) denote the set of complex lines (1-dimensional subspaces) in \( \mathbb{C}^{n+1} \). To each \((z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}\) we can assign the line

\[
[z_0, z_1, \ldots, z_n] = \{(tz_0, tz_1, \ldots, tz_n) : t \in \mathbb{C}^\times\},
\]

and every line possesses such a description. We refer to \([z_0, z_1, \ldots, z_n]\) as the homogeneous coordinates. Thus we have a map

\[
\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n; \quad (z_0, z_1, \ldots, z_n) \mapsto [z_0, z_1, \ldots, z_n].
\]

We use this to give \( \mathbb{CP}^n \) a topology called the quotient topology: a subset \( U \subset \mathbb{CP}^n \) is open iff \( \pi^{-1}(U) \) is open. This automatically makes \( \pi \) a continuous map.

(a) Let \( S^{2n+1} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \ldots + |z_n|^2 = 1\} \).

Show that \( \pi(S^{2n+1}) = \mathbb{CP}^n \) and deduce that \( \mathbb{CP}^n \) is compact.

(b) For each \( j = 0, \ldots, n \) define

\[
W_j = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n : z_j \neq 0\}.
\]

Define also

\[
\varphi_j : W_j \to \mathbb{C}^n; \quad [z_0, \ldots, z_n] \mapsto \left( \frac{z_0}{z_j}, \ldots, \frac{z_n}{z_j} \right),
\]

(2)
omitting \(z_j/z_j\) in the last \(n\)-tuple. Show that \(\{(W_j, \varphi_j) : j = 0, \ldots, n\}\) is a complex atlas for \(\mathbb{CP}^n\). Deduce that \(\mathbb{CP}^n\) is a compact complex \(n\)-manifold.

**Remark.** \(\mathbb{CP}^1\) is called the complex projective line; \(\mathbb{CP}^2\) is called the complex projective plane.

(3) A “branch cut” construction of the Riemann surface for \(w^2 = z\). Take two copies of the Riemann sphere \(C_\infty\) and cut each along the line \(0 < \text{Re}(z), \text{Im}(z) = 0\) to obtain cut spheres \(S_1, S_2\). By “cut” we mean this line is doubled, so that the map

\[
\{w = re^{i\theta} \in \mathbb{C} : 0 \leq \theta \leq \pi, 0 \leq r\} \to S_j; \quad w \mapsto w^2,
\]

is injective. On each sheet \(S_j\), label the image of \(\theta = 0\) (together with the point \(\infty_j\) at infinity) by \(C^+_j\) and the image of \(\theta = \pi\) (together with the point \(\infty_j\)) by \(C^-_j\). Now glue \(C_1\) to \(C_2\) so that \(C^-_1 = C^+_2\) and \(C^+_1 = C^-_2\) to obtain an oriented surface \(S(\sqrt{z})\) covering \(C_\infty\). Show how to give this a complex atlas with two charts.

(4) For \(S^2, \mathbb{CP}^1\) and \(S(\sqrt{z})\), with the complex structures given in the previous exercises, find a biholomorphic map between each of these and \(C_\infty\). Deduce that

\[S^2 \simeq \mathbb{CP}^1 \simeq S(\sqrt{z}) \simeq C_\infty.\]

(5) Following the outline given in example 2.4(4) of the notes, describe explicitly a complex atlas for the planar algebraic curve

\[C = \{(z, w) \in \mathbb{C}^2 : w^2 - z(z - 1)(z - a) = 0\}, \quad a \neq 0, 1.\]

In particular, show \(f(z, w) = w^2 - z(z - 1)(z - a)\) has \(f_w \neq 0\) when \(z \neq 0, 1, a\) and hence the function \(z : C \to \mathbb{C}\) provides a coordinate map when restricted to a suitable neighbourhood of any point for which \(z \neq 0, 1, a\). Show that at about the points where \(z = 0, 1, a, f_z \neq 0\) and therefore \(w : C \to \mathbb{C}\) can be used locally to get coordinates. Verify that your atlas has holomorphic change of coordinates on overlapping charts.

