The harmonic map equation is one amongst many examples of “integrable partial differential equations”, that is, p.d.e. which can be written as “zero curvature equations”. It is a relatively recent example, having been recognised as such (by geometers) only in the 1980’s.

Since there is, as yet, no universally accepted definition of integrability, the obvious question that arises with any “integrable p.d.e.” is: how integrable is it? Classically, the ideal situation was a p.d.e. whose solutions can be represented by concrete formulae involving elementary functions. Of course, this goal is usually neither attainable (except in very special cases) nor desirable (concrete formulae are often too complicated to be useful). A more realistic goal is the concept of a completely integrable Hamiltonian system, but many interesting cases are not of this type either.

The purpose of this article is to review the equations for harmonic maps of finite uniton number, in a manner which demonstrates that these equations are integrable in a very strong sense. In fact the equations are integrable in the most naive sense that all the solutions can be written down explicitly in terms of “known” functions, but we shall not make too much of this fact. We are more interested in describing the solutions in ways likely to lead to new results and geometrical insights. It is this aspect which may be useful in other problems, and which is indicative of the far-reaching relations between integrable systems and geometry.

Since this particular equation is rather easy to describe (and solve), in comparison with other “integrable” equations currently receiving the attention of geometers, it provides a very instructive example. Most of the previous work on harmonic maps of finite uniton number has been published in conventional differential geometric contexts; we shall take a somewhat unorthodox point of view, which may be more palatable to researchers in integrable systems theory. We present this material in §1.

A secondary aim of this article is to discuss the current state of research concerning harmonic maps of finite uniton number. The focus of recent articles in the differential
geometry literature on harmonic maps (from Riemann surfaces into Lie groups) has been on maps which specifically do not have finite uniton number; indeed, harmonic maps of finite uniton number are sometimes dismissed on the vague grounds that such maps have been “done”, for example in [Uh]. We would like to make the point that the latter paper, justly regarded as a milestone in the theory of harmonic maps, was not in fact the end of the story. It would be more accurate to say that it was the beginning, for its main achievement was to provoke geometers into recognizing the zero curvature point of view, thus establishing a very productive link with the world of integrable systems. For the convenience of the reader, in Appendix A we give a brief review of the definitions and basic theory of harmonic maps of finite uniton number, up to and including [Uh].

In §2, we give several specific instances where this “integrable systems approach” leads to concrete results which were either unattainable or left obscure by earlier methods. There are five topics, the first being the most fundamental:

1. Canonical forms of complex extended solutions for arbitrary compact Lie groups
2. A Frenet frame construction for the case $G = U_n$
3. Deformations of harmonic maps
4. Topological properties of complex extended solutions
5. Harmonic maps of finite uniton number into symmetric spaces

Of these, (1) and (2) show how to solve the equation for harmonic maps (of finite uniton number) explicitly, in terms of “holomorphic data”. The method of (1) works for any compact Lie group $G$, but involves integrations of meromorphic functions and hence troublesome residue considerations; that of (2) involves only derivatives, but seems to work only for the case $G = U_n$. Items (3) and (4) give applications to the structure of the space of harmonic maps. Since we restrict ourselves to maps into Lie groups in (1)-(4), item (5) explains how these results extend to the case of maps into symmetric spaces.

To obtain these results, various refinements of the basic zero curvature equation are necessary, as the pioneering concepts of [Uh] (such as uniton factorizations) have turned out to be awkward computational tools. Some of our examples are previously unpublished; in keeping with the expository style of the article we postpone proofs of these new results to Appendices B and C.

The survey article [Do] deals with harmonic maps which are not necessarily of finite uniton number (primarily for maps into the symmetric space $S^2$), but the reader will recognise several common threads. In particular, our canonical form (1) can be regarded as an implementation of the “DPW method” for maps of finite uniton number.

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§1 Solving the harmonic map equation

We shall consider harmonic maps \( \phi : U \rightarrow G \), where \( U \) is a simply connected open subset of a compact Riemann surface \( \Sigma \) and \( G \) is a compact Lie group with a bi-invariant Riemannian metric. It will be convenient to start with the following (local) formulation of the harmonic map equation:

\[
H^{-1}H' \quad \text{is of the form} \quad \frac{1}{\lambda} \times \text{a meromorphic function} \quad U \rightarrow g^C
\]

where \( H : U \times S^1 \rightarrow G^C \) is assumed to be a meromorphic function of \( z \in U \) and a smooth function of \( \lambda \in S^1 \) (the complex numbers of unit length). We are writing \( G^C \) for the complexification of \( G \) here, and \( g^C \) for its Lie algebra (the complexification of the Lie algebra \( g \) of \( G \)). The notation \( H' \) means \( dH/\lambda \). Equation (1.1) imposes a strong condition on \( H \): if we expand \( H \) as a Laurent series in \( \lambda \) then \( H^{-1}H' \) will also have such an expansion, but (1.1) requires that all coefficients except that of \( \lambda^{-1} \) are zero. Thus, (1.1) is a system of differential equations for the coefficients of \( H \), even though the right hand side has not been specified.

In what sense is (1.1) a “zero curvature equation”, and how is it related to other definitions of harmonic maps? In general, if \( \omega \) is a connection form, its curvature form is \( d\omega + \omega \wedge \omega \). The zero curvature equation is the equation \( d\omega + \omega \wedge \omega = 0 \). Locally, this condition is equivalent to the existence of a function \( H \) such that \( \omega = H^{-1}dH \).

(A detailed explanation of this equivalence can be found in [Sh], where it is called “the fundamental theorem of calculus”.) In the language of differential geometry, the columns of the matrix \( H^{-1} \) are a local basis of covariant constant vector fields for the connection \( d + \omega \). An important property is that such vector fields can be found successively, by solving ordinary differential equations.

It is a general principle, supported by many examples, that if \( \omega \) is required to have some particular form, then the zero-curvature equation is equivalent to some partial differential equation (for a function related to \( H \)). Equation (1.1) is of this type, and the p.d.e. in this case turns out to be the well known harmonic map equation of differential geometry. We shall explain this briefly in a moment (and in more detail in Appendix A). Right from the start, however, we wish to emphasize that the above formulation has several advantages, the principal one being this: (1.1) is a system of first order linear meromorphic ordinary differential equations. It is susceptible, in principle, to classical methods of solution.
Moreover, it is a system depending (in a very simple way) on the parameter $\lambda$, and in some cases this makes solving the system very easy.

Consider, for definiteness, the equation $H^{-1}H' = \frac{1}{\lambda}A$ where $A : U \to \mathfrak{g}^\mathbb{C}$ is a given meromorphic function, with $G^\mathbb{C} = GL_n^\mathbb{C}$ (all invertible $n \times n$ complex matrices). Then we seek a "fundamental solution matrix" $H(z, \lambda)$ for

$$(y'_1(z), \ldots, y'_n(z)) = \frac{1}{\lambda}(y_1(z), \ldots, y_n(z))A(z),$$

i.e. a matrix whose rows are $n$ linearly independent solutions of this system.

A formal solution of $H^{-1}H' = \frac{1}{\lambda}A$ of the form

$$H(z, \lambda) = \sum_{i \leq 0} H_i(z)\lambda^i$$

may be obtained immediately by substitution, since the differential equation is equivalent to the equations

$$H'_0 = 0, \quad H'_{-1} = H_0A, \quad H'_{-2} = H_{-1}A, \ldots$$

which may be solved recursively for $H_0, H_{-1}, \ldots$.

Let us assume in addition that $A$ is nilpotent. Then by taking $H_0 = I$ and all constants of integration zero, we obtain a solution $H(z, \lambda)$ which is a polynomial in $\lambda^{-1}$. In particular this formal solution converges, although it is not necessarily meromorphic. When such a solution is meromorphic, it corresponds to a harmonic map $\Sigma \to G$ of finite uniton number, and we claim that all harmonic maps $\Sigma \to G$ of finite uniton number are of this form.

In other words, we claim that the equation for harmonic maps $\Sigma \to G$ of finite uniton number is equivalent to a system of differential equations which is solvable by quadrature — it is integrable in the most naive sense of the word. This statement applies to the case of any compact Lie group $G$, not just $G = U_n$, as we shall explain.

In principle the same method may be used to study arbitrary harmonic maps (not just those of finite uniton number), but with formidable technical difficulties; the general problem is discussed in [Do].

**Example 1.2:** Let us solve the equation $H^{-1}H' = \frac{1}{\lambda}A$ for $H = \sum_{i \leq 0} H_i\lambda^i$, where

$$A = \begin{pmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix}$$

$(u, v, w$ are given meromorphic functions of $z)$. Choosing $H_0 = I$ and all constants of integration zero we obtain

$$H = I + \frac{1}{\lambda} \begin{pmatrix} 0 & f_u & f_v \\ 0 & 0 & f_w \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{\lambda^2} \begin{pmatrix} 0 & 0 & f(wf_u) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad \square$$
Keeping this example in mind, we return now to the general theory, to look at the harmonic map equation in the context of the zero curvature equation and some of the standard machinery of integrable systems.

We define a complex extended solution to be a map $H : U \times S^1 \to G^C$ which is holomorphic for $z \in U$ and smooth for $\lambda \in S^1$, and is such that the Fourier series of $H^{-1}H'$ is of the form

\[(1.3)\quad H^{-1}H' = \sum_{i \geq -1} A_i \lambda^i.\]

In the language of Appendix A, a complex extended solution is simply a $\Lambda G^C$-valued map which represents an extended solution $F : U \to \Omega G$, via the identification $\Omega G = \Lambda G^C/\Lambda_+ G^C$.

Let us briefly review this notation. The based loop group $\Omega G$ is the space of (smooth) loops $\gamma : S^1 \to G$ such that $\gamma(1) = e$, where $e$ is the identity element of $G$. The complex loop group $\Lambda G^C$ is the space of all smooth maps $\gamma : S^1 \to G^C$, and $\Lambda_+ G^C$ is its subgroup consisting of maps such that $\gamma$ extends holomorphically to the unit disk. We refer to [Pr-Se] for further information on loop groups. However, the only fact we really need in this section (and then only in order to explain the relation with harmonic maps) is the above identification and the equivalent statement that $\Lambda G^C = \Omega G \Lambda_+ G^C$ with $\Omega G \cap \Lambda_+ G^C = \{e\}$ (the Iwasawa decomposition). This implies that any loop $\gamma : S^1 \to G^C$ may be factored uniquely as

$$\gamma = \gamma_u \gamma_+,$$

where $\gamma_u \in \Omega G$, $\gamma_+ \in \Lambda_+ G^C$.

The factorization is a generalization of the Gram-Schmidt procedure of linear algebra, the latter being the factorization of an invertible complex matrix $A$ in the form $A = A_u A_+$, where $A_u$ is unitary and $A_+$ is upper triangular.

The connection between harmonic maps and complex extended solutions is as follows. If $H$ is a complex extended solution, we regard it as a map $H : U \to \Lambda G^C$, then define $F = H_u$, i.e. the first factor of $H = H_u H_+$. The map $\phi(z) = F(z, -1)$ is a harmonic map from $U$ to $G$. Conversely, if $\phi : U \to G$ is harmonic, we obtain a corresponding extended solution $F$ as in Appendix A, and hence\(^1\) a complex extended solution $H$.

A complex extended solution $H$ (or any corresponding harmonic map) is said to have finite uniton number if $H$ is a finite Laurent series in $\lambda$, i.e. a polynomial in $\lambda$ and $\lambda^{-1}$. It is a result of [Uh] (see Appendix A) that every harmonic map $S^2 \to G$ has finite uniton number. By analogy with soliton theory, harmonic maps of finite uniton number are sometimes called unitons.

\[^1\]To justify the passage from $F$ to $H$, some technical arguments are needed, and these can be found in [Do-Pe-Wu]. In the case of maps of finite uniton number, however, the existence of $H$ is elementary.
The concept of a complex extended solution originates from several sources. In [Wa], harmonic maps are identified with certain holomorphic vector bundles, and complex extended solutions arise as clutching functions. (These clutching functions provided our original motivation; the vector bundle point of view was developed further in [An].) In [Se], harmonic maps are identified with certain holomorphic maps into an infinite-dimensional Grassmannian, and complex extended solutions arise (see [Gu]) as representatives of such maps with respect to natural coordinate charts. In [Do-Pe-Wu], complex extended solutions\(^2\) arise in the above manner. All these approaches aim to exploit the underlying complex geometry, a point of view suggested by twistor theory.

