LINEARIZATION OF THE UNIVERSAL TEICHMÜLLER SPACE: RIGIDITY AND FIXED POINT PROBLEM
普遍タイヒミュラー空間の線形化：剛性と固定点問題

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1. Statement of rigidity theorems

We consider certain rigidity of the deformation of a subgroup Γ of the group $\mathrm{M\ddot{o}b}(\mathbb{S}) \cong \mathrm{PSL}(2,\mathbb{R})$ of the projective transformations in a group of circle diffeomorphisms

$$\mathrm{Diff}_+^{1+\alpha}(\mathbb{S}) = \{g : \mathbb{S} \to \mathbb{S} \mid g : \text{diffeomorphism},\ g' : \alpha\text{-Hölder continuous}\}.$$

Here α-Hölder continuous condition for $0 < \alpha < 1$ means that there is a constant $c > 0$ such that

$$|g'(x) - g'(y)| \leq c|x - y|^\alpha$$

for any $x, y \in \mathbb{S} = \mathbb{R}/\mathbb{Z}$. More precisely, g is identified with its lift $g : \mathbb{R} \to \mathbb{R}$ and the condition is given for it.

We refer to rigidity when a weaker equivalence implies a stronger equivalence. A typical example is given as follows. For a triangle group $\Gamma \subset \mathrm{M\ddot{o}b}(\mathbb{S})$, if $f\Gamma f^{-1}$ is in $\mathrm{M\ddot{o}b}(\mathbb{S})$ for a homeomorphism $f : \mathbb{S} \to \mathbb{S}$, then f actually belongs to $\mathrm{M\ddot{o}b}(\mathbb{S})$. Hence the conjugation by a homeomorphism implies the conjugation by a Möbius transformation. In this sense, Γ is rigid.

We formulate rigidity problems in the theory of Teichmüller spaces. Consult monographs by Lehto [13] and Nag [17] for the theory of Teichmüller spaces. An orientation-preserving homeomorphism w is said to be quasiconformal if partial derivatives in the distribution sense exist and if the complex dilatation $\mu_w(z) = w_{\bar{z}}/w_z$ satisfies $\|\mu_w\|_\infty < 1$. Let

$$\mathrm{Bel}(\mathbb{D}) = \{\mu \in L^\infty(\mathbb{D}) \mid \|\mu\|_\infty < 1\}$$

be the space of such complex dilatations, which are called Beltrami coefficients. Set the group of all quasiconformal automorphisms of $\mathbb{D}$ by $\mathrm{QC}(\mathbb{D})$. By the measurable Riemann mapping theorem, for every $\mu \in \mathrm{Bel}(\mathbb{D})$, there is $w \in \mathrm{QC}(\mathbb{D})$ satisfying $\mu_w = \mu$ uniquely up to the post-composition of a Möbius transformation of $\mathbb{D}$. This gives the identification

$$\mathrm{M\ddot{o}b}(\mathbb{D}) \backslash \mathrm{QC}(\mathbb{D}) \cong \mathrm{Bel}(\mathbb{D}).$$

Every $w \in \mathrm{QC}(\mathbb{D})$ extends continuously to a quasisymmetric automorphism of $\mathbb{S} = \partial\,\mathbb{D}$. Here an orientation-preserving self-homeomorphism $g : \mathbb{S} \to \mathbb{S}$ is called quasisymmetric if there is $M \geq 1$ such that the quasisymmetric quotient $m_g(x,t)$ satisfies

$$\frac{1}{M} \leq m_g(x,t) = \frac{g(x+t)-g(x)}{g(x)-g(x-t)} \leq M$$

for every $x \in \mathbb{S} = \mathbb{R}/\mathbb{Z}$ and for every $t > 0$. Let QS be the group of all quasisymmetric automorphism of $\mathbb{S}$. We denote the boundary extension map by

$$q : \mathrm{QC}(\mathbb{D}) \to \mathrm{QS},$$

which is known to be a surjective homomorphism. The universal Teichmüller space is defined by

$$T = \mathrm{M\ddot{o}b}(\mathbb{S}) \backslash \mathrm{QS}\,.$$

Then the boundary extension q induces the Teichmüller projection $\pi : \mathrm{Bel}(\mathbb{D}) \to T$. The quotient topology of T is induced from $\mathrm{Bel}(\mathbb{D})$ by π. Actually the Teichmüller distance can be defined by using the norm on $\mathrm{Bel}(\mathbb{D})$.