(6) **Projective algebraic planar curves.** A complex polynomial \(F(z_0, z_1, z_2)\) is **homogeneous of degree \(k\)** if

\[F(tz_0, tz_1, tz_2) = t^kF(z_0, z_1, z_2), \quad \text{for all } t \in \mathbb{C}.\]

It follows that

\[S = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 : F(z_0, z_1, z_2) = 0\} \subset \mathbb{CP}^2,\]

is well-defined. Further, it is a closed subset (and therefore compact) since

\[\pi^{-1}(S) = \{(z_0, z_1, z_2) \in \mathbb{C}^3 \setminus \{0\} : F(z_0, z_1, z_2) = 0\}\]

is closed.
(a) Using the charts (2) on $\mathbb{CP}^2$ assign to $S$ three subsets $C_j \subset \mathbb{C}^2$ defined by $C_j = \varphi_j(S \cap W_j)$. Show that each $C_j$ is an affine algebraic curve and that $\varphi$ is one-to-one on each $C_j$. Deduce that $C_j \simeq S \cap W_j$.

$C_0 = \{(z, w) \in \mathbb{C}^2 : F(1, z, w) = 0\}$.

(b) Show that each $C_j$ is non-singular provided the gradient $dF = (F_{z_0}, F_{z_1}, F_{z_2})$ does not vanish on $C_j$. Deduce that if $dF$ does not vanish on $S$ then $S$ is a compact Riemann surface.

(c) Show that for $a \neq 0, 1$ the homogeneous polynomial

\[ F(z_0, z_1, z_2) = z_0 z_2^2 - z_1 (z_1 - z_0)(z_1 - az_0) \]

determines a compact Riemann surface for which $C_0$ is the affine curve $C$ in question (5).

(7) Let $S$ be a compact Riemann surface and $\varphi : S \to \mathbb{C}_\infty$ a holomorphic map. Show that the number of points on $S$ lying over any fixed $p \in \mathbb{C}_\infty$ is at most $\nu(\varphi)$, the topological degree of $\varphi$. In other words

\[ \# \varphi^{-1}(p) \leq \nu(\varphi). \]

(8) Now suppose $\varphi : S \to \mathbb{C}_\infty$ is a holomorphic map with topological degree 1. Show that $\varphi$ must be a bijection. Any holomorphic bijection is a biholomorphism (by the holomorphic inverse function theorem) and therefore $S$ is the Riemann sphere.

(9) Show that every meromorphic function of degree 1 on the Riemann sphere is a fractional linear transformation, i.e.,

\[ f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0. \]

Conclude that the group of biholomorphisms of $\mathbb{C}_\infty$ is isomorphic to $PSL(2, \mathbb{C})$ (i.e., $SL(2, \mathbb{C})/\{\pm I\}$).

(10) Let $g = u(x, y) + iv(x, y)$ be a smooth complex-valued function on (an open subset of) $\mathbb{R}^2$. Show that the Cauchy-Riemann equations for $g$ are equivalent to the equation $\partial g / \partial \bar{z} = 0$. This is why $\partial / \partial \bar{z}$ is referred to as the Cauchy-Riemann operator. Now consider $g$ as a map

\[ g : \mathbb{R}^2 \to \mathbb{R}^2; \quad (x, y) \mapsto (u, v), \]

with total derivative at each point $p = (x, y)$ given by

\[ dg_p : \mathbb{R}^2 \to \mathbb{R}^2; \quad dg_p = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}. \]

Let $J : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map representing multiplication by $i$ using the identification $\mathbb{R}^2 \simeq \mathbb{C}$. Show that $g$ satisfies the Cauchy-Riemann equations if and only if at each point

\[ dg \circ J = J \circ dg. \]
(11) Let \( f \in \mathcal{O}_U \). Show that \( \tilde{f} \) is anti-holomorphic (i.e., holomorphic in \( \bar{z} \)). hence \( \mathcal{O}_U \) is the algebra of anti-holomorphic functions on \( U \).

(12) (a) Let \( \xi \in T_p\mathbb{R}^n \) be a real tangent vector at some point \( p \in \mathbb{R}^n \), i.e., \( \xi : \mathcal{E}_p^0 \to \mathbb{R} \) is \( \mathbb{R} \)-linear and \( \xi(fg) = (\xi f)g(p) + f(p)(\xi g) \) for \( f, g \in \mathcal{E}_p^0 \). Show that \( \xi k = 0 \) for any constant function \( k \in \mathbb{R} \).