Equation (1.3) (for complex extended solutions) has several theoretical advantages, in addition to the undeniable practical advantage of being just a system of linear meromorphic ordinary differential equations. We discuss two such aspects next; both of them are fundamental in the theory of integrable systems.

(1) *Wide availability of gauge transformations*

Although there is already some freedom in the choice of extended solution for a given harmonic map, there is much greater freedom in the choice of a complex extended solution:

**Proposition 1.4.** Let \(H\) be a complex extended solution. Let \(M\) be a \(C^\infty\)-valued map which is holomorphic in \(z\) and holomorphic in \(\lambda\) for \(0 \leq |\lambda| \leq 1\). Then the product \(HM\) is a complex extended solution.

*Proof.* It suffices to check that \((HM)^{-1}(HM)' = M^{-1}H^{-1}H'M + M^{-1}M'\) has (at worst) a simple pole at \(\lambda = 0\). This is obvious, since \(H^{-1}H'\) has at worst a simple pole at \(\lambda = 0\), and \(M\) is holomorphic there. \(\square\)

The harmonic map associated to the complex extended solution \(HM\) is exactly the same as the one associated to \(H\), since \((HM)_u = H_u\). Therefore we gain the flexibility of choosing \(M\) to suit our purposes.

**Example 1.5:** Let \(F\) be the extended solution associated to \(H\), i.e. \(F = H_u\) where \(H = H_uH_+\). Then we can say that \(F\) is obtained by applying the gauge transformation \(H \mapsto H(H_+)^{-1} = F\). \(\square\)

**Example 1.6:** Assume that there is a factorization \(H = H_-H_+\) where \(H_+, H_-\) (not necessarily the same as in the previous example) are holomorphic for \(0 \leq |\lambda| \leq 1, 1 \leq 2\).

\(^2\)For the convenience of the reader, we note that our \(F\) is the extended frame \(F\) of [Do], section 2.1, and our \(H\) is the holomorphic extended frame \(C\) of [Do], section 2.3. From now on however we allow \(H\) to be meromorphic, as, for harmonic maps of finite uniton number on a compact Riemann surface, there is no reason to distinguish between holomorphic and meromorphic extended frames. Thus, our \(H\) will serve either as \(C\) (section 2.3) or as \(F_-\) (section 2.4) of [Do]. In particular, a solution of (1.1) is an example of a complex extended solution.
\(|\lambda| \leq \infty\), respectively. Then the gauge transformation \(H \mapsto H(H_+)^{-1} = H_-\) produces a complex extended solution \(H_-\) such that \((H_-)^{-1}(H_-)'\) is linear in \(\lambda^{-1}\). (The latter has a Laurent series with only non-positive powers of \(\lambda\) by definition, but the complex extended solution property means that all terms except those in \(\lambda^0, \lambda^{-1}\) are zero.) Now, it follows from the Birkhoff decomposition of the loop group \(\Lambda G_C\) (see [Pr-Se]) that such a factorization exists, perhaps after translation of \(H\) by a constant loop, providing we allow \(H_-, H_+\) to have poles in \(z\). By multiplying \(H_-\) on the right by \(H_-(z, \infty)^{-1}\), we can assume that \((H_-)^{-1}(H_-)'\) is zero at \(\lambda = \infty\), and hence is of the form \(\lambda^{-1}A(z)\), for some meromorphic map \(A\). Thus we obtain a complex extended solution having the special form of (1.1).

(2) **Evident symmetry groups**

Symmetries are a prominent feature of integrable systems. For complex extended solutions, two symmetry groups arise very naturally:

**Proposition 1.7.** Let \(H\) be a complex extended solution of finite uniton number.

(i) Let \(\alpha \in \mathbb{C}^*\) (a nonzero complex number). Then the map \(\alpha \cdot H(z, \lambda) = H(z, \alpha \lambda)\) is a complex extended solution of finite uniton number.

(ii) Let \(\gamma : S^1 \to G_C\) be a map such that \(\gamma\) and \(\gamma^{-1}\) are finite Laurent series in \(\lambda\). Then the product \(\gamma H\) is a complex extended solution of finite uniton number.

**Proof.** (i) is obvious, and (ii) follows immediately from the formula \((\gamma H)^{-1}(\gamma H)' = H^{-1}H'\).

Thus we obtain actions of the group \(\mathbb{C}^*\) and the “algebraic loop group” \(\Lambda^{\text{alg}} G_C\) (the set of maps \(\gamma\) which satisfy the hypotheses of (ii) above) on the set of complex extended solutions — and hence on the set of harmonic maps — of finite uniton number. While the effect on each of \(H\) and \(H^{-1}H'\) is evident, the effect on the corresponding extended solution \(F = H_u\) and the harmonic map \(\phi(z) = F(z, -1)\) is in general nontrivial. These actions coincide with the “circle action” and the “dressing action” of [Uh], as was shown in [Gu-Oh]. The two actions do not commute, but they combine to give a natural action of a semi-direct product group \(\mathbb{C}^* \ltimes \Lambda^{\text{alg}} G_C\).

Having introduced the main properties of complex extended solutions, we turn to the task of using them to study harmonic maps. The following questions represent some reasonable goals:

**Q1** For a given class of harmonic maps, how are such maps characterized in terms of their complex extended solutions?

Harmonic maps of finite uniton number were originally defined in [Uh] without reference
to complex extended solutions; it follows from [Uh] that the original definition is equivalent to ours. Thus, our definition can be regarded as a characterization of harmonic maps of finite uniton number in terms of their complex extended solutions, i.e. that the complex extended solutions are finite Laurent series in $\lambda$. Another example is the class of harmonic maps arising from the “twistor construction” — the complex extended solutions of such maps can be characterized by a scaling condition (see part (1) of §2).

(Q2) Is there a canonical form amongst the complex extended solutions associated to a given harmonic map?

The large gauge freedom (and symmetry group) suggests the problem of choosing a canonical representative, and hence parametrizing the space of harmonic maps (or symmetry group orbits of harmonic maps) by specific meromorphic functions. We shall in fact do this, later, for harmonic maps of finite uniton number, and it will be necessary to make full use of both gauge transformations and the symmetry group. For the moment, we give two simple examples.

Example 1.8: Let $f : \Sigma \rightarrow \mathbb{C}^n$ be a meromorphic function whose derivatives $f, f', \ldots, f^{(n-1)}$ are linearly independent at almost all points of $\Sigma$. Let

$$H(z, \lambda) = \begin{pmatrix} \lambda^2 & \ldots & \lambda^2 & \lambda & 1 & \ldots & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f^{(n-1)} & \ldots & f' & f \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \lambda^2 & \ldots & \lambda^2 & \lambda & 1 & \ldots & 1 \end{pmatrix} \text{diag}(\lambda^2, \ldots, \lambda^2, \lambda, 1, \ldots, 1)$$

where \text{diag}(\lambda^2, \ldots, \lambda^2, \lambda, 1, \ldots, 1) is the $n \times n$ diagonal matrix with the first $n - i - 1$ entries $\lambda^2$ and the last $i$ entries 1 (with $0 \leq i \leq n - 1$). Then $H$ is a complex extended solution associated to a harmonic map $\Sigma \rightarrow \mathbb{C}P^{n-1}$. It is well known (see Appendix A) that all harmonic maps $S^2 \rightarrow \mathbb{C}P^{n-1}$ arise this way, so the above formula for $H$ may be regarded as a canonical form for such maps.

Example 1.9: Let $p : \Sigma \rightarrow \mathbb{C}$ be a meromorphic function. Let

$$H(z, \lambda) = \exp \left( \frac{1}{\lambda} \begin{pmatrix} 0 & p(z) \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 1 & \frac{1}{\lambda} p(z) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & p(z) \\ 0 & 1 \end{pmatrix} \left( \begin{pmatrix} 1 & \frac{1}{\lambda} \\ 0 & 1 \end{pmatrix} \right).$$

Then $H$ is a complex extended solution associated to the holomorphic map $\Sigma \rightarrow \mathbb{C}P^1$ whose homogeneous coordinate expression is $z \mapsto [p(z); 1]$. This example is taken from [Wa]. It is equivalent to the case $n = 2, i = 0$ of the previous example, and it represents a (different) canonical form for such maps.

(Q3) How do geometrical properties of harmonic maps translate into properties of complex extended solutions?

For differential geometric properties in general, see [Do] and the references therein. An instructive special case is that of holomorphic maps $\Sigma \rightarrow S^2$, i.e. Gauss maps of minimal
surfaces in $\mathbb{R}^3$ (section 3 of [Do]). Other important special cases have been investigated in detail from the point of view of differential geometry in [Br1], [Br2], [Br3], [Hu], [CFW], [Hs]. In (3) and (4) of the next section we shall study a topological property, using complex extended solutions.

To demonstrate the effectiveness of complex extended solutions, we conclude this section by stating a very simple canonical form for complex extended solutions corresponding to arbitrary harmonic maps $\Sigma \to U_n$ of finite uniton number (in particular for arbitrary harmonic maps $S^2 \to U_n$). In essence, this is just a systematic generalization of Example 1.2.

Let $k \in \{0, 1, \ldots, n-1\}$ ($k$ will be the uniton number of the harmonic map, as defined in Appendix A). Let $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ with $k = v_1 \geq v_2 \geq \cdots \geq v_n = 0$ and $v_i - v_{i+1} \geq 0$ or 1 for all $i$. We refer to $v$ as the “type” of the harmonic map. Associated to $v$ there is a flag $F_v$ of subspaces of $\mathbb{C}^n$, and a parabolic subgroup $P_v$ of $GL_n \mathbb{C}$, namely the group of all invertible linear transformations of $\mathbb{C}^n$ which fix $F_v$. The group $GL_n \mathbb{C}$ acts transitively on the space $\Omega_v$ of all flags of type $v$, with isotropy subgroup $P_v$ at $F_v$. It will be convenient to identify $F_v$ with the homomorphism $\gamma_v : S^1 \to U_n, \lambda \mapsto \text{diag}(\lambda^{v_1}, \ldots, \lambda^{v_n})$, and $\Omega_v$ with the conjugacy class of $\gamma_v$. Let the number of occurrences of $\lambda^k, \lambda^{k-1}, \ldots, 1$ in $\text{diag}(\lambda^{v_1}, \ldots, \lambda^{v_n})$ be (respectively) $a_0, a_1, \ldots, a_k$.

Let $p_v$ be the Lie algebra of $P_v$, and let $p^0_v$ be the nil-radical of $p_v$; explicitly, this means that $p_v$ consists of all complex matrices of the form

$$
\begin{pmatrix}
A_{0,0} & A_{0,1} & \ldots & A_{0,k-1} & A_{0,k} \\
0 & A_{1,1} & \ldots & A_{1,k-1} & A_{1,k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & A_{k-1,k-1} & A_{k-1,k} \\
0 & 0 & \ldots & 0 & A_{k,k}
\end{pmatrix}
$$

where $A_{i,j}$ is an $a_i \times a_j$ sub-matrix, and $p^0_v$ consists of all complex matrices of the form

$$
\begin{pmatrix}
0 & A_{0,1} & \ldots & A_{0,k-1} & A_{0,k} \\
0 & 0 & \ldots & A_{1,k-1} & A_{1,k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & A_{k-1,k} \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}
$$

We have the descending central series of $p^0_v$,

$$
p^0_v \supseteq p^1_v \supseteq \cdots \supseteq p^{k-1}_v \supseteq p^k_v = \{0\},
$$

defined by $p^i_v = [p^{i-1}_v, p^0_v]$. The Lie algebra $p^i_v$ consists of all $n \times n$ complex matrices in “block form” $(A_{\alpha,\beta})_{1 \leq \alpha,\beta \leq k}$, with $A_{\alpha,\beta} = 0$ for $\alpha \geq \beta - i$. 

