The Petersson series vanishes at infinity

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Abstract. The Petersson series with respect to a simple closed geodesic \( c \) on a hyperbolic Riemann surface \( R \) is the relative Poincaré series of the canonical holomorphic quadratic differential on the annular cover of \( R \) and it defines a holomorphic quadratic differential \( \varphi_c(z)dz^2 \) on \( R \). For the hyperbolic metric \( \rho(z)|dz| \) on \( R \), we give an upper estimate of \( \rho^{-2}(z(p))|\varphi_c(z(p))| \) in terms of the hyperbolic length of \( c \) and the distance of \( p \in R \) from \( c \).

1. Introduction

Let \( \Gamma \) be a torsion-free Fuchsian group acting on a upper half-plane model \( \mathbb{H} = \{ \zeta = \xi + i\eta \mid \eta > 0 \} \) of the hyperbolic plane. Throughout this paper, we always assume that a Riemann surface \( R \) is represented by \( \mathbb{H}/\Gamma \). A holomorphic quadratic differential \( \varphi(z)dz^2 \) on \( R \) can be identified with a holomorphic function \( \varphi(\zeta) \) on \( \mathbb{H} \) that satisfies \( \varphi(\gamma(\zeta)) = \varphi(\zeta)\gamma'(\zeta)^2 \) for every \( \gamma \in \Gamma \). We call such a holomorphic function \((2,0)\)-automorphic form for \( \Gamma \). A holomorphic \((2,0)\)-automorphic form \( \varphi(\zeta) \) is integrable if the integral of \( |\varphi(\zeta)| \) over a fundamental domain of \( \Gamma \) is finite. This is equivalent to saying that the integral \( \int_R |\varphi(z)|dx\,dy \) is finite. We denote the space of all integrable holomorphic \((2,0)\)-automorphic form on \( \mathbb{H} \) for \( \Gamma \) by \( Q^1(\mathbb{H},\Gamma) \). This can be identified with the space of all integrable holomorphic quadratic differentials on \( R \) which is a complex Banach space with the norm \( ||\varphi||_1 = \int_R |\varphi(z)|\,dx\,dy \). If \( \Gamma \) is the trivial group \( 1 \), then \( Q^1(\mathbb{H},1) \) is nothing but the Banach space of all integrable holomorphic functions on \( \mathbb{H} \).

An integrable holomorphic \((2,0)\)-automorphic form for \( \Gamma \) is produced from an integrable holomorphic function \( f \) by the Poincaré series

\[
\Theta_\Gamma(f(\zeta)) = \sum_{\gamma \in \Gamma} f(\gamma(\zeta))\gamma'(\zeta)^2.
\]

It is known that \( \Theta_\Gamma : Q^1(\mathbb{H},1) \to Q^1(\mathbb{H},\Gamma) \) is a surjective bounded linear operator with the operator norm not greater than 1 for every Fuchsian group \( \Gamma \). See Kra [7] for details on automorphic forms and the Poincaré series.

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Let $\rho(\zeta) = 1/\text{Im} \zeta$ be the hyperbolic density on $\mathbb{H}$. It induces the hyperbolic metric $\rho(z)|dz|$ on a Riemann surface $R = \mathbb{H}/\Gamma$. For a hyperbolic element $\gamma_c \in \Gamma$ corresponding to a simple closed geodesic $c$ on the hyperbolic Riemann surface $R$, we consider the annulus $A = \mathbb{H}/(\gamma_c)$ which covers $R$. We may assume that $\gamma_c(\zeta) = e^{i(c)}\zeta$ where $\ell(c)$ denotes the hyperbolic length of $c$.

For an integrable holomorphic $(2, 0)$-automorphic form $\phi$ for $\langle \gamma_c \rangle$, the relative Poincaré series

$$\sum_{[\gamma] \in \langle \gamma_c \rangle \Gamma} \phi(\gamma(\zeta))\gamma'(\zeta)^2$$

also defines an integrable holomorphic $(2, 0)$-automorphic form for $\Gamma$. Here the sum is taken over all representatives of the cosets $\langle \gamma_c \rangle \Gamma$. Then $\Theta(\gamma_c)_\Gamma : Q^1(\mathbb{H}, \langle \gamma_c \rangle) \rightarrow Q^1(\mathbb{H}, \Gamma)$ is also a surjective bounded linear operator with norm not greater than 1.

We choose $\phi(\zeta) = \zeta^{-2}$, which is an integrable holomorphic $(2, 0)$-automorphic form for $\langle \gamma_c \rangle$. The polar coordinates $(t, \theta) \in \mathbb{H} \times (0, \pi)$ for $\zeta = \exp(t + i\theta) \in \mathbb{H}$ induce an euclidean metric $\sqrt{dt^2 + d\theta^2}$ on the annulus $A = \mathbb{H}/(\gamma_c)$ and this coincides with the euclidean metric $|d\zeta/\zeta|$ induced by the holomorphic quadratic differential on $A$ corresponding to $\phi(\zeta) = \zeta^{-2}$. In particular, the area form $|d\zeta|/d\theta$ is equal to $dl/dt$ and hence $\int_A \phi(\zeta)|d\zeta/d\theta| = \pi \ell(c)$. The relative Poincaré series

$$\varphi_c(\zeta) = \Theta(\gamma_c)_\Gamma(\phi(\zeta)) = \sum_{[\gamma] \in \langle \gamma_c \rangle \Gamma} \gamma'(\zeta)^2$$

called the Petersson series with respect to $c$, which defines the holomorphic quadratic differential $\varphi_c(z)dz^2$ on $R$. The norm $\|\varphi_c\|_1$ is bounded by $\|\phi\|_1 = \pi \ell(c)$. This plays an important role on the variation of the hyperbolic length $\ell(c)$ under a quasiconformal deformation of $R$ (cf. Gardiner [5]) and the Weil-Petersson geometry on Teichmüller spaces (cf. Wolpert [15]).