(b) Let \( x^1, \ldots, x^n \) be the canonical coordinates on \( \mathbb{R}^n \). A version of Taylor’s theorem in several variables says that if \( f \) is a smooth function on some open \( U \subset \mathbb{R}^n \) then it can be expanded around a point \( p \in U \) in the form

\[
f(x) = f(p) + \sum_{j=1}^n \frac{\partial f}{\partial x_j}(p)(x_j - x_j(p)) + \sum_{j,k=1}^n h_{jk}(x)(x_j - x_j(p))(x_k - x_k(p)),
\]

where the \( h_{jk} \) are smooth functions. Show that for any \( \xi \in T_p\mathbb{R}^n \)

\[
\xi f = \sum_{j=1}^n (\xi x_j) \frac{\partial f}{\partial x_j}(p).
\]

Deduce from this that \( T_p\mathbb{R}^n \) is spanned by the tangent vectors \( X_1, \ldots, X_n \) where \( X_j = (\partial x_j)_p \), i.e.,

\[
X_j : \mathcal{E}_p^0 \to \mathbb{R}, \quad X_jf = \frac{\partial f}{\partial x_j}(p).
\]

(c) Let \( M \) be a complex \( m \)-manifold. Given \( p \in M \) let \( (U, z_1, \ldots, z_m) \) be a chart about \( p \) and write \( z_j = x_j + iy_j \). Prove that \( T^C_p M \simeq T_p M \otimes_{\mathbb{R}} \mathbb{C} \) and

\[
T_p M = \mathbb{R}\langle X_1, Y_1, \ldots, X_m, Y_m \rangle,
\]

where \( X_j = (\partial x_j)_p, Y_j = (\partial y_j)_p \). Deduce that

\[
\dim_{\mathbb{R}}(T_p M) = \dim_{\mathbb{C}}(T^C_p M) = 2m.
\]

(d) Deduce from the pervious part that in any coordinate chart \( (U, z_1, \ldots, z_m) \) on \( M \)

\[
\mathcal{T}_U = \mathcal{E}_U^0 \langle \partial z_1, \ldots, \partial z_m, \partial \bar{z}_1, \ldots, \partial \bar{z}_m \rangle,
\]

\[
\Theta_U = \mathcal{O}_U \langle \partial z_1, \ldots, \partial z_m \rangle.
\]

(e) Let \( \xi \in T_p M \) be a complex tangent vector at a point \( p \in M \). Show that for every chart \( U \subset M \) about \( p \) there is a (non-unique) complex vector field \( Z \in \mathcal{T}_U \) such that \( \xi = Z_p \). Deduce that \( \mathcal{T}_U \to T^C_p M; Z \mapsto Z_p \) is onto. Deduce that this holds also for any open neighbourhood \( U \) of \( p \).

(13) Let \( M \) be a complex \( m \)-manifold. Given a chart \( (U, z_1, \ldots, z_m) \) about \( p \in M \) define \( z_j = x_j + iy_j \) and \( X_j = \partial x_j, Y_j = \partial y_j \). Define also the \( \mathbb{R} \)-linear map

\[
J_p : T_p M \to T_p M, \quad J_p(X_j) = Y_j, \quad J_p(Y_j) = -X_j.
\]
Then $J^2_p = -\text{id}$, hence it is a complex structure on the $\mathbb{R}$-space $T_p M$. Prove that $J_p$ is independent of the choice of chart about $p$.

(14) Let $S = \mathbb{C}/\Lambda$ be a complex torus and $\pi : \mathbb{C} \to S$ the covering map $z \mapsto z + \Lambda$. We say a vector field $Z \in T_c \mathbb{C}$ is $\Lambda$-periodic if for each $\lambda \in \Lambda$ the translation map $\tau_\lambda : \mathbb{C} \to \mathbb{C}$, $\tau_\lambda(z) = z + \lambda$ has $(\tau_\lambda)_* Z = Z$. Similarly, a 1-form $\omega$ on $\mathbb{C}$ is $\Lambda$-periodic if $\tau_\lambda^* \omega = \omega$ for every $\lambda \in \Lambda$.