9
For each \( i \in \{1, \ldots, k\} \), let \( B_i \) be a \( p_i \)-valued meromorphic function. Consider the map
\[
H(z, \lambda) = \exp B(z, \lambda) \quad \text{where} \quad B(z, \lambda) = \frac{1}{\lambda} B_1(z) + \frac{1}{\lambda^2} B_2(z) + \cdots + \frac{1}{\lambda^k} B_k(z).
\]

Then \( H^{-1}H' \) is a polynomial in \( \lambda^{-1} \) with no constant term, and \( H \) is a complex extended solution if and only if the only nonzero coefficient in this polynomial is that of \( \lambda^{-1} \) itself. The coefficient of \( \lambda^{-1} \) is evidently \( B'_1 \), so we can say that \( H \) is a complex extended solution if and only if
\[
H^{-1}H' = \frac{1}{\lambda} B'_1.
\]
By the well known formula for the derivative of the exponential map ([He], Chapter 2, Theorem 1.7), the complex extended solution condition is therefore
\[
B'_i - \frac{1}{2!} (\text{ad}\, B) B'_i + \frac{1}{3!} (\text{ad}\, B)^2 B'_i - \frac{1}{4!} (\text{ad}\, B)^3 B'_i + \cdots = \frac{1}{\lambda} B'_1
\]
where \( \text{ad}\, B X \) means \( BX - XB \). This condition is — as expected — a system of meromorphic ordinary differential equations for \( B_1, \ldots, B_k \) which can be integrated recursively.

Indeed, for \( i = 2, \ldots, k \), the equation arising from the coefficient of \( \lambda^i \) expresses \( B'_i \) as a polynomial in terms of \( B_1, \ldots, B_{i-1} \) and \( B'_1, \ldots, B'_{i-1} \). For example, the first equation, for \( i = 2 \), is
\[
B'_2 = \frac{1}{2} (B_1 B'_1 - B'_1 B_1).
\]

Example 1.10: For the group \( U_3 \) and the homomorphism \( \gamma_v(\lambda) = \text{diag}(\lambda^2, \lambda, 1) \), \( B \) is of the form
\[
B = \frac{1}{\lambda} B_1 + \frac{1}{\lambda^2} B_2 = \frac{1}{\lambda} \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{\lambda^2} \begin{pmatrix} 0 & 0 & d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]
where \( a, b, c, d \) are meromorphic functions of \( z \). The complex extended solution condition is
\[
B'_2 = \frac{1}{2} (B_1 B'_1 - B'_1 B_1),
\]
which is just
\[
d' = \frac{1}{2} (ac' - a'c).
\]
This can be solved directly, by choosing \( a, b, c \) arbitrarily and then integrating to obtain \( d \). To compare this complex extended solution with the one in Example 1.2, we compute
\[
H = \exp B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{\lambda^2} \begin{pmatrix} 0 & 0 & d + \frac{1}{2} ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
This is exactly the formula of Example 1.2, with \( u = a', v = b', w = c' \). (Note that the derivative of \( d + \frac{1}{2} ac \) is then \( d' + \frac{1}{2} (a'c + ac') = \frac{1}{2} (ac' - a'c) + \frac{1}{2} (a'c + ac') = ac' = w f u.) \]

It turns out that all harmonic maps \( \Sigma \to U_n \) of finite uniton number arise this way:
Theorem 1.11. (i) Let $H$ be a complex extended solution, corresponding to a harmonic map $\phi : \Sigma \to U_n$ of finite uniton number. Then there exists a gauge transformation (in the sense of Proposition 1.4) which converts $H$ to the above canonical form $\exp B$, for some $v$ and some meromorphic $B_1, \ldots, B_k$.

(ii) Conversely, let $B_1 : \Sigma \to p^0_v$ be any meromorphic function. Let $B_2, \ldots, B_k$ be functions obtained by solving recursively the system of equations obtained from equating coefficients of powers of $\lambda$ in

$$B' - \frac{1}{2!}(\text{ad} B)B' + \frac{1}{3!}(\text{ad} B)^2B' - \frac{1}{4!}(\text{ad} B)^3B' + \ldots = \frac{1}{\lambda} B'_1.$$

If $B_2, \ldots, B_k$ are meromorphic, then $B$ defines a complex extended solution $H = \exp B$ (corresponding to a harmonic map $\phi : \Sigma \to U_n$ of finite uniton number).

(iii) The effect of changing the constants of integration in (ii) is to change $H$ by the (dressing) action of the group $\Lambda^{alb}G^C$ in the sense of Proposition 1.7 (ii).

Proof. (i) is a special case of the results of [Bu-Gu]; we shall review this in the next section. (ii) is immediate from the construction above. (iii) follows from two facts. First, changing the constants of integration does not change $H - 1H'$. Second, $H^{-1}H'$ determines $H$ up to multiplication on the left by a map $\gamma : S^1 \to G^C$, i.e. up to the dressing action. □

The above construction may be regarded as an Ansatz which, when used in (1.1), happens to produce all harmonic maps of finite uniton number. We shall explain (geometrically) why it works in the next section. From the point of view of integrable systems, it is notable that the canonical form (with all constants of integration taken to be zero) can be achieved by a combination of gauge transformations and dressing transformations.

It is important to note that the passage from the complex extended solution $H$ to the corresponding extended solution $F = H_u$ and harmonic map $\phi = F|_{\lambda=-1}$ is a purely (real) algebraic operation. This is because the factorization $\gamma = \gamma_u \gamma_+$ of an algebraic loop $\gamma$ is a purely algebraic operation. Geometrically, the factorization is an infinite-dimensional version of the Gram-Schmidt orthogonalization procedure, but when $\gamma$ is algebraic the procedure takes place in a finite-dimensional vector space (this can be seen from the Grassmannian model of the loop group $\Omega U_n$).

Finally, we remark that the construction shows that the number of meromorphic functions $\Sigma \to C$ required to produce a harmonic map $\Sigma \to U_n$ of type $v$ is (at most) the complex dimension of the vector space $p^0_v$. We have therefore obtained a very precise and complete solution of the harmonic map equation in this situation. The only blemish is the fact that arbitrary meromorphic functions are not allowed, only those which lead to meromorphic functions in the integration procedure described above. (For example, given a meromorphic $B_1$, the equation $B'_2 = \frac{1}{2}(\text{ad} B_1)B'_1$ gives rise to a meromorphic $B_2$ if and
only if \((\text{ad } B_1)B_1'\) has no periods or residues.) This blemish will be removed, in the case of the unitary group, in (2) of the next section.

\[\text{§2 Explanations, applications, generalizations}\]

(1) Canonical forms of complex extended solutions for arbitrary compact Lie groups

Theorem 1.11 has a rather surprising origin; surprising, that is, from the point of view of differential geometry, but one that is typical of the integrable systems approach. This is explained in [Bu-Gu], where a canonical form was given for complex extended solutions \(H : \Sigma \times S^1 \to G^C\) of finite uniton number, for any compact Lie group \(G\). To avoid introducing further Lie algebraic notation, we shall just discuss the case \(G = U_n\) here.

The key ingredient is the loop group \(\Omega U_n\). This has a well known Morse-theoretic decomposition into stable manifolds with respect to the energy functional \(E : \Omega U_n \to \mathbb{R}\), \(\gamma \mapsto \int_{S^1} |d\gamma|^2\). The critical points of \(E\) are the geodesic loops, i.e. the homomorphisms. The connected critical manifolds are the conjugacy classes \(\Omega_v\) of the homomorphisms \(\gamma_v(\lambda) = \text{diag}(\lambda^{v_1}, \ldots, \lambda^{v_n})\) with \(v_1 \geq \cdots \geq v_n\). Although \(\Omega U_n\) and each stable manifold is infinite-dimensional, it turns out that each unstable manifold \(U_v\) is finite-dimensional (with the structure of a vector bundle of rank \(r(v)\) over \(\Omega_v\), where \(r(v)\) is the index of the geodesic \(\gamma_v\)). The union of these unstable manifolds is known ([Pr]) to be the algebraic loop group \(\Omega \text{alg} U_n\), a proper subset of \(\Omega U_n\). Without loss of generality ([Uh], [Se]), for the purpose of studying harmonic maps of finite uniton number, it suffices to consider extended solutions \(F : \Sigma \to \Omega \text{alg} U_n\).

Since \(F\) is a holomorphic map of a Riemann surface, its image must be contained in the closure of a single unstable manifold \(U_v\), for some particular \(v\). The main result of [Bu-Gu] is that it suffices to consider only a finite number of such \(v\), namely those satisfying the additional conditions \(v_i - v_{i+1} = 0\) or \(1\) for all \(i\), and \(v_n = 0\). A proof of this fact was given in Chapter 20 of [Gu], using the Grassmannian model of \(\Omega U_n\), whereas the proof of [Bu-Gu] uses the loop group \(\Omega G\) directly, and works for any \(G\).

Since each \(v_i - v_{i+1}\) can be 0 or 1, there are \(2^{n-1}\) possible “types” of harmonic maps, and in fact they all occur, i.e. no further reduction is possible. For \(n = 3\) the types are \((2, 1, 0), (1, 1, 0), (1, 0, 0),\) and \((0, 0, 0)\). The first case gives the harmonic maps of Example 1.10. The second and third types correspond to holomorphic maps into \(Gr_2(C^3)\) and \(CP^2\), respectively, the latter spaces being regarded as totally geodesic submanifolds of \(U_3\). The fourth type corresponds to constant maps.

Now, the structure of the unstable manifold \(U_v\) is well known, and in particular it has a “big cell” consisting of loops of the form

\[
[\exp (P_0 + \lambda P_1 + \cdots + \lambda^{k-1} P_{k-1})] \cdot \gamma_v
\]
where $P_i \in \mathfrak{p}_i^v$, and where the dot indicates the action $\epsilon \cdot \delta = (\epsilon \delta)_u$ of $\Lambda G^C$ on $\Omega U_n$. The canonical form arises by expressing the complex extended solution equation in the “new coordinates” $P_0, \ldots, P_{k-1}$, as follows. For each $i \in \{0, 1, \ldots, k-1\}$, let $C_i$ be a $\mathfrak{p}_i^v$-valued meromorphic function. Consider the map

$$H(z, \lambda) = \left[ \exp C(z, \lambda) \right] \gamma_v(\lambda)$$

where $C(z, \lambda) = C_0(z) + \lambda C_1(z) + \cdots + \lambda^{k-1} C_{k-1}(z)$.

Using the formula for the derivative of the exponential map, it is easy to verify that $H$ is a complex extended solution if and only if, for each $i = 0, 1, \ldots, k-2$, the coefficient of $\lambda^i$ in

$$\sum_{n \geq 0} \frac{(-1)^n}{(n+1)!} (\text{ad} \ C)^n C'$$

has zero component in $\mathfrak{p}_i^{i+1}$. We obtain a system of meromorphic differential equations which can be solved recursively for $C_0, \ldots, C_{k-1}$. So far this appears somewhat different from the canonical form $\exp B$ given at the end of the last section; however, one has

$$\exp B = \gamma_v^{-1} \exp C\gamma_v = \exp \gamma_v^{-1} C\gamma_v,$$

i.e. the two versions differ only by a “trivial” dressing transformation.

