Higher solutions of Hitchin’s self-duality equations

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Recapitulation

(equivariant) harmonic map $f : X \to M(\pm 1)$

associated family of flat connections

$$\nabla^\lambda = \lambda^{-1}\Phi + \nabla \mp \lambda \Phi^*$$

$\lambda$-connection: scales the $(1,0)$-part

associated families are given by holomorphic sections of the moduli space of $\lambda$-connections on $X$ glued with the moduli space of $\lambda$-connections on $\bar{X}$
Sections of $\mathcal{M}_{DH}$

sections are locally (in $\mathbb{C}^*$) given by families of flat connections

Globally:

- a family of flat connection $\lambda \mapsto + \nabla^\lambda = \lambda^{-1} \Phi + \nabla + \lambda \ldots$ for $\lambda \in \mathbb{C}$;
- a family of flat connection $\lambda \mapsto - \nabla^\lambda = \lambda \Psi + \tilde{\nabla} + \lambda^{-1} \ldots$ for $\lambda \in \mathbb{C}P^1 \setminus \{0\}$;
- on the intersection $\mathbb{C}^*$ we have a (unique up to sign) family of gauge transformations $g(\lambda)$:

$$-\nabla^\lambda = + \nabla^\lambda . g(\lambda)$$

Example

For $\nabla$ a flat unitary connection and $\Phi$ nilpotent Higgs field we get a section via

$$-\nabla^\lambda = \nabla + \lambda^{-2} \Phi.$$
Real sections

A section $s$ is invariant under $\rho$ if and only if

$$+
\nabla - \bar{\lambda}^{-1} = + \nabla^\lambda \cdot g(\lambda).$$

Important property:

$$g(\lambda)g(-\bar{\lambda}^{-1}) = gg^* = \pm \text{Id}$$

▶ in the following we want the negative sign!

Remark

$+ \nabla^\lambda$ provides by complex conjugation a lift over $\mathbb{C}P^1 \setminus \{0\}$. 
The wrong sign: +

joint work with I. Biswas, M. Röser:

- there are real sections which satisfy $gg^* = \text{Id}$;
- they correspond to harmonic maps into the space of oriented circles in the 2-sphere;
- they are counter-examples to Simpson’s question only in a strict sense;

Remark

*Their connected component can be distinguished from the component of the self-duality equations within the framework of $\mathcal{M}_{DH}$.*
Main idea for constructing real sections with — sign

Flows:
- describe evolution of systems;
- can be used to go in desired directions.

Idea: Deform the moduli spaces $\mathcal{M}_{DH}$ and flow the real sections therein.
Babich-Bobenko 93:

- finite gap solutions of the $cosh$ – Gordon equation for odd spectral genus;
- give rise to minimal cylinder in hyperbolic 3-space;
- extend nicely through the boundary at $\infty$;

**Theorem (L. Heller, H.)**

There exists families of flat connections on tori $M$ satisfying $gg^* = -Id$ and giving rise to sections of $\mathcal{M}_{DH}$. They do not correspond to solutions of the self-duality equation globally on $M$. 
Construction of the initial data

- let $\Sigma = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ be a rectangular torus;
- $\rho([\xi]) = [\bar{\xi} + \frac{1+\tau}{2}]$;
- $\lambda: \Sigma \to \mathbb{C}P^1$ be a 2-fold covering s.t. $\lambda \circ \rho = -\bar{\lambda}^{-1}$;

For $X = \mathbb{C}/(2\mathbb{Z} + 2\bar{\tau}\mathbb{Z})$ we get a map into its Jacobian via $\chi$ defined by

$$d\chi = \frac{\pi i}{2\bar{\tau}}(a\rho(\xi - \frac{1+\tau}{2}) + b)d\xi \quad \text{and} \quad \chi(0) = 0$$

for

$$a = -\frac{\tau}{\pi i} \quad \text{and} \quad b = -2\eta_3\pi i$$

where $\eta_3 = \zeta(\frac{\tau}{2})$. With

$$\alpha(\xi) = -\chi(\bar{\xi} - \frac{1+\tau}{2}) - \frac{\pi i}{2\bar{\tau}}$$

we get a family of line bundle connections

$$\xi \mapsto d + \alpha(\xi)dz + \chi(\xi)d\bar{z}$$

which give rise to the family of $\text{SL}(2,\mathbb{C})$-connections on $X$. 
The deformation direction