For a quadratic differential $\varphi(z)dz^2$ on $R$, $\rho^{-2}(z(p))|\varphi(z(p))|$ is well-defined for $p \in R$ independent of a local parameter $z$ around $p$ and hence $\rho^{-2}|\varphi|$ gives a function on $R$. For the $(2, 0)$-automorphic form $\varphi(\zeta)$ and for a point $\zeta \in \mathbb{H}$ over $p \in R$, the function $\rho^{-2}(\zeta)|\varphi(\zeta)|$ is the lift of $\rho^{-2}|\varphi|$ to the universal cover $\mathbb{H}$. We provide the supremum norm $\|\varphi\|_\infty = \sup_{\zeta \in \mathbb{H}} \rho^{-2}(\zeta)|\varphi(\zeta)|$ for a holomorphic $(2, 0)$-automorphic form $\varphi(\zeta)$ for $\Gamma$ (and for a holomorphic quadratic differential) and call it bounded if $\|\varphi\|_\infty$ is finite. The space of all bounded holomorphic $(2, 0)$-automorphic forms for $\Gamma$ is denoted by $Q^\infty(\mathbb{H}, \Gamma)$. This is a complex Banach space with the norm $\|\varphi\|_\infty$.

In this paper, we will give an estimate of the function $\rho^{-2}|\varphi_c|$ of $p \in R$ for $\varphi_c(z)dz^2$ defined by the Petersson series with respect to a simple closed geodesic $c$ on $R$ in terms of the hyperbolic distance $d(p, c)$ of $p$ from $c$. Our main theorem can be stated as follows.

**The Main Theorem.** Let $\varphi_c(z)dz^2$ be a holomorphic quadratic differential on a hyperbolic Riemann surface $R$ given by the Petersson series with respect to a simple closed geodesic $c$ on $R$. Then, for a sufficiently small $r_0 > 0$, there is a positive constant $B$ depending only on $r_0$ such that

$$\rho(z(p))^{-2}|\varphi_c(z(p))| \leq B \ell(c)e^{-d(p, c)/3}$$

for every $p \in R$ with $d(p, c) > r_0$ such that there is no closed curve based at $p$ and freely homotopic to $c$ with length less than $2r_0$. In particular, $\varphi_c(z)dz^2$ is bounded and it vanishes at infinity.
Here we say that a holomorphic quadratic differential \( \varphi(z)dz^2 \) on \( R \) vanishes at infinity if, for every \( \varepsilon > 0 \), there is a compact subset \( V \) of \( R \) such that \( \sup_{p \in R - V} \rho(z(p))^{-2}|\varphi(z(p))| < \varepsilon \). The corresponding holomorphic \((2,0)\)-automorphic form on \( \mathbb{H} \) is called similarly. We denote the subspace of \( Q^\infty(\mathbb{H}, \Gamma) \) consisting of all holomorphic \((2,0)\)-automorphic forms vanishing at infinity by \( Q^\infty_0(\mathbb{H}, \Gamma) \). This space has an importance in the theory of asymptotic Teichmüller spaces developed by Earle, Gardiner and Lakic (see \([6]\) and \([3]\)).

The assumption on the point \( p \in R \) in the statement of the Main Theorem eliminates the case where \( c \) is very short and \( p \) is in a collar neighborhood of \( c \). An estimate of \( \rho(z(p))^{-2}|\varphi_c(z(p))| \) in this case has been given in \([9]\).

We remark that, if the injectivity radii of \( R \) are uniformly bounded away from zero, then the conclusion of the Main Theorem easily follows from a basic estimate given in the next section. For a point \( p \) on \( R \), the injectivity radius \( \tau(p) \) is defined to be the radius of a maximal hyperbolic open disk centered at \( p \) that is embedded in \( R \). However, the existence of a cusp does not make the problem difficult even if \( \tau(p) \) tends to zero as \( p \) gets closer to a cusp: the essential problem occurs in the case where \( R \) has a sequence of simple closed geodesics whose lengths tend to zero.

Note that, it has been proved by Niebur and Sheingorn \([10]\) that \( Q^1(\mathbb{H}, \Gamma) \) is contained in \( Q^\infty(\mathbb{H}, \Gamma) \) if and only if \( R = \mathbb{H}/\Gamma \) has no such sequence of short simple closed geodesics whose lengths tend to zero. Moreover, it is shown in \([8]\) that the operator norm of the inclusion map \( Q^1(\mathbb{H}, \Gamma) \hookrightarrow Q^\infty(\mathbb{H}, \Gamma) \) is given in terms of the infimum of the lengths of simple closed geodesics on \( R \) (see also Sugawa \([14]\)). On the other hand, when \( R \) has a sequence of simple closed geodesics whose lengths tend to zero, examples of integrable but not bounded holomorphic quadratic differentials have been constructed in Pommerenke \([12]\) and Ohsawa \([11]\) as well as in \([9]\).

We further remark that, only to show that \( \varphi_c(z)dz^2 \) vanishes at infinity in the Main Theorem, there is a simpler argument. This can be done by transferring the Petersson series to the unit disk \( \mathbb{D} \) by biholomorphic conjugation and relying on a technique due to Ahlfors \([1]\). These arguments as well as the density of \( Q^\infty_0(\mathbb{H}, \Gamma) \) in \( Q^1(\mathbb{H}, \Gamma) \) will be discussed in the last section.