All this is valid for maps into a general compact Lie group $G$. More precisely, Theorem 1.11 is valid for such groups, providing the Lie algebras $\mathfrak{p}_i^v$ are defined Lie algebraically as in [Bu-Gu]. Thus, we have a method of constructing all harmonic maps of finite uniton number into any compact Lie group. An immediate consequence is an upper bound on the (minimal) uniton number. For $G = U_n$ it is $n - 1$; for general $G$ the upper bound can be expressed root-theoretically, and the results for the simple groups are as follows:

<table>
<thead>
<tr>
<th>$G$</th>
<th>upper bound on uniton number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU_n$</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$SO_{2n+1}$</td>
<td>$2n - 1$</td>
</tr>
<tr>
<td>$Sp_{2n}$</td>
<td>$2n - 1$</td>
</tr>
<tr>
<td>$SO_{2n}$</td>
<td>$2n - 3$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>5</td>
</tr>
<tr>
<td>$F_4$</td>
<td>11</td>
</tr>
<tr>
<td>$E_6$</td>
<td>11</td>
</tr>
<tr>
<td>$E_7$</td>
<td>17</td>
</tr>
<tr>
<td>$E_8$</td>
<td>29</td>
</tr>
</tbody>
</table>

Another consequence of the canonical form (and the algebraic nature$^3$ of the Iwasawa decomposition) is the fact that all harmonic maps $\Sigma \to G$ of finite uniton number are

$^3$The algebraic nature of the factorization can be seen from the Grassmannian model in the case $G = U_n$; in the general case it follows by taking a faithful unitary representation of the compact Lie group $G$. 

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(real) algebraic functions of meromorphic functions on $\Sigma$. This answers a question raised at the end of [Wo], where a construction of harmonic maps $S^2 \to U_n$ was distilled from the unitor factorization of [Uh].

It should be noted, incidentally, that the above construction provides a very satisfactory implementation of the “DPW method” (introduced in [Do-Pe-Wu] and surveyed in [Do]) for the case of harmonic maps of finite uniton number. The DPW meromorphic potential (or normalized potential, in the terminology of [Do], Remark 2.4.4) is simply $B'_i dz$.

There are special families of harmonic maps of finite uniton number, studied by many authors, including the maps sometimes known as superminimal or isotropic, which have been given a general formulation via the “twistor construction” in [Bu-Ra], [Br3] and [Sa]. These maps have a simple characterization in terms of extended solutions: they correspond to extended solutions $F$ which are $S^1$-invariant, where $S^1$ acts on the loop group $\Omega G$ by $\alpha \cdot \gamma(\lambda) = \gamma(\alpha \lambda) \gamma(\alpha)^{-1}$. Geometrically, this means that the image of $F$ lies in a conjugacy class of homomorphisms (a connected component of the set of critical points of the function $\gamma \mapsto \int_{S^1} |d\gamma|^2$), i.e. a generalized flag manifold of the group $\Gamma$. In the case $\Gamma = U_n$, it means that $F$ takes values in $\Omega v$ for some $v$. The corresponding harmonic map $\phi$ always takes values in a symmetric space of $\Gamma$ (embedded totally geodesically in $\Gamma$). The relation between $F$ and $\phi$ is particularly simple in this situation: $\phi$ is obtained by composing $F$ with a natural projection from the generalized flag manifold to the symmetric space. Such projection maps are called (generalized) twistor fibrations. (The twistor fibration $CP^3 \to S^4$ of Yang-Mills theory is an example; here $\Gamma = SO_5$ or $Sp_2$.) Harmonic maps of this type were originally studied without reference to loop groups, although the loop group formulation provides a unifying framework.

Let us consider now the harmonic maps whose extended solutions take values in the conjugacy class $\Omega$ of a particular homomorphism $\gamma \in \Omega G$. The complex extended solutions $H$ of such maps are characterized by the following scaling condition:

$$H(z, \alpha \lambda) = H(z, \lambda) \gamma(\alpha) \quad \text{for all } \alpha \in S^1.$$  

This fact is an immediate consequence of the canonical form $H(z, \lambda) = [\exp C(z)] \gamma(\lambda)$. Here $C$ is a meromorphic function taking values in the nil-radical of the parabolic subalgebra determined by $\gamma$, i.e. in the “big cell” of the generalized flag manifold $\Omega$. (In other words, we have $C_1 = \cdots = C_{k-1} = 0$ and $C = C_0$.) A feature of this scaling condition is that it distinguishes the particular twistor fibration. The dressing action of Proposition 1.7 (ii) preserves the generalized flag manifold $\Omega$; in fact it reduces to the natural action of $G^C$ on this space.

The complex extended solution equation for harmonic maps of this special type can be integrated directly, exactly as in the case of general maps of finite uniton number. Indeed, this was done by Bryant in [Br3], long before the introduction of extended solutions, at least in the special case of uniton number 2. Several authors have studied the meromorphic
(2) A Frenet frame construction\textsuperscript{4} for the case $G = U_n$

The canonical form of (1), together with the “integrable” meromorphic o.d.e. for its coefficient functions, shows that harmonic maps $\Sigma \rightarrow G$ of finite uniton number are parametrized by collections of meromorphic functions on $\Sigma$. However, it is not clear from this description which meromorphic functions are disallowed on the grounds that they give rise to logarithms during the integration process.

It turns out that, in the case $G = U_n$, this difficulty may be avoided: there is an alternative parametrization of (complex extended solutions corresponding to) harmonic maps $\Sigma \rightarrow U_n$ of finite uniton number, in which the initial data consists of arbitrary meromorphic functions and their derivatives; no integrations are needed. One way to obtain this new parametrization is to make a change of variable in the meromorphic o.d.e. for complex extended solutions:

\textit{Example 2.1:} Consider the o.d.e. $d' - \frac{1}{2}(ac' - a'c) = 0$ for complex extended solutions of the type $G = U_3, v = (2, 1, 0)$ (Example 1.10). Instead of choosing $a, b, c$ and integrating to obtain $d$, let us introduce new variables

$$
\alpha = d + \frac{1}{2}ac, \quad \beta = c, \quad \gamma = a, \quad \delta = b.
$$

Then the o.d.e. simplifies to $\alpha' = \beta'\gamma$. \textit{This may be solved without integration:} choose meromorphic functions $\alpha, \beta, \delta$, then obtain $\gamma$ as $\gamma = \alpha'/\beta'$. Thus, all harmonic maps of this type may be constructed by choosing as initial data three meromorphic functions $\alpha, \beta, \delta$ and then performing a series of derivatives and algebraic operations. \hfill $\square$

It can be shown that the same phenomenon occurs for complex extended solutions of type $v$ for $U_n$, in general. We shall not prove this here, because there is a more geometrical, though less economical, version of the parametrization, in which the initial data consists of a collection of meromorphic maps $\Sigma \rightarrow \mathbb{C}^n$. This is in the spirit of the Eells-Wood description of harmonic maps into $\mathbb{C}P^{n-1}$ (see Theorem A.1 of Appendix A), and our result can be viewed as the natural generalization to the case where the target manifold is $U_n$. A similar result holds for $Gr_k(\mathbb{C}^n)$.

The appropriate context for this version — and this is why our argument is restricted to the case $G = U_n$ — is the Grassmannian model $Gr^{(n)} = \Omega U_n$. As explained in Appendix A, the extended solution equation for a holomorphic map $W: \Sigma \rightarrow Gr^{(n)}$ is

\begin{equation}
(2.2) \quad \lambda W' \subseteq W.
\end{equation}

\textsuperscript{4}The results in (2) have been obtained in joint work with Francis Burstall.
For each \( z \in \Sigma \), \( W(z) \) is a linear subspace of the Hilbert space

\[
H^{(n)} = L^2(S^1, C^n) = \bigoplus_{i \in \mathbb{Z}} \lambda^i C^n.
\]

However, for maps of uniton number \( k \), it suffices (by [Se]) to consider the case

\[
\lambda^k H_+^{(n)} \subseteq W \subseteq H_+^{(n)}
\]

where

\[
H_+^{(n)} = \bigoplus_{i \geq 0} \lambda^i C^n.
\]

Thus, \( W \) is a holomorphic map to a finite-dimensional Grassmannian manifold. Although this means that the original Hilbert space \( H^{(n)} \) will play no essential role, we continue to use it for notational convenience; the reader should bear in mind that we always work in the finite-dimensional vector space \( H_+^{(n)}/\lambda^k H_+^{(n)} \cong \bigoplus_{i=0}^{k-1} \lambda^i C^n \cong C^{kn} \). If \( W \) is spanned locally by \( C^{kn} \)-valued holomorphic functions \( s_1, \ldots, s_r \) (mod \( \lambda^k H_+^{(n)} \)), we write \( W = [s_1] + \cdots + [s_r] + \lambda^k H_+^{(n)} \). With this convention, the notation \( W' \) means

\[
W' = [s_1]' + \cdots + [s_r]' = [s_1] + \cdots + [s_r] + [s_1'] + \cdots + [s_r'] + \lambda^k H_+^{(n)}.
\]

Equation (2.2) imposes a nontrivial condition on the holomorphic map \( W \), but all such maps may be generated from arbitrary holomorphic maps in a simple way:

**Proposition 2.3.** Let \( X : \Sigma \to Gr_s(\bigoplus_{i=0}^{k-1} \lambda^i C^n) \) be any holomorphic map. Define \( W \) by

\[
W = X + \lambda X' + \lambda^2 X'' + \cdots + \lambda^{k-1} X^{(k-1)} + \lambda^k H_+^{(n)}.
\]

Then \( W \) is a solution of (2.2), and all solutions \( W : \Sigma \to Gr^{(n)} \) (of uniton number \( k \)) of (2.2) arise this way.

**Proof.** Given \( X \), it is obvious that \( W \) satisfies (2.2). Conversely, given any solution \( W \) of (2.2) with \( \lambda^k H_+^{(n)} \subseteq W \subseteq H_+^{(n)} \) (see above), we may take any \( X \) such that \( W = X + \lambda^k H_+^{(n)} \). \( \square \)

As it stands, this result is quite trivial, although it demonstrates already that harmonic maps of finite uniton number may be constructed from “unconstrained” holomorphic data using only derivatives and algebraic operations. It is of more interest to know how to choose holomorphic data for a particular type of solution, perhaps. As a concrete example, we shall do this below for the cases \( G = U_3 \) and \( U_4 \).
We need the following additional notation. Any holomorphic map $X : \Sigma \to Gr_s(\oplus_{i=0}^{k-1} \lambda^i C^n)$ may be written in the form

$X = X_0 + \lambda X_1 + \lambda^2 X_2 + \cdots + \lambda^{k-1} X_{k-1}$,

where $X_i : \Sigma \to Gr_{s_i}(\oplus_{j=0}^{k-i-1} \lambda^j C^n)$ for some $s_i$; we may assume that the image of $X_i$ is not contained in $Gr_{s_i}(\oplus_{j=0}^{k-i-2} \lambda^j C^n)$. (In other words, we take an echelon form of $X$ with respect to the flag given by the subspaces $\oplus_{j=0}^{k-i-1} \lambda^j C^n$.)

We shall specify suitable maps $X_0, X_1, \ldots$ for each type of (non-constant) harmonic map of finite uniton number into $U_3$ and $U_4$ in the tables below. In each case, $l, m, \ldots : \Sigma \to C^n$ are arbitrary vector-valued meromorphic functions. We regard these functions as “Frenet frame data”, analogous to the $C^n$-valued meromorphic function $f$ in Theorem A.1 of Appendix A.

**Frenet frame data for maps $\Sigma \to U_3$:**

| $(2, 1, 0)$ | $X_0 = [l + \lambda m]$ | $X_1 = 0$ |
| $(1, 1, 0)$ | $X_0 = [l]$ | |
| $(1, 0, 0)$ | $X_0 = [l] + [m]$ | |

**Frenet frame data for maps $\Sigma \to U_4$:**

| $(3, 2, 1, 0)$ | $X_0 = [l + \lambda m + \lambda^2 n]$ | $X_1 = 0$ | $X_2 = 0$ |
| $(2, 2, 1, 0)$ | $X_0 = [l + \lambda m]$ | $X_1 = 0$ | |
| $(2, 1, 1, 0)$ | $X_0 = [l + \lambda m]$ | $X_1 = [n]$ | |
| $(2, 1, 0, 0)$ | $X_0 = [l + \lambda m] + [l' + \lambda n]$ | $X_1 = 0$ | |
| $(1, 1, 1, 0)$ | $X_0 = [l]$ | | |
| $(1, 1, 0, 0)$ | $X_0 = [l] + [m]$ | | |
| $(1, 0, 0, 0)$ | $X_0 = [l] + [m] + [n]$ | | |

**Theorem 2.4.** All harmonic maps $\Sigma \to U_n$ of finite uniton number, for $n = 3$ or $4$, arise through the above construction. That is, for any choice of $C^n$-valued meromorphic functions $l, m, \ldots$, the map $W = X + \lambda X' + \lambda^2 X'' + \cdots + \lambda^{k-1} X^{(k-1)} + \lambda^k H^{(n)}$ (where $X = X_0 + \lambda X_1 + \lambda^2 X_2 + \cdots + \lambda^{k-1} X_{k-1}$) is an extended solution corresponding to a harmonic map $\Sigma \to U_n$ of finite uniton number, and all such maps arise this way.