Instead of working on $X$ given by

$$x^2 = \frac{w^2 - m}{w^2 + 1}$$

(for some $m \in S^1$) we consider the Riemann surface given by

$$x^\beta = \frac{w^2 - m}{w^2 + 1}.$$ 

geometric visualization (via harmonic maps into $S^3$):
Singular connections

consider:

- $\mathbb{Z}_{g+1}$ acts on $X$, $g = g(X)$ odd;
- $\mathbb{Z}_{g+1}$ has 4 fix points of branch order $g$;
- $X/\mathbb{Z}_{g+1}^2 = \text{torus}$;
Singular connections

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- $\mathbb{Z}_{g+1}$ acts on $X$, $g = g(X)$ odd;
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Equivariant flat connections can be described by flat connections on the torus with singularities.
Abelianization

On $\mathbb{C}/(\mathbb{Z} + \tilde{\tau}\mathbb{Z})$ we consider singular flat $SL(2, \mathbb{C})$-connections with local monodromy around $[0]$ in the conjugacy class of

$$\begin{pmatrix}
\exp(2\pi i \gamma) & 0 \\
0 & \exp(-2\pi i \gamma)
\end{pmatrix}.$$ 

Deligne extension gives a holomorphic bundle $V$ with meromorphic connection. Atiyah’s classification of vector bundles over elliptic curves implies:

- $V = L \oplus L^*$ for $L \in \text{Jacobian} \setminus \{\text{spin}\}$;
- or a non-trivial extension of a spin-bundle by itself.

$SL(2, \mathbb{C})$-connections given by

$$d + \begin{pmatrix}
-\chi d\bar{z} + \alpha dz \\
\beta^+ dz
\end{pmatrix} \begin{pmatrix}
\beta^- dz \\
\chi d\bar{z} - \alpha dz
\end{pmatrix}$$

- $\beta^\pm$ meromorphic sections determined by $\chi$ and $\gamma$
Theorem (L.Heller, H.; 2014)

The moduli space of singular flat $\text{SL}(2, \mathbb{C})$-connections is doubly covered by the moduli space of flat line bundles away from spin bundles. The $2 : 1$-correspondence extends in a specific way through spin bundles.
Spectral data for higher genus

Parametrize $\nabla^\lambda$ via spectral data $(\Sigma, \chi, \alpha)$:

- double covering $\Sigma$ of the $\lambda$-plane;
- meromorphic odd map $(\chi, \alpha): \Sigma \to A^1(T^2)$;

Two conditions:

Asymptotic behaviour:
$\chi$ holomorphic, $\alpha$ with first order pole at $\lambda^{-1} \{0\}$;

Reality:
$(\chi, \alpha)$ satisfy a special relation along $\lambda^{-1}(S^1) \subset \Sigma \Rightarrow \alpha$ is determined by $\chi$;

For harmonic maps into $SU(2)$ $(\chi, \alpha)$ coincides with the Narasimhan-Seshadri section along $\lambda^{-1}(S^1) \subset \Sigma$.
Deformation theory

Deformation of local monodromy (via $\gamma$) induces deformation of spectral data:

- consider $\chi + x : \Sigma^0 \to \text{Jac}(T^2)$;
- reality + asymptotic behavior gives rise to a non-linear equation on $x$ (depending on $\gamma$);
- for initial condition on tori: differential invertible;
- application of implicit function theorem yields deformation.