A quasiconformal automorphism $w \in \mathrm{QC}(\mathbb{D})$ is called asymptotically conformal if the complex dilatation vanishes at the boundary, that is, $\mu_w(z) \to 0$ $(|z| \to 1)$. The space of all Beltrami coefficients vanishing at the boundary is denoted by $\mathrm{Bel}_0(\mathbb{D})$ and the group of all asymptotically conformal automorphisms of $\mathbb{D}$ is denoted by $\mathrm{AC}(\mathbb{D})$. On the other hand, a quasisymmetric automorphism $g \in \mathrm{QS}$ is called symmetric if the quasisymmetric quotient $m_g(x,t)$ tends to 1 as $t \to 0$ uniformly with respect to $x \in \mathbb{S}$. Note that a circle diffeomorphism is symmetric, but a symmetric automorphism is not necessarily differentiable nor bi-Lipschitz. The group of all symmetric automorphisms of $\mathbb{S}$ is denoted by Sym. Then the restriction of the boundary extension to $\mathrm{AC}(\mathbb{D})$ gives a surjective homomorphism $q : \mathrm{AC}(\mathbb{D}) \to \mathrm{Sym}$. The small universal Teichmüller space is defined by

$$T_0 = \mathrm{M\ddot{o}b}(\mathbb{S}) \backslash \mathrm{Sym} = \pi(\mathrm{Bel}_0(\mathbb{D})).$$

The universal asymptotic Teichmüller space is given by $AT = \mathrm{Sym} \backslash \mathrm{QS}$. Similarly, for an integer r with $2 \leq r \leq \infty$ or $r = 1+\alpha$ $(0 < \alpha < 1)$, we define $DT^r = \mathrm{Diff}^r_+(\mathbb{S}) \backslash \mathrm{QS}$.

Our rigidity theorems can be stated as follows.

Theorem 1. *Let* Γ *be a non-abelian, infinite subgroup of* $\mathrm{M\ddot{o}b}(\mathbb{S})$. *If* $f\Gamma f^{-1} \subset \mathrm{M\ddot{o}b}(\mathbb{S})$ *for* $f \in \mathrm{Sym}$, *then* $f \in \mathrm{M\ddot{o}b}(\mathbb{S})$.

Theorem 2. *Let* Γ *be a non-abelian, infinite subgroup of* $\mathrm{M\ddot{o}b}(\mathbb{S})$. *If* $f\Gamma f^{-1} \subset \mathrm{Diff}^{1+\alpha}_+(\mathbb{S})$ *for* $f \in \mathrm{Sym}$, *then* $f \in \mathrm{Diff}^{1+\alpha}_+(\mathbb{S})$.

As a corollary to Theorem 2 and a result by Ghys and Tsuboi [11], we see that, if Γ has a dense orbit in $\mathbb{S}$ and $f\Gamma f^{-1} \subset \mathrm{Diff}^r_+(\mathbb{S})$ $(r \geq 2)$ for $f \in \mathrm{Sym}$, then $f \in \mathrm{Diff}^r_+(\mathbb{S})$.

We mention some problems related to these theorems. Let $\Gamma \subset \mathrm{M\ddot{o}b}(\mathbb{S}) = \mathrm{PSL}(2,\mathbb{R})$ be a non-elementary Fuchsian group. The deformation space of Γ in $\mathrm{M\ddot{o}b}(\mathbb{S})$ is given as the Teichmüller space of Γ, which is defined by

$$T(\Gamma) = \mathrm{M\ddot{o}b}(\mathbb{S}) \backslash \{ f \in \mathrm{QS} \mid f\Gamma f^{-1} \subset \mathrm{M\ddot{o}b}(\mathbb{S})\} \subset T.$$

In a similar manner, the deformation space of Γ in Sym is given as the asymptotic Teichmüller space of Γ, which is defined by

$$AT(\Gamma) = \mathrm{Sym}\backslash\{f \in \mathrm{QS} \mid f\Gamma f^{-1} \subset \mathrm{Sym}\} \subset AT.$$

We also define the deformation space of Γ in $\mathrm{Diff}^r_+(\mathbb{S})$:

$$DT^r(\Gamma) = \mathrm{Diff}^r_+(\mathbb{S})\backslash\{f \in \mathrm{QS} \mid f\Gamma f^{-1} \subset \mathrm{Diff}^r_+(\mathbb{S})\} \subset DT^r.$$