We relegate the proof, which is an explicit computation based on the canonical form given in part (1), to Appendix B. The proof shows that generic choices of $l, m, \ldots$ produce extended solutions of the indicated type, while for certain special choices an extended solution of simpler type may be obtained. It also shows that $l, m, \ldots$ are expressable in terms of the data of the canonical form via a change of variable like that in Example 2.1.
The same method works for $U_n$ in general. As the definitions of $X_0, X_1, \ldots$ in the general case are increasingly complicated (and non-canonical), we just remark that the number of $\mathbb{C}^n$-valued meromorphic functions $l, m, \ldots$ needed to construct a harmonic map $\Sigma \to U_n$ of finite uniton number is at most $n - 1$. To see this, let the number of occurrences of $\lambda^k, \lambda^{k-1}, \ldots, 1$ in $\gamma_v(\lambda) = \text{diag}(\lambda^{v_1}, \ldots, \lambda^{v_n})$ be (respectively) $a_0, a_1, \ldots, a_k$. An upper bound on the number of $\mathbb{C}^n$-valued functions required can be computed by considering the “worst case” $a_k \geq a_{k-1} \geq \cdots \geq a_0$. The number in this case is at most

$$a_0 k + (a_1 - a_0)(k - 1) + \cdots + (a_k - a_{k-1})0 = a_0 + a_1 + \cdots + a_{k-1} = n - a_k \leq n - 1$$

as claimed. Whether the method can be extended to other compact Lie groups $G$ is an open problem.

Let us try to make an honest assessment of the various methods introduced so far. First, the method of (1) gives a solution in terms of the minimum number of meromorphic functions $\Sigma \to \mathbb{C}$, i.e. the nonzero component functions of $B_1$, and this holds for any compact Lie group $G$. However, one has to exclude meromorphic functions which produce nonzero residues in the integration process (and it does not seem to be easy to characterize such functions). For $G = U_n$, the change of variables avoids this problem, and allows us to express the solution in terms of unconstrained holomorphic data, but specifying the change of variable needed for each type of harmonic map is complicated. Finally, the method of Theorem 2.4 gives a more systematic solution using unconstrained initial data, again in the case $G = U_n$, but this data is certainly not minimal. For example, six meromorphic functions (the components of $l$ and $m$) are needed for maps into $U_3$ of type $(2, 1, 0)$, whereas we know from Example 2.1 that four are enough. This kind of problem was visible already in Examples 1.8 and 1.9, in the description of holomorphic maps $S^2 \to S^2$. It seems, in conclusion, that what is the “best” method depends on what one is trying to do.

(3) Deformations of harmonic maps

“Dressing actions” of (infinite-dimensional) Lie groups exist on the solution spaces of many well known “integrable” partial differential equations. A dressing action produces many new solutions from a given solution, and suggests the problem of describing the orbits of the action. In the case of harmonic maps of finite uniton number (in contrast to examples like the KdV equation), neither of these leads to significant new results. The reason is that the orbits of the dressing action are too small (and too numerous): they are finite-dimensional. On the other hand, the dressing action may be used to “move solutions around”, and it turns out that this is sufficient to obtain some basic global results on spaces of harmonic maps.

In the general context of zero curvature equations, the dressing action is defined using a factorization of matrix-valued functions (the type of factorization depending on the particular problem). It is usually very difficult to carry out such factorizations explicitly. In the case of harmonic maps of finite uniton number, a drastic simplification is possible.
(see [Gu-Oh]): the dressing action on harmonic maps $\Sigma \to G$ is equivalent to the natural action of the complex loop group $\Lambda_+ G^\mathbb{C}$ on the target space $\Omega G = \Lambda G^\mathbb{C}/\Lambda_+ G^\mathbb{C}$ of the corresponding extended solutions $F : \Sigma \to \Omega G$. In terms of complex extended solutions, this is the action defined earlier in Proposition 1.7 (ii). If we restrict attention to harmonic maps of uniton number at most $k$, then the dressing action reduces, roughly speaking, to the action of elements of $\Lambda_+ G^\mathbb{C}$ of the form $\sum_{i=0}^{k} A_i \lambda^i$ (assuming a suitable matrix representation of $G$). This shows that the dressing orbits are finite-dimensional.

Further reductions are possible in the case of special types of maps. For example, in the case of harmonic maps associated to $S^1$-invariant extended solutions, the dressing action reduces to the natural action of the finite-dimensional Lie group $G^\mathbb{C}$ on the “twistor space” $\Omega$. In [Gu-Oh], this concrete realization of the dressing action was used to enumerate the connected components of the space Harm($S^2$, $G/K$) of harmonic maps, for $G/K = S^n$ and $G/K = \mathbb{CP}^n$.

The method involves continuously deforming a general harmonic map, through a family of harmonic maps, to one of simpler type. In each case a suitable deformation is obtained by choosing a one parameter subgroup of $G^\mathbb{C}$ and applying it (via the dressing action) to an extended solution. In order to understand the effects of such deformations, it is very helpful to interpret the action of a one parameter subgroup of $G^\mathbb{C}$ on the generalized flag manifold $\Omega$ as the flow of a Morse-Bott function.

A similar method was used in [Fu-Gu-Ko-Oh] to compute the fundamental group of Harm($S^2$, $S^n$). Results on connected components of $S^1$-invariant extended solutions for other $G/K$ have been given in [Mu1], [Mu2]. (The special feature of the cases $G/K = S^n$ or $\mathbb{CP}^n$, however, is that all harmonic maps $S^2 \to G/K$ correspond to $S^1$-invariant extended solutions.) Dong ([Du]) used this method to obtain an estimate relating the energy and the uniton number of a harmonic map $S^2 \to U_n$.

For harmonic maps which do not necessarily correspond to $S^1$-invariant extended solutions, the same method of Morse-theoretic deformations is available, working now in the loop group $\Omega G$, but these do not seem powerful enough to determine the connected components of Harm($S^2$, $G/K$) or Harm($S^2$, $G$). We shall give a method in (4) for the case Harm($S^2$, $Gr_k(\mathbb{C}^n)$) or Harm($S^2$, $U_n$), which makes use of the canonical form for complex extended solutions.

Finally we note that the space of harmonic maps of finite uniton number and fixed energy, which (from the canonical form) is an algebraic variety, has singularities in general. However, for $n = 1, 2$, (each connected component of) the space Harm($S^2$, $\mathbb{CP}^n$) is actually a smooth manifold. For $n = 1$ this is elementary, and for $n = 2$ the manifold structure was established and studied in [Cr], [Le-Wo1], [Le-Wo2].

(4) Topological properties of complex extended solutions
As in (2) we shall restrict attention to the group $G = U_n$, in order to make use of the Grassmannian model $Gr^{(n)} = \Omega U_n$ and the corresponding version $\lambda W' \subseteq W$ of the harmonic map equation. The great advantage of this formulation, for harmonic maps of finite uniton number, is that $W : \Sigma \rightarrow Gr_r(H^{(n)}_+ / \lambda^k H^{(n)}_+)$ is a holomorphic map to a finite-dimensional Grassmannian. We shall also restrict attention to the case $\Sigma = S^2$. The homotopy class $[W]$ of $W$ is then a non-negative integer, and it is well known that this is (a constant multiple of) the energy of the corresponding harmonic map. We shall use complex extended solutions to prove Theorem 2.9 below, that any two harmonic maps with the same energy may be connected by a continuous path in the space of harmonic maps.

We begin with some remarks on general holomorphic maps $V : S^2 \rightarrow Gr_k(C^n)$. Using the Schubert decomposition of $Gr_k(C^n)$, and the fact that $S^2$ is a one-dimensional complex algebraic variety, we may write $V = HE$ for some $k$-plane $E$, where $H : S^2 - \Gamma \rightarrow GL_n \mathbb{C}$ is a holomorphic map defined on the complement of a finite set $\Gamma$. Moreover, we may write $H = \exp B$ where $B : S^2 - \Gamma \rightarrow \mathfrak{n}$ is a holomorphic map into some nilpotent Lie subalgebra $\mathfrak{n}$ of $\mathfrak{gl}_n \mathbb{C}$. (The affine space $(\exp \mathfrak{n})E$ is a Schubert cell in $Gr_k(C^n)$, biholomorphically equivalent to $\mathfrak{n}$ itself, and the image of $V|_{S^2 - \Gamma}$ lies in this cell.) The map $B$ is globally defined on $S^2$ if and only if it (and hence $V$) is constant. How, then, does (the locally defined) $B$ reflect the (global) invariant $|V|$? The answer is provided by classical Schubert calculus:

**Proposition 2.5.** Let $Z$ be a linear subspace of $C^n$ of codimension $k$. Then $|V|$ is the number of points $z \in S^2$ (counted with multiplicities) such that $\dim V(z) \cap Z \geq 1$, whenever this number is finite.

If there exist points $z_1, \ldots, z_d$ in the domain $S^2 - \Gamma$ of $B$, such that $d = |V|$ and $\dim (H(z_i)E) \cap Z \geq 1$ for all $i$, then we must have \{ $z_1, \ldots, z_d$ \} = \{ $z \in S^2$ | $\dim V(z) \cap Z \geq 1$ \} (assuming that the latter set is finite). By general position arguments, it is clear that, for any $B$, there exist $Z$ and $z_1, \ldots, z_d$ with these properties. In this sense, $B$ carries sufficient information to determine $|V|$. We shall use later on the following partial converse of this statement. For any $V$ and any $Z$, it is possible to deform $V$ continuously to a holomorphic map $\tilde{V} = HE$ such that there exist $\tilde{z}_1, \ldots, \tilde{z}_d$ with \{ $\tilde{z}_1, \ldots, \tilde{z}_d$ \} = \{ $z \in S^2$ | $\dim \tilde{V}(z) \cap Z \geq 1$ \}, and \{ $\tilde{z}_1, \ldots, \tilde{z}_d$ \} $\subseteq S^2 - \Gamma$.

**Example 2.6:** In the case $k = 1$ and $E = \{(x, 0, \ldots, 0) \mid x \in \mathbb{C}\} = \mathbb{C}$, a holomorphic map $V = HE$ is the same thing as a map $V : S^2 \rightarrow \mathbb{C}P^{n-1}$ which can be written in homogeneous coordinates as $V(z) = [1; a_1(z), \ldots, a_{n-1}(z)]$, for some rational functions $a_1, \ldots, a_{n-1}$. In this case the finite set $\Gamma$ is the union of the poles of $a_1, \ldots, a_{n-1}$. Let us choose the codimension one subspace $Z = \{(x_1, \ldots, x_{n-1}, 0) \mid x_1, \ldots, x_{n-1} \in \mathbb{C}\}$. Then, assuming that $a_{n-1}$ is not identically zero, the homotopy class of $V$ is measured by the zeros of $a_{n-1}$ together with certain poles of $a_1, \ldots, a_{n-1}$. By a continuous deformation of
$a_{n-1}$ — and this applies even in the case where $a_{n-1}$ is identically zero — we may deform $V$ to $\tilde{V}$ in such a way that the homotopy class of $\tilde{V}$ is measured exactly by the zeros of $\tilde{a}_{n-1}$. □

Now we return to the case of an extended solution $W$. Let

$$W = (\exp C)\gamma_v H_+^{(n)} = (A_0 + \lambda A_1 + \cdots + \lambda^{k-1} A_{k-1}) \gamma_v H_+^{(n)}$$

be the canonical form from (1). We have

$$\gamma_v H_+^{(n)} = E_0 \oplus \lambda (E_0 \oplus E_1) \oplus \cdots \oplus \lambda^{k-1} (E_0 \oplus \cdots \oplus E_{k-1}) \oplus \lambda^k H_+^{(n)}$$

where $C^n = E_k \oplus E_{k-1} \oplus \cdots \oplus E_0$ is the common eigenspace decomposition of the matrices $\gamma_v(\lambda)$ (for all $\lambda$); thus $\dim E_i = a_i$. Let

$$X_0 = (\exp C)\gamma_v E_0 = (A_0 + \lambda A_1 + \cdots + \lambda^{k-1} A_{k-1}) E_0.$$ 

This defines a holomorphic map $X_0 : S^2 \to Gr_{a_0}(C^{kn})$, which can be taken as the first of a sequence $X_0, X_1, \ldots, X_{k-1}$ producing $W$ in the manner of Theorem 2.4.