2. Basic estimate

We will review an integral estimate of the hyperbolic supremum norm of a holomorphic function and apply it to the Poincaré series. This also shows that injectivity radius is the issue that we should manage.

**Proposition 2.1.** Let \( \varphi(z)dz^2 \) be a holomorphic quadratic differential on a hyperbolic Riemann surface \( R \), \( \tau(p) \) the injectivity radius at \( p \in R \) and \( U(p, \tau(p)) \) the hyperbolic disk of radius \( \tau(p) \) centered at \( p \). Then

\[
\rho^{-2}(z(p))|\varphi(z(p))| \leq \frac{1}{4\pi \tanh^2(\tau(p)/2)} \int_{U(p, \tau(p))} |\varphi(z)| \, dx \, dy
\]

for a local coordinate \( z = x + iy \) around \( p \).

**Proof.** By lifting \( \varphi(z)dz^2 \) to the unit disk \( \mathbb{D} \), we have a holomorphic \((2,0)\)-automorphic form \( \varphi(\zeta) \) on \( \mathbb{D} \). We may assume that \( p \in R \) corresponds to the origin \( 0 \in \mathbb{D} \), that is, \( \zeta = \xi + i\eta \) gives a local coordinate such that \( \zeta(p) = 0 \). Let
\( \rho_D(\zeta) = 2/(1 - |\zeta|^2) \) denote the hyperbolic density on \( \mathbb{D} \). Then

\[
\rho^{-2}(z(p))|\varphi(z(p))| = \rho_D^{-2}(0)|\varphi(0)| = \frac{|\varphi(0)|}{4}
\]

and

\[
\varphi(0) = \frac{1}{\pi a^2} \int_{|\zeta| \leq a} \varphi(\zeta) \, d\xi d\eta,
\]

where \( U(p, r(p)) \) lifts to the euclidean disk \( \{|\zeta| \leq a\} \) of radius \( a = \tanh(r(p)/2) \).

Hence

\[
|\varphi(0)| \leq \frac{1}{\pi a^2} \int_{|\zeta| \leq a} |\varphi(\zeta)| \, d\xi d\eta = \frac{1}{\pi \tanh^2(r(p)/2)} \int_{U(p, r(p))} |\varphi(z)| \, dx dy,
\]

which yields the desired inequality. \( \square \)

It is well known that there is a constant \( r_0 > 0 \) (related to the Margulis constant) independent of the choice of a hyperbolic Riemann surface \( R \) such that if \( r(p) < r_0 \) then the disk neighborhood \( U(p, r(p)) \) of \( p \) is entirely contained either in the canonical cusp neighborhood or in the canonical collar of a short simple closed geodesic on \( R \). Here the canonical cusp neighborhood is a horocyclic cusp neighborhood of hyperbolic area \( 2 \) and the canonical collar of a simple closed geodesic \( \alpha \) is its neighborhood of width

\[
\omega = \arcsinh \frac{1}{\sinh(\ell(\alpha)/2)}.
\]

Note that, in this latter case, \( \omega \geq r(p) \) and \( 2r(p) \geq \ell(\alpha) \) are satisfied. From these conditions, the upper bound of the hyperbolic length of \( \alpha \) is known as \( \ell(\alpha) \leq 2 \arcsinh 1 \).

Fix such a constant \( r_0 > 0 \). We define the cut-off injectivity radius at \( p \in R \) as \( \tau(p) = \min\{r(p), r_0\} \). Then Proposition 2.1 implies that

\[
\rho^{-2}(z(p))|\varphi_c(z(p))| \leq \frac{r_0^2}{4\pi \tanh^2(r_0/2) \tau(p)^2} \int_{U(p, \tau(p))} |\varphi(z)| \, dx dy
\]

for any holomorphic quadratic differential \( \varphi(z)dz^2 \) on \( R \). We apply this formula for the quadratic differential \( \varphi_c(z)dz^2 \) on \( R \) induced by the Petersson series with respect to a simple closed geodesic \( c \). By setting \( b(r_0) = r_0^2/[4 \tanh^2(r_0/2)] \), we have

\[
\rho^{-2}(z(p))|\varphi_c(z(p))| \leq \frac{b(r_0)}{\pi \tau(p)^2} \sum_{|\gamma| \in \langle \gamma_c \rangle \setminus \Gamma} \int_{U(\zeta(\gamma), \tau(p))} \frac{|\gamma'(\zeta)|^2}{|\gamma'(\zeta)|^2} \, d\xi d\eta
\]

\[
= \frac{b(r_0)}{\pi \tau(p)^2} \sum_{|\gamma| \in \langle \gamma_c \rangle \setminus \Gamma} \int_{U(\zeta(\gamma), \tau(p))} \frac{1}{|\zeta|^2} \, d\xi d\eta.
\]

**Lemma 2.2.** For every \( p \) with \( d(p, c) > r_0 \),

\[
\rho^{-2}(z(p))|\varphi_c(z(p))| \leq \frac{2e^{\tau(p)}b(r_0)}{\tau(p)^2} \ell(c) e^{-d(p, c)}
\]

is satisfied.
Hence the Main Theorem is verified in this case, which is the desired inequality.

Suppose that the point \( p \in R \) satisfies \( r(p) \geq r_0 \). Then, by \( r(p) = r_0 \), Lemma 2.2 immediately shows that

\[
\rho^{-2}(z(p))|\varphi_\epsilon(z(p))| \leq \frac{2\epsilon^n b(r_0)}{r_0^2} \ell(c) e^{-d(p, c)}.
\]

Hence the Main Theorem is verified in this case.