(a) Show that $d\pi : T\mathbb{C} \to TS$ gives a bijective correspondence between $T_S$ and $\Lambda$-periodic vector fields on $\mathbb{C}$, and $\pi^*$ identifies 1-forms on $S$ with $\Lambda$-periodic 1-forms on $\mathbb{C}$.

(b) Show that $\partial_z, \partial_{\bar{z}}$ are both $\Lambda$-periodic vector fields on $\mathbb{C}$. Deduce that $T_S = \mathcal{E}_S^0(\partial_z, \partial_{\bar{z}})$, $\Theta_S = \mathcal{C}(\partial_z)$, where we have identified $\partial_z$ with its push-forward $\pi_* \partial_z$.

(c) Show that $dz, d\bar{z}$ are $\Lambda$-periodic, hence $\mathcal{E}_S^1 = \mathcal{E}_S^0(dz, d\bar{z})$, $\Omega_1^S = \mathcal{C}(dz)$, where we have identified $dz, d\bar{z}$ with 1-forms on $S$. Hence $S$ has genus 1.

(d) Let $\omega = fdx + gdy$ be a smooth real differential on $S$, where $z = x + iy$ on $\mathbb{C}$ and we think of $f, g$ as $\Lambda$-periodic smooth real-valued functions on $\mathbb{C}$. Show that $\omega$ is harmonic iff $f, g$ satisfy the Cauchy-Riemann equations and therefore are harmonic conjugates. Conclude that $f, g$ must be constant functions, hence $H^1(S) = \mathcal{C}(dx, dy) = \mathcal{C}(dz, d\bar{z})$.

(15) Use the definitions from the lectures to verify the exterior differentiation formulas: for any functions $f, g$ and 1-form $\omega$,

\[
\begin{align*}
    d(fg) &= gdf + fdg, \\
    d(f\omega) &= df \wedge \omega + fd\omega.
\end{align*}
\]

(16) (a) Let $\varphi : \mathbb{C}_\infty \to \mathbb{C}/\Lambda$ be a holomorphic map from the Riemann sphere to a complex torus. Use the fact that $\varphi^* dz \in \Omega^1_{\mathbb{C}_\infty} = \{0\}$ to show that the tangent map $d\varphi$ vanishes everywhere. Deduce that $\varphi$ is constant.

(b) Now suppose $S$ is a compact Riemann surface of genus $g > 0$, hence $\dim(\Omega_1^S) > 0$. Use a similar argument to the previous part to show that a holomorphic map $\varphi : \mathbb{C}_\infty \to S$ must be constant (you may assume every non-zero $\omega \in \Omega^1_S$ has $\omega_p \neq 0$ at all but finitely many points $p \in S$).

(17) Let $\omega = f dx + g dy$ be a smooth 1-form over the standard 2-simplex $\Delta_2 \subset \mathbb{R}^2$, where $f, g \in \mathcal{E}_\mathbb{R}^0$. Show that Stokes’ theorem is just Green’s theorem in the plane in this case.
(18) Let \( S = C \cup \{ \infty_c \} \) be the one point compactification of the affine elliptic curve
\[
C = \{ (x, y) \in \mathbb{C}^2 : y^2 = x(x-1)(x-a) \}
\]
where the real constant \( a \) has \( a > 1 \). From Q5 we can equip this with an atlas which puts about \( P \in C \) the coordinate \( y \) if \( y(P) = 0 \) and \( x - x(P) \) if \( y(P) \neq 0 \). About the point \( \infty_c \) at infinity we use \( 1/y \) on a sufficiently small neighbourhood. Let \( R_0 = (0,0) \), \( R_1 = (1,0) \) and \( R_2 = (a,0) \) be the unique points on \( S \) for which \( y = 0 \). For any meromorphic function \( f \) on \( S \) let \( (f) \) denote its divisor of zeroes and poles, i.e., the formal sum
\[
(f) = \sum_{P \in S} \nu_P(f)P,
\]
where we recall that \( \nu_f(P) \in \mathbb{Z} \) is the degree of \( f \) at \( P \).