Since $X_0$ is part of the unconstrained holomorphic data defining $W$, any deformation of $X_0$ through holomorphic maps gives rise to a deformation of $W$ through extended solutions. In other words, given a family $X_0^t$ with $0 \leq t \leq 1$, we define $W_t = X_t + \lambda X_1 + \lambda^2 X_2 + \cdots + \lambda^{k-1} X_{k-1} + \lambda^k H_+^{(n)}$, where $X_t = X_0^t + \lambda X_1 + \lambda^2 X_2 + \cdots + \lambda^{k-1} X_{k-1}$. (It turns out that we shall not need to deform $X_1, \ldots, X_{k-1}$.) Our objective is to find a deformation $X_0^t$ which gives rise to a continuous deformation $W_t$ of $W$, with the additional property that $W_1$ has lower uniton number than $W_0 (= W)$. We shall obtain such a deformation by deforming continuously the (coefficients of the) component functions of $H$. To verify the continuity, we need the following proposition and its proof.

**Proposition 2.7.** $|W| = |X_0|$. 

A proof is given in Appendix C.

We can use this in the following way to determine when the deformation $X_0^t$ gives rise to a continuous deformation of $W$. Since we consider only deformations obtained by changing the coefficient functions continuously, any discontinuity will force $|W_1| < |W_0|$. (This may be seen by regarding $W_0$ as a holomorphic map $S^2 \to CP^N$ for some large $N$; for maps into projective space the assertion is clear.) Therefore, if $|W_t|$ remains constant, the deformation must be continuous. Let $z_1, \ldots, z_d$ be the points where the first entry of the last column of $A_{k-1}$ vanishes (without loss of generality we may assume these points are in the domain of $H$, and, by the proof of Proposition 2.7, we may assume $d = |W| = |X_0|$). If these points are unchanged during the deformation, then we must have $|W_t| \geq |W_0|$ for all $t$, hence $|W_t| = |W|$, and hence any deformation of this type is continuous. Our strategy will be to deform $X_0$ by deforming only $A_0 E_0$ (and we will assume $k \geq 2$, so that $A_0 \neq A_{k-1}$). Then $z_1, \ldots, z_d$ will indeed remain unchanged.
Proposition 2.8. Assume $k \geq 2$. Then there exist (a sequence of) continuous deformations of the above type such that the resulting extended solution has uniton number less than $k$.

A proof is given in Appendix C.

It follows that all non-constant extended solutions can be deformed continuously to extended solutions of uniton number 1. The latter correspond to holomorphic maps into Grassmannian submanifolds of $U_n$, and (using the method of (3) above) it is easy to show that any two such may be connected by a continuous path, if they have the same energy. Let $\text{Harm}_c(S^2, U_n)$ be the space of harmonic maps from $S^2$ to $U_n$ which have energy $e$. (We may assume that the energy has been normalized so that $e = |W|$, where $W$ is any corresponding extended solution.) Then the above method gives:

Theorem 2.9. For any $e \geq 0$, and any $n \geq 1$, $\text{Harm}_c(S^2, U_n)$ is path-connected.

In Appendix C, in addition to the omitted proofs, we shall give a much more explicit method for the case $n = 3$. We remark that Anand ([An]) has obtained independently this and other results on the space $\text{Harm}_c(S^2, U_3)$ for low values of $e$, by using the vector bundle approach.

(5) Harmonic maps of finite uniton number into symmetric spaces

If $G/K$ is a symmetric space of $G$, harmonic maps $\Sigma \to G/K$ may be considered as special examples of harmonic maps $\Sigma \to G$, by the remarks in Appendix A. To what extent can the results of (1)-(4) be adapted to such maps? A partial answer to this question is that all the above results can be extended to the case of inner symmetric spaces, i.e. where the involution of the symmetric space is an inner automorphism of $G$.

A canonical form for the case of maps into an inner symmetric space $G/K$ was given in [Bu-Gu]; this is a straightforward modification of (1). By making use of this, the Frenet frame construction of (2) may be extended to the case of maps into any inner symmetric space associated with $G = U_n$, i.e. any Grassmannian $G/K = \text{Gr}_k(C^n)$. The Morse-theoretic deformations of (3) were used primarily in the case of maps into symmetric spaces in the first place, so no further comment is needed here. Finally, in the case of (4), one can prove the analogue of Theorem 2.9 for the space $\text{Harm}(S^2, \text{Gr}_k(C^n))$, again by studying the associated complex extended solutions. That is, if $\text{Harm}_{d,e}(S^2, \text{Gr}_k(C^n))$ is the space of harmonic maps from $S^2$ to $\text{Gr}_k(C^n)$ which have energy $e$ and homotopy class $d$, then this space is path-connected. The method of proof of Theorem 2.9 permits a reduction to the case $k = 1$, where the result is given by combining [Cr] and [Gu-Oh].

The situation for general (compact) non-inner symmetric spaces has not been investigated. Since any such space admits a totally geodesic Cartan embedding into a corresponding compact Lie group, the situation is in principle covered by the results for Lie
groups. However, a more precise analysis would be desirable.

**APPENDIX A: REVIEW OF HARMONIC MAPS, FROM FRENET FRAMES TO THE ZERO CURVATURE EQUATION**

These brief historical comments provide the differential geometric context for the zero curvature equation method, and go as far as the 1989 articles of Uhlenbeck ([Uh]) and Segal ([Se]) which we take as our starting point in §1. There are at least two deficiencies in our exposition. One is that we ignore a vast amount of exploratory work in differential geometry which, while not central to our story, was instrumental in shaping attitudes to research on harmonic maps from Riemann surfaces to Lie groups or homogeneous spaces. For this we refer to the comprehensive survey articles [Ee-Le1], [Ee-Le2]. The other is that we omit references to previous work in integrable systems which provided the foundation for [Uh] and [Se].

Harmonic maps from the two-sphere $S^2$ to the complex Grassmannian manifold $Gr_k(C^n)$ or the unitary group $U_n$ are “model examples” for two important problems: (1) Minimal immersions of (compact) Riemann surfaces into symmetric spaces (differential geometry), and (2) The nonlinear $\sigma$-model or chiral model (mathematical physics). The case $G/K = Gr_k(C^n)$ is closely related to the case $G = U_n$, so we shall concentrate on the latter. Remarks on general Lie groups $G$ and general symmetric spaces $G/K$, as well as general Riemann surfaces $\Sigma$ in the role of the domain of $\phi$, will appear later on.

The harmonic map equation, for maps $\phi : S^2 \to U_n$, is the second order partial differential equation

$$(\phi^{-1}\phi_\bar{z})_\bar{z} + (\phi^{-1}\phi_z)_{\bar{z}} = 0$$

where $z$ is the usual local coordinate on $S^2 = C \cup \infty = CP^1$. It is the Euler-Lagrange equation for the functional $\int ||d\phi||^2$, and is therefore a very natural generalization of the geodesic equation in Riemannian geometry. Harmonic maps $S^2 \to Gr_k(C^n)$ constitute a special case, in the following sense. The manifold $Gr_k(C^n)$ can be embedded totally geodesically in $U_n$ by sending a complex $k$-plane $V$ to the unitary linear transformation $\pi_V - \pi_V^\perp$, where $\pi_V : C^n \to C^n$ is orthogonal projection onto $V$ (with respect to usual the Hermitian inner product of $C^n$), and a map $\phi : S^2 \to Gr_k(C^n)$ is harmonic if and only if the composition $S^2 \to Gr_k(C^n) \to U_n$ is harmonic.

The above equation admits many solutions, the simplest (non-constant) ones being the holomorphic maps $S^2 \to Gr_k(C^n)$. The space $\text{Hol}(S^2, Gr_k(C^n))$ of all such holomorphic maps has connected components $\text{Hol}_d(S^2, Gr_k(C^n))$ indexed by $d = 0, 1, 2, \ldots$, where $d$ is the homotopy class $[\phi] \in \pi_2 Gr_k(C^n)$. Each connected component has the structure of a complex manifold of dimension $nd + k(n - k)$ (noncompact if $d \geq 1$), which has been thoroughly studied from the point of view of algebraic topology. The case $k = 1, n = 2$
(hence $Gr_k(C^n) = S^2$) is particularly simple, since holomorphic maps $S^2 \to S^2$ are just the same as rational functions of the complex variable $z$.

In contrast, the larger class of (not necessarily holomorphic) harmonic maps is much less easy to describe. The first major result was for $k = 1$, i.e. harmonic maps $S^2 \to CP^n$, and in the related case of harmonic maps $S^2 \to S^m$. From the point of view of differential geometry, this originated from work of Calabi and of Chern on the Cartan moving frame of a minimal immersion, and was achieved by several people simultaneously in the late 1970’s; the following version is that of Eells and Wood ([Ee-Wo]). Let $[f] : S^2 \to CP^{n-1}$ be any holomorphic map, and let $i \in \{0, 1, \ldots, n-1\}$. Define $\phi : S^2 \to CP^{n-1}$ by

\[ (*) \quad \phi(z) = ([f(z)] + [f'(z)] + \cdots + [f^{(i)}(z)]) \ominus ([f(z)] + [f'(z)] + \cdots + [f^{(n-1)}(z)]). \]

Here we regard $f, f', \ldots$ as $C^n$-valued rational functions, and $[f(z)] + [f'(z)] + \cdots$ means the vector space spanned by the lines $[f(z)], [f'(z)], \ldots$. The notation $A \ominus B$ denotes the (Hermitian) orthogonal complement of $B$ in $A$. Formula ($*$) defines a $CP^{n-1}$-valued function which is smooth except at a finite number of singular points; it can be shown that the singularities are removable. (If any of the $C^n$-valued functions appearing here are identically zero, an obvious modification of the construction produces maps into lower dimensional complex projective spaces.) The result is:

**Theorem A.1.** The map $\phi$ defined by formula ($*$) is harmonic, and all harmonic maps $S^2 \to CP^{n-1}$ arise this way.

This is a very satisfactory solution to the problem: all harmonic maps $S^2 \to CP^{n-1}$ are obtained by performing a series of algebraic operations and derivatives on rational functions. Rational functions are “known” objects and the operations are mechanical (in particular, programmable on a computer), so the solution is as explicit as one could hope for. The computation of the orthogonal complement is somewhat messy, of course, and so the formula for the homogeneous components of $\phi$ (in terms of the components of $f$) may be unpleasant to look at. But this is not a defect of the method; on the contrary, it tells us that the formula ($*$) is the heart of the matter.

The simplicity of this result has been both a blessing and a curse. It inspired great efforts to find generalizations to other situations, and it demonstrated important themes in mathematical physics such as twistor theory and the role of instanton solutions of field equations. On the other hand, generalizations to the case of other target spaces were frustratingly slow in coming. For several important manifolds $M$, such as Grassmannians, quadrics, real and quaternionic projective spaces, various results pointed to a description of all harmonic maps $S^2 \to M$ in terms of “holomorphic data”, but always lacked the simplicity of the $CP^{n-1}$ case. Surveys of these purely differential geometric developments can be found in [Ee-Le1], [Ee-Le2].