Now we investigate the case where \( r(p) < r_0 \). Then \( r(p) = r(p) \) and \( p \) is either in the canonical cusp neighborhood or in the canonical collar. For the moment, suppose that \( p \) is in the canonical cusp neighborhood \( \Omega \subset R \). Note that \( \Omega \) is disjoint from \( c \). We can represent \( \Omega \) as a quotient space of \( \{ \zeta \in \mathbb{H} \mid \text{Im} \zeta > 1/2 \} \) by the parabolic element \( \zeta \mapsto \zeta + 1 \); we may assume that \( \Gamma \) contains this element. Then \( \Omega = \{ 0 < |w| < e^{-\pi} \} \) by using the local parameter \( w = \exp(2\pi i \zeta) \). Also the hyperbolic density is given by \( \rho(w) = (-|w| \log |w|)^{-1} \). It is known that a larger punctured disk \( \bar{\Omega} = \{ 0 < |w| < e^{-\pi/2} \} \) is also embedded in \( R \) (see Seppälä and Sorvali [13]).

Proposition 2.3. Let \( \varphi(z)dz^2 \) be an integrable holomorphic quadratic differential on \( R \) and \( p \) a point in the canonical cusp neighborhood \( \Omega \subset R \) with the local parameter \( w = \exp(2\pi i \zeta) \). Then

\[
\rho^{-2}(w(p))|\varphi(w(p))| \leq \frac{2\epsilon^n |w(p)| (\log |w(p)|)^2}{\pi} \|\varphi\|_1
\]

is satisfied.

Proof. It is easy to see that \( \varphi(w) \) has at most a simple pole at the puncture \( w = 0 \). Hence \( w\varphi(w) \) is a holomorphic function of \( w = u + iv \) and satisfies

\[
w(p)\varphi(w(p)) = \frac{1}{\pi a^2} \int_{|w-w(p)| \leq a} w\varphi(w) dudv
\]

for \( a = e^{-\pi} \). Then

\[
\rho^{-2}(w(p))|\varphi(w(p))| \leq \frac{|w(p)| (\log |w(p)|)^2}{\pi a^2} \int_{|w-w(p)| \leq a} |w| |\varphi(w)| dudv \\
\leq \frac{2|w(p)| (\log |w(p)|)^2}{\pi a} \int_{R} |\varphi(z)| dxdy,
\]

which is the desired inequality. \( \square \)
Assume that \( p \in \Omega \) is at distance \( d \geq d(p, c) \) from the boundary \( \partial \Omega \). Then \( \text{Im} \, \zeta(p) = e^{d/2} \) and hence
\[
|w(p)| = \exp(-\pi e^{d}) \leq \exp(-\pi(1 + d)).
\]
Recall that the quadratic differential \( \varphi_c(z)dz^2 \) on \( \mathbb{R} \) determined by the Petersson series satisfies \( ||\varphi_c||_1 \leq \pi \ell(c) \). From Proposition 2.3, we have
\[
\rho^{-2}(w(p))|\varphi_c(w(p))|\leq 2\pi^2 \ell(c) \exp(\pi + 2d - \pi(1 + d)).
\]
In particular,
\[
\rho^{-2}(w(p))|\varphi_c(w(p))| \leq 2\pi^2 \ell(c) e^{-d(p, c)},
\]
which satisfies the condition of the Main Theorem. This means that we do not have to take care of the case where \( p(r(p) < r_0) \) is in the canonical cusp neighborhood.

3. Comparison of euclidean areas

In what follows, we investigate the case where the point \( p \) satisfying \( r(p) < r_0 \) is in the canonical collar of some short simple closed geodesic \( \alpha \). Recall that \( \ell(\alpha) \leq 2 \arcsinh 1 \) is satisfied in this case. Since we assume in the Main Theorem that there is no closed curve based at \( p \) that is freely homotopic to \( c \) with its length less than \( 2r_0 \), we know that \( \alpha \) is distinct from \( c \). Moreover, we see that \( \alpha \) is disjoint from \( c \). Indeed, if not, then every point of injectivity radius less than \( r_0 \) in the collar of \( \alpha \) is within distance \( r_0 \) from \( c \), but this violates the assumption \( d(p, c) > r_0 \).

Since we assume that \( \Gamma \) contains the element \( \gamma_c(\zeta) = e^{\ell(c)} \zeta \) corresponding to \( c \), every element \( \gamma_{\alpha} \in \Gamma \) corresponding to a simple closed geodesic \( \alpha \) different from \( c \) has the axis \( \tilde{\alpha} \) in \( \mathbb{H} \) whose end points are on the real axis \( \mathbb{R} \). We take the neighborhood \( \tilde{C}(\tilde{\alpha}) \) of \( \tilde{\alpha} \) that is the lift of the canonical collar \( C(\alpha) \) of \( \alpha \) and consider a part of \( \tilde{C}(\tilde{\alpha}) \) that contains the lifts of \( U(p, r(p)) \). In this section, we compare the euclidean areas of these regions as subsets of \( \mathbb{R}^2 \). To describe a signed distance from \( \tilde{\alpha} \), we use an angle parameter \( \theta \in (-\pi/2, \pi/2) \) representing the sector angle, which is given by \( \theta = \arctan \sinh \omega \) for the signed distance \( \omega \) from \( \tilde{\alpha} \).