Remark

*For rational $\gamma > 0$ we get sections of $\mathcal{M}_{DH}$ for compact immersed surfaces of genus $g > 1$.***
Details

Spectral curve $\Sigma$:

$$y^2 = \lambda(\lambda - r)(\lambda + \frac{1}{r})$$

real involution covering $\rho$:

$$(y, \lambda) \mapsto (\bar{y}\bar{\lambda}^{-2}, -\bar{\lambda}^{-1})$$

deforestation of $\chi$ by

$$x = y f(\lambda)$$

changes $\alpha$ by

$$-y \lambda^{-2} f(-\bar{\lambda}^{-1})$$
Theorem (L. Heller, H.)

For \( g \gg 1 \) there exists compact Riemann surfaces \( X \) with real sections \( s \) of \( \mathcal{M}_{DH} \) such that \( gg^* = -Id \) and which are not given by solutions of Hitchin’s self-duality equations.

Question

What is the geometric relevance of these sections?
Loop group factorization

\[ LSL_n(\mathbb{C}) = \{ \gamma : S^1 \to \text{SL}(2, \mathbb{C}) \mid \gamma \text{ is smooth} \} \]

\[ LSL_n^+(\mathbb{C}) = \{ \gamma \in LSL_n(\mathbb{C}) \mid \gamma \text{ extends holomorphically to } D \} \]

\[ LSL_n^- , \text{Id}(\mathbb{C}) = \]
\[ \{ \gamma \in LSL_n(\mathbb{C}) \mid \gamma \text{ extends holomorphically to } \mathbb{C}P^1 \setminus \bar{D}; \gamma(\infty) = \text{Id} \} \]

Theorem (Birkhoff)

\[ LSL_n^+(\mathbb{C}) \times LSL_n^- , \text{Id}(\mathbb{C}) \to LSL_n(\mathbb{C}) \]

is a diffeomorphism onto an open dense subset – the so-called big cell.
Relations to self-duality equations

Consider $\nabla^\lambda = \lambda^{-1}\Phi + \nabla + \lambda$... with

$$\nabla^{-\bar{\lambda}^{-1}} = \nabla^\lambda \cdot g(\lambda)$$

and with $gg^* = -\text{Id}$.

Assume, for $x \in U \subset X$, $g_x$ lies in the big cell, then

$$g = hH(h^*)^{-1}$$

on $U$, where $h$ is positive, and $H$ is constant in $\lambda$ with $H\bar{H} = -\text{Id}$.

Then, there is $h_0 : U \rightarrow \text{SL}(2, \mathbb{C})$ with

$$h_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{h}_0^{-1} = H$$

and

$$\nabla^\lambda \cdot (hh_0)$$

is a solution to the self-duality equations on $U$. 
Theorem (L. Heller, H.)

There are Riemann surfaces $X$ of genus $g > 1$ and real sections of $\mathcal{M}_{DH}$ which correspond to a solution of the self-duality equation away from a 1d submanifold $Y$ of $X$. These solutions grow to $\infty$ in first order for $x \to Y$, and give rise to equivariant Willmore surfaces in $S^3$. 

Higher solutions to Hitchin’s self-duality equations
Idea of proof

- on torus $Y$ corresponds to a line;
- smooth dependence on local eigenvalue $\gamma \Rightarrow g$ leaves big cell only along a smooth curve $Y \subset X$;
- use a normal form

\[
\begin{pmatrix}
\lambda^{-1} & p(x) \\
0 & \lambda
\end{pmatrix}
\]

to compute the behavior of the solution when $g$ leaves the big cell $\Leftrightarrow p(x) = 0$;
Open problems

- investigate the moduli space of real sections → smooth, hyper-Kähler, complete?
- geometric interpretation for non-nilpotent Higgs fields?
- is there an algebraic way to go from rank 2 to rank 4?
- when does loop group factorizations work globally, and when not?
- applications in math. physics?
- is it possible to glue solutions?
- can all (smooth) solutions of the self-duality equations be obtained from the complex geometry of $\mathcal{M}_{DH}$?