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The next major progress came from the theory of integrable systems. It had been known for some time that the harmonic map equation, like several other important equations, could be written as a zero curvature equation. To do this, one introduces the matrix-valued complex 1-form
\[ \omega = \frac{1}{2} \left( \frac{1}{\lambda} \right) \phi^{-1} \phi_z dz + \frac{1}{2} (1 - \lambda) \phi^{-1} \phi_z d\bar{z} \]
where \( \lambda \) is a complex parameter. Then an easy computation shows that the harmonic map equation is equivalent to the condition
\[ d\omega + \omega \wedge \omega = 0 \text{ for all } \lambda. \]
(This is the condition that the curvature tensor of the connection \( d + \omega \) is identically zero.) We refer to [Uh], [Se], [Hi-Se-Wa] for background information on this geometrically mysterious formulation.

From this point of view, the first significant advance was made by Uhlenbeck in [Uh], where the concept of “extended solution” was introduced. An extended solution associated to a harmonic map \( \phi : S^2 \to U_n \) is any map \( F : S^2 \times \mathbb{C}^* \to U_n \) such that \( \omega = F^{-1} dF \).

The existence of such a map \( F \), smooth in the variable \( z \in S^2 \) and holomorphic in the variable \( \lambda \in \mathbb{C}^* = \mathbb{C} - \{0\} \), is an elementary fact (see, for example, [Sh]). Conversely, if \( F \) is a smooth map with the properties
\[ F^{-1} F_z = \frac{1}{2} (1 - \frac{1}{\lambda}) A \]
\[ F^{-1} F_{\bar{z}} = \frac{1}{2} (1 - \lambda) B \]
where \( A, B \) are smooth \( n \times n \) matrix-valued functions of \( z \), then the map \( \phi(z) = F(z, -1) \) satisfies the zero curvature equation and is therefore harmonic.

This reformulation of the harmonic map equation has two specific advantages. First, by restricting \( F \) to the circle \( |\lambda| = 1 \), we may regard \( F \) as a map from \( S^2 \) into the (based) loop group \( \Omega U_n = \{ \gamma : S^1 \to U_n \mid \gamma(1) = I \} \), and the second condition implies that this \( F : S^2 \to \Omega U_n \) is holomorphic with respect to the standard complex structures (see [Pr-Se] for the complex structure of \( \Omega U_n \)). Thus, the harmonic condition is immediately translated to holomorphic one, and from now on one works entirely with holomorphic objects. The second advantage is based on the (at first sight inconvenient) fact that the correspondence between \( \phi \) and \( F \) is not one to one. If \( F \) is an extended solution associated to \( \phi \), then \( \gamma F \) is obviously an extended solution associated to \( \gamma(-1) \phi \), and it is another elementary fact that this is essentially the only ambiguity. The freedom to pre-multiply \( F \) by a loop \( \gamma \) may be used to seek a canonical representative associated to a given harmonic map \( \phi \). Uhlenbeck used this freedom to prove (in [Uh]) the following statement:
Theorem A.2. For any harmonic map $\phi : S^2 \to U_n$, there exists an associated extended solution $F$ of the form $F(z, \lambda) = \sum_{i=0}^{k} A_i(z) \lambda^i$, with $k \leq n - 1$.

The least such (non-negative) integer $k$ is called the (minimal) uniton number of $\phi$. This is a measure of the complexity of $\phi$; for example, constant maps have uniton number zero, and holomorphic maps $S^2 \to Gr_k(C^n)(\to U_n)$ have uniton number one.

A second result of [Uh] was the existence of a multiplicative “uniton factorization” of $F$ into $k$ factors of the form $\pi V(z) + \lambda \pi V(z)^\perp$, where each $V$ is a (smooth) map into a complex Grassmannian. When $k = 1$ there is only one factor and the map $V$ is holomorphic. When $k \geq 2$ the maps $V$ are in some sense holomorphic with respect to perturbations of the standard complex structure, where the perturbation depends on the previous factors. Roughly speaking, this means that a harmonic map of uniton number $k$ can be factored as a product of $k$ holomorphic maps (but, because of the perturbations, this statement is not strictly true, which reduces the usefulness of the factorization as a practical tool).

A heuristic explanation of this factorization was given in [Uh], by using the “dressing” procedure of integrable systems theory. This procedure generally works for zero curvature equations which involve a condition on the type and location of the poles of $\omega$. (In the case of the harmonic map equation, there are two simple poles, at $\lambda = 0$ and $\lambda = \infty$.) Given a solution $F$, one obtains under certain conditions a new solution of the form $FX$, where $X$ is the solution of an associated “Riemann-Hilbert” problem for a contour consisting of two small circles around the poles. We omit further details as we use a different manifestation of the dressing procedure in this article; the version of [Uh] and its relation to the uniton factorization is discussed in more detail in [Be-Gu].

The treatment of Uhlenbeck is analytic, deriving finiteness of the uniton number from ellipticity of the harmonic map equation and compactness of $S^2$. A different version was given by Segal in [Se], using the Grassmannian model of the loop group $\Omega U_n$. Without going into details, we remark that the loop group $\Omega U_n$ may be identified with an infinite-dimensional Grassmannian manifold $Gr^{(n)}$, which is a subspace of the Grassmannian of all linear subspaces of the Hilbert space $H^{(n)} = L^2(S^1, C^n)$ (see [Pr-Se]), and the conditions that a (smooth) map $W : S^2 \to Gr^{(n)}$ corresponds to an extended solution $F : S^2 \to \Omega U_n$ are

$$W_z \subseteq W$$

$$W_{\bar{z}} \subseteq \lambda W.$$

The first condition means that the vector $\partial s(z)/\partial z$ is contained in the subspace $\lambda^{-1} W(z)$ of $H^{(n)}$, for every (smooth) map $s : S^2 \to H^{(n)}$ such that $s(z) \in W(z)$. The second condition is interpreted in a similar way; it is equivalent to saying that the map $W$ is holomorphic. Segal’s version of the finiteness of the uniton number is that the map $W$
(corresponding to the original harmonic map \( \phi : S^2 \to U_n \)) may be chosen so that its image lies in the “algebraic” Grassmannian \( G_{\text{alg}}^{(n)} \). An elementary treatment of the loop group and Grassmannian model approach to harmonic maps may be found in [Gu].

Finally, a brief comment on the situation for maps \( \Sigma \to G \) (or \( G/K \)), where \( \Sigma \) is an arbitrary compact Riemann surface and \( G \) is an arbitrary compact Lie group (or \( G/K \) a symmetric space). The harmonic map equation makes sense here, with \( z \) interpreted as a local coordinate on \( \Sigma \), and \( f^{-1}df \) interpreted as the pullback by \( f \) of the Maurer-Cartan form on \( G \), as does the definition of extended solution. An extended solution gives rise to a harmonic map in the above manner, but there is no guarantee that a harmonic map \( \Sigma \to G \) gives rise to an extended solution \( \Sigma \to \Omega G \), when \( \Sigma \) is not simply connected (i.e. when \( \Sigma \) has positive genus). The class of harmonic maps which admit extended solutions \( \Sigma \to \Omega G \) is precisely the class of harmonic maps of finite uniton number.

**Appendix B: Proof of Theorem 2.4**

We begin with extended solutions of type \((2,1,0)\). From Examples 1.10 and 2.1 we obtain

\[
W = \exp C \text{diag}(\lambda^2, \lambda, 1) H_+^{(n)} = \begin{bmatrix} 1 & \gamma & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 & \delta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{diag}(\lambda^2, \lambda, 1) H_+^{(n)}.
\]

Now, \( \text{diag}(\lambda^2, \lambda, 1) H_+^{(n)} = V_3 \oplus \lambda V_3 \oplus \lambda V_2 \oplus \lambda^2 H_+^{(n)} \), where \( C^3 = V_1 \oplus V_2 \oplus V_3 \) is the decomposition given by the directions of the standard basis vectors. If we introduce

\[
l = \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix}, \quad m = \begin{pmatrix} \delta \\ 0 \\ 0 \end{pmatrix}, \quad n = \begin{pmatrix} \gamma \\ 0 \\ 1 \end{pmatrix},
\]

then \( W \) is the span of the three vector-valued functions \( l + \lambda m, \lambda(l + \lambda m), \lambda n \) (mod \( \lambda^2 H_+^{(n)} \)). The equation \( \gamma = \alpha'/\beta' \) (the complex extended solution equation) implies that \( [l'] = [n] \). Hence:

\[
W = [l + \lambda m] + \lambda[l + \lambda m] + \lambda[l'] + \lambda^2 H_+^{(n)}
\]

\[
= [l + \lambda m] + \lambda[l + \lambda m] + \lambda[l' + \lambda m'] + \lambda^2 H_+^{(n)}
\]

\[
= [l + \lambda m] + \lambda[l + \lambda m]' + \lambda^2 H_+^{(n)}.
\]

This establishes the formula of Theorem 2.4 for harmonic maps of type \((2,1,0)\). For types \((1,1,0)\) and \((1,0,0)\) the statements of Theorem 2.4 are obvious.
In the case of $U_4$, the arguments for type $(3, 2, 1, 0)$ and $(2, 2, 1, 0)$ are exactly like that for $(2, 1, 0)$ above, so let us consider type $(2, 1, 1, 0)$. We may write

$$
\exp C = \begin{pmatrix}
1 & \gamma & \beta & \alpha_1 \\
0 & 1 & 0 & \alpha_2 \\
0 & 0 & 1 & \alpha_3 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} + \lambda \begin{pmatrix}
0 & 0 & 0 & \delta \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

(thus defining the change of variable for this situation), in terms of which the complex extended solution equation is

$$\alpha'_1 = \gamma \alpha'_2 + \beta \alpha'_3.$$

Let us define

$$l = \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
1 \\
\end{pmatrix}, \quad m = \begin{pmatrix}
\delta \\
0 \\
0 \\
0 \\
\end{pmatrix}, \quad p = \begin{pmatrix}
\beta \\
0 \\
1 \\
0 \\
\end{pmatrix}, \quad q = \begin{pmatrix}
\gamma \\
1 \\
0 \\
0 \\
\end{pmatrix}.$$

Since $\text{diag}(\lambda^2, \lambda, 1)H^{(n)}_+ = V_4 \oplus \lambda V_4 \oplus \lambda V_2 \oplus \lambda^2 H^{(n)}_+$, we have

$$W = [l + \lambda m] + \lambda[l + \lambda m] + \lambda[p] + \lambda[q] + \lambda^2 H^{(n)}_+.$$

The complex extended solution equation implies $[l'] \subseteq [p] + [q]$, so

$$W = [l + \lambda m] + \lambda[l + \lambda m]' + \lambda[n] + \lambda^2 H^{(n)}_+$$

where $[n]$ is any complementary (one-dimensional) subspace to $[l']$ in $[p] + [q]$. If we put $X_0 = [l + \lambda m]$ and $X_1 = [n]$, and $X = X_0 + \lambda X_1$, then $W = X + \lambda X' + \lambda^2 H^{(n)}_+$, as required.

The next case to consider is $(2, 1, 0, 0)$. We begin as usual with the canonical form, which is

$$
\exp C = \begin{pmatrix}
1 & \gamma & \beta_1 & \alpha_1 \\
0 & 1 & \beta_2 & \alpha_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} + \lambda \begin{pmatrix}
0 & 0 & \epsilon & \delta \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
$$

Then we express $W$ in terms of

$$l_1 = \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
0 \\
1 \\
\end{pmatrix}, \quad l_2 = \begin{pmatrix}
\delta \\
0 \\
0 \\
0 \\
\end{pmatrix}, \quad m_1 = \begin{pmatrix}
\beta_1 \\
\beta_2 \\
1 \\
0 \\
\end{pmatrix}, \quad m_2 = \begin{pmatrix}
\epsilon \\
0 \\
0 \\
0 \\
\end{pmatrix}, \quad r = \begin{pmatrix}
\gamma \\
1 \\
0 \\
0 \\
\end{pmatrix}.$$
The complex extended solution equations are \( \alpha' = \delta \alpha' \), \( \beta' = \delta \beta' \), which imply \([l'_1] = [m'_1] = [r] \).