**Proposition 3.1.** Let \( \tilde{\alpha} \) be a hyperbolic geodesic line in \( \mathbb{H} \) which is a semicircle of euclidean radius \( h > 0 \). Then the signed euclidean area of the one-sided neighborhood of \( \tilde{\alpha} \) within angle \( \theta \in (-\pi/2, \pi/2) \) is given by
\[
S(\theta) = h^2 \left\{ \frac{\pi}{2} \tan^2 \theta + \theta \tan^2 \theta + \theta + \tan \theta \right\}.
\]
Here we assume that the one-sided neighborhood is outside the semicircle and its area is positive if \( \theta > 0 \) and it is inside the semicircle and its area is negative if \( \theta < 0 \).

**Proof.** We assume \( \theta > 0 \). The one-sided neighborhood of \( \tilde{\alpha} \) in question is the crescent-shaped region in the euclidean disk \( D \) of radius \( h/\cos \theta \) as in Figure 1. The area of the sector in \( D \) with angle \( \pi + 2\theta \) is \( (h/\cos \theta)^2(\pi + 2\theta)/2 \) and the area of the triangle with base length \( 2h \) is \( h^2 \tan \theta \). Since \( S(\theta) \) is the area of the chordal region in \( D \) over \( \mathbb{R} \) minus the area \( \pi h^2/2 \) of the semi-disk of radius \( h \), we have
\[
S(\theta) = \left( \frac{h}{\cos \theta} \right)^2 \left( \frac{\pi}{2} + \theta \right) + h^2 \tan \theta - \frac{\pi h^2}{2}.
\]
This is equivalent to the required formula above. The case where $\theta < 0$ can be treated similarly and we obtain the same formula. \hfill \Box

An easy computation (omitted) also gives the derivative of $S(\theta)$ as follows.

**Proposition 3.2.** The derivative of the function $S(\theta)$ is given by

$$S'(\theta) = \frac{h^2}{\cos^3 \theta} \{ (\pi + 2\theta) \sin \theta + 2 \cos \theta \},$$

which satisfies

$$0 < S'(\theta) < \frac{2\pi h^2}{\cos^3 \theta}$$

for $-\pi/2 < \theta < \pi/2$.

We are dealing with the case where $r(p) < r_0$ and $U(p, r(p))$ is contained in the canonical collar $C(\alpha)$ of some simple closed geodesic $\alpha$ of $R$. The width of $C(\alpha)$ is $\arcsinh(\sinh(\ell(\alpha)/2))^{-1}$, which is represented by an angle

$$\tilde{\theta} = \arctan \frac{1}{\sinh(\ell(\alpha)/2)} > 0.$$

Then a connected component of the inverse image of $C(\alpha)$ under the universal cover $\mathbb{H} \to R$ is the two-sided neighborhood $\tilde{C}(\tilde{\alpha})$ of a geodesic line $\tilde{\alpha}$ within the angle $\tilde{\theta}$. By Proposition 3.1, its euclidean area is given by

$$S(\tilde{\theta}) - S(-\tilde{\theta}) = 2h^2 (\tilde{\theta} \tan^2 \tilde{\theta} + \tan \tilde{\theta} + \tilde{\theta}),$$

where $h$ is the euclidean radius of the semicircle $\tilde{\alpha}$. Here, we note that the condition $\ell(\alpha) \leq 2 \arcsinh 1$ is equivalent to $\tilde{\theta} \geq \pi/4$. Then the euclidean area of $\tilde{C}(\tilde{\alpha})$ is estimated from below by

$$2h^2 (\tilde{\theta} \tan^2 \tilde{\theta} + \tan \tilde{\theta} + \tilde{\theta}) \geq 2h^2 (\pi/4) \tan^2 \tilde{\theta} = \frac{\pi h^2}{2} \frac{1}{\sinh^2 (\ell(\alpha)/2)}.$$

Assume that the point $p$ is on the level curve of angle $\theta_0$ in the collar $C(\alpha)$ and $U(p, r(p))$ is between $\theta_1$ and $\theta_2$ for $\theta_1 < \theta_0 < \theta_2$. Since $U(p, r(p))$ is contained in $C(\alpha)$, we have $-\theta \leq \theta_1$ and $\theta_2 \leq \theta$. Lifting $C(\alpha)$ to $\mathbb{H}$, we consider a subregion
$C_{[\theta_1, \theta_2]}(\tilde{\alpha})$ of $C(\tilde{\alpha})$ between the angles $\theta_1$ and $\theta_2$ and estimate its euclidean area $S(\theta_2) - S(\theta_1)$ from above. By Proposition 3.2, we have

$$S(\theta_2) - S(\theta_1) = \int_{\theta_1}^{\theta_2} S'(\theta) \, d\theta \leq 2\pi h_2^2 \int_{\theta_1}^{\theta_2} \frac{d\theta}{\cos^2 \theta}.$$ 

We assume that $\theta_0 \geq 0$ for the sake of simplicity. The case where $\theta_0 < 0$ can be treated similarly. Since $\cos \theta_1 \geq \cos \theta_2$ under this assumption, we have

$$\int_{\theta_1}^{\theta_2} \frac{d\theta}{\cos^2 \theta} \leq \frac{1}{\cos^2 \theta_2} \int_{\theta_1}^{\theta_2} \frac{d\theta}{\cos \theta} = \frac{2r(p)}{\cos^2 \theta_2}.$$ 

Here the last equality is a consequence from the following formula between the hyperbolic distance $\omega$ from the core geodesic $\alpha$ and the angle parameter $\theta$:

$$\omega = \mathrm{arcsinh}(\tan \theta) = \int_0^\theta \frac{d\theta}{\cos \theta}.$$ 