(a) Show that \( (x) = 2R_0 - 2\infty_c \) while \( (y) = R_0 + R_1 + R_2 - 3\infty_c \).

(b) Show that the meromorphic differential \( dx \) has simple zeroes at \( R_1, R_2, R_3 \) and a degree 3 pole at \( \infty_c \).

(c) Conclude that the meromorphic differential \( \frac{1}{y}dx \) is actually holomorphic with neither zeroes nor poles.

(d) Now conclude, using lemma 8.2, that every holomorphic differential on \( S \) is a constant scalar multiple of \( \frac{1}{y}dx \) and therefore \( S \) has genus 1.

(19) Now choose real numbers \( 0 < a_1 < a_2 < \ldots < a_{2g} \) and let \( S = C \cup \{ \infty_c \} \) be the one point compactification of the affine curve
\[
C = \{ (x, y) \in \mathbb{C}^2 : y^2 = x(x-a_1)(x-a_2)\ldots(x-a_{2g}) \}.
\]
As above \( y \) is a local coordinate about each of the points \( R_0, R_1, \ldots, R_{2g} \) at which \( y = 0 \), with \( R_0 = (0,0) \) and \( R_j = (a_j,0) \), while \( x - x(P) \) is a local coordinate about every other point on \( C \) and \( 1/y \) is a local coordinate about \( \infty_c \). It is a fact that every meromorphic function on \( S \) can be written in the form
\[
f(x,y) = \frac{a(x) + by(x)}{c(x) + yh(x)}, \quad a, b, c, h \in \mathbb{C}[x].
\]
(a) Show that \( (x) = 2R_0 - 2\infty_c \) while
\[
(y) = R_0 + R_1 + \ldots + R_{2g} - (2g + 1)\infty_c.
\]
(b) Show that the meromorphic differential \( \frac{1}{y}dx \) has neither zeroes nor poles on \( C \) and has degree \( 2g - 2 \) at \( \infty_c \). Deduce that it is holomorphic when \( g \geq 1 \).

(c) For \( g \geq 1 \) show that if \( \omega \) is a holomorphic differential on \( S \) then
\[
\omega = \frac{p(x)}{y}dx
\]
where \( p(x) \in \mathbb{C}[x] \) is a polynomial of degree \( \deg(p) \leq g - 1 \). Deduce that \( S \) has genus \( g \).

Thus we have examples of compact Riemann surfaces of every genus \( g \). These examples are called hyperelliptic surfaces. In general a Riemann surface is hyperelliptic if it admits a meromorphic function of topological degree 2.

(20) Let \( S \) be the genus 1 surface used in Q18 and \( \Pi_S \) its fundamental domain, which we can view as a quadrilateral in the complex plane.

![Figure 1. Fundamental polygon for an elliptic curve.](image)

Let \( \omega \) be the unique holomorphic differential on \( S \) for which \( \int_a \omega = 1 \). Fix any point \( O \in \Pi_S \) in the interior and define \( f(z) = \int^z_O \omega \) for \( z \in \Pi_S \).

(a) Show that in \( \Pi_S \)

\[
i \int_{\Pi_S} \omega \wedge \bar{\omega} = 2 \int_{\Pi_S} |f'(z)|^2 dx \wedge dy > 0,
\]

where \( dz = dx + idy \).

(b) Show that along any curve \( \gamma(t) \) in \( S \),

\[
\int_{\gamma} \bar{\omega} = \int_{\gamma} \omega.
\]

(c) Use arguments like those in the proof of theorem 5.9 and 5.10 to show that, for \( B = \int_b \omega \),

\[
\int_{\partial \Pi_S} f \bar{\omega} = -2i \text{Im}(B).
\]

Deduce from Stokes’ theorem that the period integral \( B \) has \( \text{Im}(B) > 0 \).
(21) Let \( S \) be compact Riemann surface of genus \( g \) and show that the Hermitian form
\[
(\ ,\ ) : \Omega_S^1 \times \Omega_S^1 \to \mathbb{C}; \quad (\omega, \eta) \mapsto i \int_S \omega \wedge \bar{\eta},
\]
is positive definite. By applying this to a normalised basis \( \omega_1, \ldots, \omega_g \) of \( \Omega_S^1 \) with period integrals \( B_{jk} = \int_{b_j} \omega_k \) show that the period matrix \( B = (B_{jk}) \) is symmetric and with positive definite imaginary part \( \text{Im}(B) = (\text{Im}(B_{jk})) \).