At this point we do not appear to have unconstrained holomorphic data, because the four vector-valued functions \( l_1, l_2, m_1, m_2 \) are required to satisfy the condition \([l'_1] = [m'_1] \). There are two ways of dealing with this. The first is to observe that \( X = X_0 = [l_1 + \lambda l_2] + [m_1 + \lambda m_2] \) always generates a solution \( W = \lambda W' + \lambda^2 H^{(n)} \) of the equation \( \lambda W' \subseteq W \), which coincides with one of type \((2, 1, 0, 0)\) in the particular case \([l'_1] = [m'_1] \).

The second point of view is to use the fact that two vector-valued functions \( l_1, m_1 \) satisfy \( \dim[l'_1]' + [m'_1]' = 3 \) if and only if there exists some \( l \) such that \( [l] \subseteq [l_1] + [m_1], [l]' = [l_1] + [m_1] \), and \([l]' = [l_1]' + [m_1]' \). Using this we can rewrite the above formula for \( W \) as

\[
W = [l_1 + \lambda l_2] + \lambda [l_1 + \lambda l_2] + [m_1 + \lambda m_2] + \lambda [m_1 + \lambda m_2] + \lambda [r] + \lambda^2 H^{(n)} \\
= [l_1 + \lambda l_2] + \lambda [l_1] + [m_1 + \lambda m_2] + \lambda [m_1] + \lambda [r] + \lambda^2 H^{(n)} \\
= [l_1 + \lambda l_2] + [m_1 + \lambda m_2] + \lambda ([l_1] + [m_1] + [r]) + \lambda^2 H^{(n)} \\
= [al + bl' + \lambda l_2] + [cl + dl' + \lambda m_2] + \lambda ([l] + [l'] + [l'']) + \lambda^2 H^{(n)}
\]

(writing \( l_1 = al + bl' \) and \( m_1 = cl + dl' \) where \( a, b, c, d \) are meromorphic functions). This can be written in the form \([l + \lambda m] + [l' + \lambda n] + \lambda ([l] + [l'] + [l'']) + \lambda^2 H^{(n)}\) as stated in Theorem 2.4.

For the remaining types \((1, 1, 1, 0), (1, 1, 0, 0)\) and \((1, 0, 0, 0)\), the statements of Theorem 2.4 are obvious.

**Appendix C: Proof of Propositions 2.7 and 2.8**

**Proof of Proposition 2.7.** We regard \( W \) as a holomorphic map \( S^2 \rightarrow Gr_r(\mathbb{C}^{kn}) \), where

\[
r = \dim \ E_0 \oplus \lambda E_0 \oplus E_1 \oplus \cdots \oplus \lambda^{k-1}(E_0 \oplus \cdots \oplus E_{k-1}) \\
= a_0 + (a_0 + a_1) + \cdots + (a_0 + \cdots + a_{k-1}).
\]

To measure \(|W|\) (using Proposition 2.5) we need a linear subspace \( Z \) of codimension \( r \) in \( \mathbb{C}^{kn} \). We choose

\[
Z = (E_k \oplus \cdots \oplus E_1) \oplus V_n \oplus \lambda(E_k \oplus \cdots \oplus E_2) \oplus \cdots \oplus \lambda^{k-2}(E_k \oplus E_{k-1}) \oplus \lambda^{k-1}(E_k \oplus V_1)
\]
where (as usual) $V_1, \ldots, V_n$ are the standard basis directions in $\mathbb{C}^n$ (so $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_n$ and $V_n \subseteq E_0, V_1 \subseteq E_k$). On the other hand, $X_0$ defines a holomorphic map $S^2 \to Gr_{a_0}(\mathbb{C}^{kn})$, and in fact a holomorphic map into the Grassmannian of $a_0$-dimensional subspaces of

$$\mathbb{C}^n \oplus \lambda(E_2 \oplus \cdots \oplus E_2) \oplus \lambda^2(E_3 \oplus \cdots \oplus E_3) \oplus \cdots \oplus \lambda^{k-1} E_k.$$ 

Now, $Z$ is a subspace of this as well, and it has codimension $a_0$. So $Z$ may be used to measure $|X_0|$. The proof of the proposition is completed by noting that, by construction, $\dim W(z) \cap Z \geq 1$ if and only if $\dim X_0(z) \cap Z \geq 1$, since both conditions are equivalent to the vanishing of the first entry of the last column of the matrix $A_{k-1}$. □

**Proof of Proposition 2.8.** We take a deformation of the form described in (4) of §2 which renders $A_0 E_0$ constant. For the resulting extended solution we claim that

1. $\lambda^k H^{(n)}_+ + \lambda^{k-1} A_0 E_0 \subseteq W$, and
2. $W \subseteq H^{(n)}_+ \ominus (A_0 E_0)^\perp$

(where the orthogonal complement is taken in $\mathbb{C}^n$). To establish these, we shall use the formula

$$W = (A_0 + \cdots + \lambda^{k-1}A_{k-1})(E_0 \oplus \lambda(E_0 \oplus E_1) \oplus \cdots \oplus \lambda^{k-1}(E_0 \oplus \cdots \oplus E_{k-1}) \oplus \lambda^k H^{(n)}_+).$$

It follows that $W$ contains

$$(A_0 + \cdots + \lambda^{k-1}A_{k-1})(\lambda^{k-1}(E_0 \oplus \cdots \oplus E_{k-1}) \oplus \lambda^k H^{(n)}_+) = \lambda^{k-1} A_0(E_0 \oplus \cdots \oplus E_{k-1}) \oplus \lambda^k H^{(n)}_+.$$

Since $A_0 E_0$ is constant, we obtain (1). Similarly, by considering $W + \lambda H^{(n)}_+$, we see that $W$ is orthogonal to $(A_0 E_0)^\perp$, and this is (2). Any $W$ satisfying (1) and (2) must have unpton number less than $k$ (after translation by a constant loop), since $\dim[(H^{(n)}_+ \ominus (A_0 E_0)^\perp)/[\lambda^k H^{(n)}_+ \oplus \lambda^{k-1} A_0 E_0] = (k-1)n$. □

As a concrete illustration we return to the case $n = 3$, where the underlying geometry is more transparent. We have seen in (1) of §2 that there are three types of non-constant harmonic map $S^2 \to U_3$, namely $(2, 1, 0), (1, 1, 0)$, and $(1, 0, 0)$. The last two correspond to holomorphic maps from $S^2$ into totally geodesic submanifolds $Gr_2(\mathbb{C}^3)$ and $\mathbb{C}P^2$ of $U_3$, and (as stated earlier) all such maps are in the same path-component of $\text{Harm}_c(S^2, U_3)$. The proof of Theorem 2.9 is carried out by showing that any harmonic map of type $(2, 1, 0)$ (hence with unpton number at most 2) may be deformed continuously through harmonic maps to a harmonic map of unpton number 1. Let us see how this works, explicitly.
Without loss of generality we begin with an extended solution of the form

\[ W = (A_0 + \lambda A_1) \text{diag}(\lambda^2, \lambda, 1) H_+^{(3)} \]

\[ = \begin{pmatrix}
1 & \alpha' / \beta' & \alpha \\
0 & 1 & \beta \\
0 & 0 & 1
\end{pmatrix} + \lambda \begin{pmatrix}
0 & 0 & \delta \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} V_3 \oplus \lambda (V_3 \oplus V_2) \oplus \lambda^2 H_+^{(3)} \]

(cf. Appendix B). Here, \( \alpha, \beta, \delta \) are arbitrary rational functions, and the zeros of \( \delta \) are certain points \( z_1, \ldots, z_e \), where \( e = |W| \). These are the points \( z \) such that \( \dim W(z) \cap Z \geq 1 \), with \( Z = \mathbb{C}^3 \). (If \( \delta \) has less than \( e \) zeros in the domain of \( A_0 + \lambda A_1 \), or if \( \delta \) is identically zero, then we deform \( \delta \) continuously until we are in the above situation.) We have \( E_2 = V_1, E_1 = V_2, E_0 = V_3 \), and so

\[ \lambda (V_2 \oplus V_3) \oplus \lambda^2 H_+^{(3)} = W \subseteq V_3 \oplus \lambda H_+^{(3)} \]

Multiplying on the left by \( \text{diag}(\lambda^{-1}, 1, 1) \) (a trivial dressing transformation, which does not change the energy of the harmonic map) shows that

\[ \lambda H_+^{(3)} \subseteq \text{diag}(\lambda^{-1}, 1, 1) W \subseteq \text{diag}(1, \lambda, 1) H_+^{(3)} \]

hence \( \text{diag}(\lambda^{-1}, 1, 1) W \) has uniton number 1. In fact, \( W \) is a very special kind of “1-uniton” as it corresponds to a holomorphic map \( S^2 \to S^2 \).

So far we have merely repeated the above proof of Theorem 2.9 in this particular case. A closer examination of the Grassmanian model (the details of which we leave to the interested reader) reveals a more geometrical interpretation of this deformation. The original extended solution \( W : S^2 \to Gr_3(H_+^{(3)} / \lambda^2 H_+^{(3)}) = Gr_3(\mathbb{C}^6) \) actually maps into

\[ X = \{ \text{all linear subspaces } E \subseteq H^{(3)} \text{ such that} \]

\[ \lambda^2 H_+^{(3)} \subseteq E \subseteq H_+^{(3)}, \lambda E \subseteq E, \dim H_+^{(3)} / E = 3, \dim(\lambda^{-1} E \cap H_+^{(3)}) / E \geq 2 \}. \]
This is a four-dimensional algebraic subvariety of $Gr_3(\mathbb{C}^6)$ with one singular point, $E = \lambda H_+^{(3)}$. Moreover, $X - \{\lambda H_+^{(3)}\}$ has the structure of a holomorphic line bundle over the flag manifold $F_{1,2}(\mathbb{C}^3) = U_3/(U_1 \times U_1 \times U_1)$ (it is the unstable bundle of the conjugacy class of the geodesic $diag(\lambda^2, \lambda, 1)$ in the Morse theory decomposition of $\Omega^{alg} U_3$, and its closure — i.e. $X$ — is the corresponding Schubert variety in this algebraic loop group). To be precise, it is the bundle $\mathcal{E}_1^* \otimes \mathcal{E}_2$, where $\mathcal{E}_1, \mathcal{E}_2$ are the tautological vector bundles of ranks 1, 2 on $F_{1,2}(\mathbb{C}^3)$.

It follows from this description of the structure of $X$ that the homotopy class $|W|$ decomposes as $|W| = d + d_1 + d_2$, where

1. $d$ is the number of points (counting multiplicities) in $W^{-1}(\lambda H_+^{(3)})$, and
2. $d_i = -c_1 W_\infty^* \mathcal{E}_i$, where $W_\infty = \pi \circ W$ and $\pi : X - \{\lambda H_+^{(3)}\} \to F_{1,2}(\mathbb{C}^3)$ is the bundle projection.

Explicitly, the map $\pi \circ W$ is represented by

$$
\begin{pmatrix}
1 & \alpha' / \beta' & \alpha \\
0 & 1 & \beta \\
0 & 0 & 1
\end{pmatrix}
V_3 \oplus \lambda (V_3 \oplus V_2) \oplus \lambda^2 H_+^{(3)}.
$$

The rational function $\delta$ represents a meromorphic section of $\pi$, with $|W|$ zeros and $d$ poles. By deforming the rational functions $\alpha, \beta$ to constants, we deform $W$ into a (compactified) fibre of $\pi$, and hence to a holomorphic map $S^2 \to S^2$. 

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References

[Do] J. Dorfmeister, Generalized Weierstrass representations of surfaces, these proceedings.
[Ha] H. Hashimoto, Weierstrass Bryant formula of super-minimal J-holomorphic curves of a
6-dimensional sphere, preprint.