**Figure 2.** Level curves

To proceed the estimate of the area of $C_{[\theta_1, \theta_2]}(\tilde{\alpha})$, we use the following:

**Claim.** $\frac{1}{\cos \theta_2} \leq \frac{6r(p)}{\ell(\alpha)}$.

**Proof.** Set $\theta_* = \max\{\theta_1, 0\}$. Then we have

$$\frac{\ell(\alpha)}{\cos \theta_2} \leq 2 \int_{\theta_*}^{\theta_2} \frac{d\theta}{\cos \theta} + \frac{\ell(\alpha)}{\cos \theta_*}.$$ 

Indeed, consider a function

$$g(\theta) = 2 \int_{\theta_*}^{\theta} \frac{d\theta}{\cos \theta} - \ell(\alpha) \left( \frac{1}{\cos \theta} - \frac{1}{\cos \theta_*} \right)$$

for $\theta_* \leq \theta \leq \hat{\theta}$. Then $g(\theta_*) = 0$ and $g'(\hat{\theta}) = 2/\cos \hat{\theta} - \ell(\alpha) \tan \hat{\theta}/\cos \hat{\theta}$. By using

$$\ell(\alpha) \tan \hat{\theta} \leq \ell(\alpha) \tan \hat{\theta} = \frac{\ell(\alpha)}{\sinh(\ell(\alpha)/2)} \leq 2,$$
we have $g'(\theta) \geq 0$ and hence $g(\theta) \geq 0$. In particular, $g(\theta_2) \geq 0$, which yields the above inequality.

If $\theta_1 \geq 0$, then $\ell(\alpha)/\cos \theta_*$ is the length of the level curve of angle $\theta_1$, which is bounded by $2r(p)$. Indeed, since $U(p, r(p))$ is located outside the level curve of angle $\theta_1$, there is a length decreasing homeomorphism from the shortest closed curve of length $2r(p)$ based at $p$ freely homotopic to $\alpha$ onto the level curve of angle $\theta_1$. See Figure 2. If $\theta_1 \leq 0$, then $\ell(\alpha)/\cos \theta_* = \ell(\alpha)$, which is also bounded by $2r(p)$. Therefore we have

$$2 \int_{\theta_1}^{\theta_2} \frac{d\theta}{\cos \theta} + \frac{\ell(\alpha)}{\cos \theta_*} \leq 2 \int_{\theta_1}^{\theta_2} \frac{d\theta}{\cos \theta} + 2r(p) = 6r(p),$$

from which the claimed inequality follows. \qed

As a consequence, we see that the euclidean area $S(\theta_2) - S(\theta_1)$ of $\tilde{C}_{[\theta_1, \theta_2]}(\tilde{\alpha})$ is bounded above by $144 \pi h^2 r(p)^3/\ell(\alpha)^2$. Recall that we have already obtained the estimate of the euclidean area of $\tilde{C}(\tilde{\alpha})$ from below.

**Proposition 3.3.** The ratio of the euclidean area of the region $\tilde{C}_{[\theta_1, \theta_2]}(\tilde{\alpha})$ to the euclidean area of $\tilde{C}(\tilde{\alpha})$ is bounded above by $288 r(p)^3$ if $\ell(\alpha) \leq 2 \arcsinh 1$.

**Proof.** The two estimates above yield

$$\frac{S(\theta_2) - S(\theta_1)}{S(\theta) - S(-\theta)} \leq \frac{144 \pi h^2 r(p)^3/\ell(\alpha)^2}{\pi h^2/(2 \sinh^2(\ell(\alpha)/2))} = 288 r(p)^3 \left(\frac{\sinh(\ell(\alpha)/2)}{\ell(\alpha)}\right)^2,$$

If $\ell(\alpha) \leq 2 \arcsinh 1$, then $\sinh(\ell(\alpha)/2)/\ell(\alpha) \leq 1/(2 \arcsinh 1) < 1$. Hence the last term of the above inequality is bounded by $288 r(p)^3$. \qed

## 4. Proof of the main theorem

Let $\gamma_c(\zeta) = e^{\ell(\zeta)} \zeta$ and consider the annulus $A = \mathbb{H}/(\gamma_c)$. The euclidean metric on $A$ is the projection of the euclidean metric on the universal cover $\mathbb{H}$ defined by the polar coordinates $(l, t)$ with $0 < l < \infty$ and $0 < t < \pi$ satisfying $\xi + i\eta = \exp(l + it)$ in $\mathbb{H}$. Then the Jacobian matrix of the coordinate change map $(l, t) \mapsto (\xi, \eta)$ is

$$\frac{\partial(\xi, \eta)}{\partial(l, t)} = \begin{pmatrix} e^l \cos t & -e^l \sin t \\ e^l \sin t & e^l \cos t \end{pmatrix},$$

and its determinant is $J(\zeta) = e^{2l} = |\zeta|^2$ for $\zeta = \xi + i\eta \in \mathbb{H}$. This shows that $d\xi d\eta/|\zeta|^2 = dldt$.