(22) For a compact Riemann surface \( S \) and any two distinct points \( P, Q \in S \) recall that \( \eta_{PQ} \) is the unique meromorphic differential satisfying:

(a) \( \eta_{PQ} \) has poles only at \( P \) and \( Q \) with \( \text{Res}_P \eta_{PQ} = -1 \) and \( \text{Res}_Q \eta_{PQ} = 1 \),

(b) \( \int_{a_j} \eta_{PQ} = 0 \).

By adapting the argument in the proof of theorem 5.9 show that if \( A, B \in S \) are also distinct points then
\[
\int_A^B \eta_{PQ} = \int_P^Q \eta_{AB},
\]
provided the paths of integration lie in the same simply connected open subset of \( S \). This is known classically as the reciprocity formula for differentials of the third kind.

(23) Let \( S \) be the one point compactification of the affine curve
\[
C = \{(z, w) \in \mathbb{C}^2 : w^2 - z(z-1)(z-a) = 0\}, \quad a \neq 0, 1.
\]

From Q18 we know \( \Omega_S^1 = \text{Sp}\{dz/w\} \). Show that for any \( P \in S \) the Abel map
\[
A_P : S \to \text{Jac}(S)
\]
is biholomorphic (it suffices to show that it is a holomorphic bijection).

(24) (Addition on an elliptic curve.) Let \( S \) be the curve in Q23 and \( \mu : S \to S \) the holomorphic involution (i.e., automorphism of order 2) given by \( \mu(z, w) = (z, -w) \) on \( C \) and \( \mu(\infty_c) = \infty_c \). From Q6, \( S \) can be identified with the projective curve
\[
\{[z_0, z_1, z_2] : z_0 z_2^2 - z_1(z_1 - z_0)(z_1 - az_0) = 0\}.
\]
Now let \( \alpha, \beta, \gamma \in \mathbb{C} \) be not all zero.

(a) Show that this projective curve intersects the projective line
\[
\ell = \{[z_0, z_1, z_2] : \alpha z_0 + \beta z_1 + \gamma z_2 = 0\}
\]
on a divisor of degree 3, \( P_1 + P_2 + P_3 \) (not necessarily distinct points).

(b) Show that the the meromorphic function
\[
f : S \to \mathbb{C}, \quad f([z_0, z_1, z_2]) = \alpha + \beta \frac{z_1}{z_0} + \gamma \frac{z_2}{z_0}
\]
has divisor \((f) = P_1 + P_2 + P_3 - 3\infty_c\) where \(\infty_c = [0, 0, 1]\). Assuming \(f\) is not constant, deduce that
\[
\mathcal{A}_\infty(P_1) + \mathcal{A}_\infty(P_2) + \mathcal{A}_\infty(P_3) = 0.
\]

(c) For any point \(P \in S\) write down a meromorphic function \(g\) on \(S\) with divisor \((g) = P + \mu(P) - 2\infty_c\). Deduce that \(\mathcal{A}_\infty(\mu(P)) = -\mathcal{A}_\infty(P)\).

Why does this not imply \(\mathcal{A}_\infty(P) = 0\) for fixed points of \(\mu\)? Deduce that, in the previous part,
\[
\mathcal{A}_\infty(P_1) + \mathcal{A}_\infty(P_2) = \mathcal{A}_\infty(\mu(P_3)).
\]

(d) Now consider a binary operation
\[
*: S \times S \to S; \quad P_1 * P_2 = \mu(P_3)
\]
where, given \(P_1, P_2, P_3\) is the third intersection point of \(S\) with a projective line \(\ell \subset \mathbb{CP}^2\) (interpreted in the sense of divisors). Show that this operation turns \(S\) into an abelian group isomorphic to \(\text{Jac}(S)\).