By Proposition 3.3, we have an estimate of the ratio of areas of $\tilde{C}_{[\theta_1, \theta_2]}(\tilde{\alpha})$ and $\tilde{C}(\tilde{\alpha})$ measured by the euclidean area element $d\xi d\eta$. Next we consider the ratio of areas of their projections onto the annulus $A$ which are measured by the euclidean area element $dldt$. Since the Jacobian is $|\zeta|^2$, we have only to look at the minimal and maximal distances $m$ and $M$ of $\tilde{C}(\tilde{\alpha})$ from the origin $0$. Since $d(p, c) > r_0 > r(p)$, the simple closed geodesic $\alpha$ is disjoint from $c$. This implies that the neighborhood $\tilde{C}(\tilde{\alpha})$ of the geodesic line $\tilde{\alpha}$ is disjoint from the imaginary axis in $\mathbb{H}$. Note also that the angle of $\tilde{C}(\tilde{\alpha})$ is not less than $\pi/4$. Then Figure 3 illustrates the extremal situation where the ratio $M/m$ should be the largest, and
an elementary geometric calculus gives that $M/m = (\sqrt{3} + \sqrt{2})^2$ in this case. From this observation, we see that

$$\frac{\max_{\zeta \in \tilde{C}(\tilde{\alpha})} J(\zeta)}{\min_{\zeta \in \tilde{C}(\tilde{\alpha})} J(\zeta)} \leq (\sqrt{3} + \sqrt{2})^4.$$
This yields one inequality
\[ \rho^{-2}(z(p))|\varphi_c(z(p))| \leq \frac{b(r_0)}{\pi r(p)^2} \sum_{|z| \in (1, \infty)} \int_{\gamma(U(\zeta(p), r(p)))} \frac{1}{|\zeta'|} d\zeta d\eta \]
\[ \leq Kb(r_0)\ell(c)r(p). \]
On the other hand, Lemma 2.2 gives another inequality
\[ \rho^{-2}(z(p))|\varphi_c(z(p))| \leq 2e^{r_0}b(r_0)\ell(c)e^{-d(p,c)}r(p)^{-2}. \]
We have obtained two estimates as
\[ \rho^{-2}(z(p))|\varphi_c(z(p))| \leq \left\{ \begin{array}{ll}
Kb(r_0)\ell(c)r(p) \\
2e^{r_0}b(r_0)\ell(c)e^{-d(p,c)}r(p)^{-2}.
\end{array} \right. \]
Now we consider the maximum of the smaller one of these values when \( r(p) \) varies in \( (0, r_0) \):
\[ \max_{r(p) \in (0, r_0)} \min \{ Kr(p), 2e^{r_0}e^{-d(p,c)}r(p)^{-2} \} b(r_0)\ell(c) \]
\[ \leq K^{2/3}(2e^{r_0})^{1/3}b(r_0)\ell(c)e^{-d(p,c)/3}. \]
This eliminates \( r(p) \) from the formula. By setting \( B = K^{2/3}(2e^{r_0})^{1/3}b(r_0) \), we have
\[ \rho^{-2}(z(p))|\varphi_c(z(p))| \leq B\ell(c)e^{-d(p,c)/3}, \]
which completes the proof of the Main Theorem. \( \square \)

5. Application to the variation of length functions

For a Beltrami differential \( \mu = \mu(z)dz/dz \) on a hyperbolic Riemann surface \( R \), consider a quasiconformal deformation \( R_\mu \) of \( R \) given by \( \mu \) and denote the geodesic length of the free homotopy class of \( c \) on \( R_\mu \) by \( \ell_\mu(c) \). Then a variational formula due to Gardiner [5] asserts that
\[ \frac{d\ell_\mu(c)}{dt} \bigg|_{t=0} = \frac{2}{\pi} \text{Re} \int_R \mu(z)\varphi_c(z) dxdy. \]
The Main Theorem can be applied to an estimate of the derivative \( d\ell_\mu(c)/dt \big|_{t=0} \) through this formula.

We say that a Beltrami differential \( \mu(z)dz/dz \) on \( R \) vanishes at infinity if, for every \( \varepsilon > 0 \), there exists a compact subset \( V \) of \( R \) such that \( |\mu(z(p))| < \varepsilon \) for almost every \( p \in R - V \). A quasiconformal homeomorphism \( f \) of \( R \) whose complex dilatation is a Beltrami differential vanishing at infinity is called asymptotically conformal.

**Theorem 5.1.** Let \( \mu(z)dz/dz \) be a Beltrami differential on a hyperbolic Riemann surface \( R \) that vanishes at infinity. Let \( \{c_n\}_{n=1}^\infty \) be a sequence of simple closed geodesics on \( R \) escaping to the infinity. Then
\[ \frac{1}{\ell(c_n)} \cdot \frac{d\ell_\mu(c_n)}{dt} \bigg|_{t=0} \rightarrow 0 \]
as \( n \to \infty \).
Proof. For arbitrary \(\varepsilon > 0\), we take a compact subset \(V\) of \(R\) such that \(|\mu(z(p))| < \varepsilon\) for almost every \(p \in R - V\). Let \(\text{Area}(V)\) be the hyperbolic area of \(V\) and \(d(V, c_n)\) the hyperbolic distance between \(V\) and \(c_n\). Then, by using the Main Theorem for the integral on \(V\), we have

\[
\int_{R} |\mu(z)\varphi_{c_n}(z)| \, dx dy = \int_{R-V} |\mu(z)\varphi_{c_n}(z)| \, dx dy + \int_{V} |\mu(z)\varphi_{c_n}(z)| \, dx dy < \varepsilon\|\varphi_{c_n}\|_{1} + \text{Area}(V)\|\mu\|_{\infty}\beta\ell(c_n)e^{-d(V,c_n)/3}
\]

\[
\leq \ell(c_n)(\varepsilon\pi + \text{Area}(V) Be^{-d(V,c_n)/3}).
\]

Since \(d(V, c_n) \to \infty\) as \(n \to \infty\), this inequality shows that

\[
\frac{1}{\ell(c_n)} \int_{R} |\mu(z)\varphi_{c_n}(z)| \, dx dy \to 0
\]

as \(n \to \infty\). Then the Gardiner variation formula yields the statement of the theorem.

Note that it has been shown by Earle, Markovic and Saric [3] that an asymptotically conformal homeomorphism \(f\) of \(R\) with the complex dilatation \(\mu(z)dz/dz\) has an asymptotically isometric homeomorphism in its homotopy class. In particular, the ratios \(\ell_{f}(c_n)/\ell(c_n)\) for a sequence of simple closed geodesics \(\{c_n\}_{n=1}^{\infty}\) escaping to the infinity tend to 1 as \(n \to \infty\). See also [4]. Theorem 5.1 can be regarded as an infinitesimal version of this property.

6. Remarks on vanishing at infinity

It was noticed by Drasin and Earle [2] that, for an arbitrary Fuchsian group \(\Gamma\), the Banach space \(Q^{1}(\mathbb{H}, \Gamma)\) of the integrable holomorphic \((2,0)\)-automorphic forms has a dense linear subspace consisting of bounded holomorphic \((2,0)\)-automorphic forms in \(Q^{\infty}(\mathbb{H}, \Gamma)\). Actually, this claim was given for holomorphic \((2,0)\)-automorphic forms for a Fuchsian group \(G\) on the unit disk \(\mathbb{D}\) by using the fact that polynomials \(\{f(z)\}\) are dense in the Banach space \(Q^{1}(\mathbb{D}, 1)\) of all integrable holomorphic functions on \(\mathbb{D}\). Then the surjectivity of the Poincaré series operator \(\Theta_{G} : Q^{1}(\mathbb{D}, 1) \to Q^{1}(\mathbb{D}, G)\) yields that \(\{\Theta_{G}(f(z))\}\) are dense in \(Q^{1}(\mathbb{D}, G)\). Also, the technique introduced by Ahlfors [1] proves that \(\Theta_{G}(z^{n})\) for all \(n \geq 0\) are bounded holomorphic \((2,0)\)-automorphic forms in \(Q^{\infty}(\mathbb{D}, G)\).

In fact, Ahlfors’ argument further shows that \(\Theta_{G}(z^{n})\) are vanishing at infinity, namely, they belong to \(Q_{0}^{\infty}(\mathbb{D}, G)\). We will explain this method below. Then, after the conjugation to the upper half-plane \(\mathbb{H}\), we can summarize the result as follows.

**Proposition 6.1.** For every Fuchsian group \(\Gamma\), \(Q_{0}^{\infty}(\mathbb{H}, \Gamma) \cap Q^{1}(\mathbb{H}, \Gamma)\) is dense in the Banach space \(Q^{1}(\mathbb{H}, \Gamma)\) with the integrable norm.

For a Fuchsian group \(G\) acting on \(\mathbb{D}\), we consider

\[
J(z) = \rho_{G}^{-2}(z) \sum_{g \in G} |g'(z)|^{2} = \frac{1}{4} \sum_{g \in G} (1 - |g(z)|^{2})^{2},
\]

where \(\rho_{G}(z) = 2/(1 - |z|^{2})\) is the hyperbolic density on \(\mathbb{D}\). Then, as in [1], \(J(z)\) is a subharmonic function outside the images of a certain disk under \(G\). Also this is an automorphic function for \(G\) and thus regarded as a function on the Riemann surface \(R = \mathbb{D}/G\). Because of the subharmonicity, the function \(J\) on \(R\) vanishes at
in\(\infty\). See [9]. However, this method does not always tell the order of its decay in terms of the hyperbolic distance.

Let \(f(z)\) be an integrable holomorphic function on \(\mathbb{D}\) with \(|f(z)| \leq M\) for some positive constant \(M\). Its Poincaré series satisfies

\[
\rho_2^{-2}(z)|\Theta_G(f(z))| \leq MJ(z).
\]

We apply this estimate for \(f(z) = z^n\). Then we see that \(\Theta_G(z^n)\) vanishes at \(\infty\).

In addition, we look at the holomorphic \((2,0)\)-automorphic form \(\phi(\zeta) = 1/\zeta^2\) for \((\gamma_c)\) on \(\mathbb{H}\), where \(\gamma_c(\zeta) = e^{\ell(c)}\zeta\) is a hyperbolic element of a Fuchsian group \(\Gamma\). By a biholomorphic map \(\mathbb{D} \to \mathbb{H}\), we pull back \(\phi\) to \(\mathbb{D}\), which we denote by \(\tilde{\phi}(z)\). This also gives the conjugation of \(\Gamma\) with \(\gamma_c\) to a Fuchsian group \(G\) with the corresponding element \(g_c\) acting on \(\mathbb{D}\). We can verify that there is a positive constant \(L\) depending on \(\ell(c)\) such that

\[
|\tilde{\phi}(z)| \leq L\ell(c) \sum_{n \in \mathbb{Z}} |(g^n_c)'(z)|^2.
\]

See [9]. Then

\[
\rho_2^{-2}(z)|\Theta_{(g_c)\Gamma}(\tilde{\phi}(z))| \leq L\ell(c)J(z).
\]

This implies that the pull-back \(\Theta_{(g_c)\Gamma}(\tilde{\phi})\) of the Petersson series vanishes at \(\infty\) and so does the Petersson series \(\varphi_c = \Theta_{(\gamma_c)\Gamma}(\phi)\). Further arguments are necessary to obtain a quantitative estimate of the decay order for \(\varphi_c\).

References


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