Consider projections

$$\alpha : T = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash \mathrm{QS} \to AT = \mathrm{Sym} \backslash \mathrm{QS};$$
$$\theta^r : DT^r = \mathrm{Diff}^r_+(\mathbb{S})\backslash \mathrm{QS} \to AT = \mathrm{Sym} \backslash \mathrm{QS}.$$

Theorem 1 implies that the restriction $\alpha|_{T(\Gamma)} : T(\Gamma) \to AT(\Gamma)$ is injective, and Theorem 2 implies that the restriction $\theta^r|_{DT^r(\Gamma)} : DT^r(\Gamma) \to AT(\Gamma)$ is injective. Note that $\alpha|_{T(\Gamma)}$ is not surjective . More precisely, if $T(\Gamma) \neq \{[\mathrm{id}]\}$, then $\alpha T(\Gamma)\ (\cong T(\Gamma)) \subsetneqq AT(\Gamma)$ (see [15]).

Under the identification by α and θ^r, we have

$$T(\Gamma) \subset DT^r(\Gamma) \subset AT(\Gamma).$$

Then a problem asks which (or both) inclusion is strict. A theorem due to Ghys [10] implies that $T(\Gamma) = DT^r(\Gamma)$ if Γ is cocompact and $r \geq 3$.

2. The Bers embedding — Linearization of Teichmüller spaces

The proofs of Theorems 1 and 2 are carried out by using the Bers embedding of the Teichmüller space; we transfer the problems to those for the linear isometric action of Möbius transformations on certain Banach spaces.

Take any $\mu \in \mathrm{Bel}(\mathbb{D})$ and extend it to a Beltrami coefficient $\hat{\mu}$ on the Riemann sphere $\widehat{\mathbb{C}}$ by setting $\hat{\mu}(z) \equiv 0$ for $z \in \mathbb{D}^* = \hat{\mathbb{C}} - \overline{\mathbb{D}}$. Take a quasiconformal automorphism $\hat{g} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\mu_{\hat{g}} = \hat{\mu}$. The measurable Riemann mapping theorem guarantees the existence of such $\hat{g}$ and the uniqueness up to the post-composition of a Möbius transformation of $\widehat{\mathbb{C}}$.

Take the Schwarzian derivative $S_{\hat{g}} : \mathbb{D}^* \to \hat{\mathbb{C}}$ of the conformal map $\hat{g}|_{\mathbb{D}^*}$. This enables us to parametrize the marked complex projective structures on $\mathbb{D}^*$. Then, by the Nehari-Kraus theorem, $S_{\hat{g}}$ belongs to the complex Banach space

$$B(\mathbb{D}^*) = \{\varphi \in \mathrm{Hol}(\mathbb{D}^*) \mid \|\varphi\|_\infty = \sup_{z\in\mathbb{D}^*} \rho_{\mathbb{D}^*}^{-2}(z)|\varphi(z)| < \infty\},$$

where $\rho_{\mathbb{D}^*}(z) = 2/(|z|^2 - 1)$ is the hyperbolic density of $\mathbb{D}^*$. By this correspondence $\mu \mapsto S_{\hat{g}}$, a holomorphic map

$$\Phi : \mathrm{Bel}(\mathbb{D}) \to B(\mathbb{D}^*)$$

is defined, which is called the Bers projection (onto the image).

Now we have two projections: the Teichmüller projection $\pi : \mathrm{Bel}(\mathbb{D}) \to T$ and the Bers projection $\Phi : \mathrm{Bel}(\mathbb{D}) \to B(\mathbb{D}^*)$. Then we can show that $\Phi \circ \pi^{-1}$ is well-defined and injective, which defines a map $\beta : T \to B(\mathbb{D}^*)$ called the Bers embedding. Actually, β is a homeomorphism onto the image $\beta(T)$ and $\beta(T)$ is a bounded domain in $B(\mathbb{D}^*)$.

The group QS of all quasisymmetric automorphisms of $\mathbb{S}$ acts on $T = \text{Möb}(\mathbb{S})\backslash \text{QS}$ canonically:

$$([f], g) \in T \times \text{QS} \mapsto g^*[f] := [f \circ g] \in T.$$

This is regarded as the mapping class group of the universal Teichmüller space T. The action is faithful and transitive. Moreover, this is isometric with respect to the Teichmüller distance on T and biholomorphic with respect to the complex structure on T. The isotropy subgroup of QS at the origin $[\text{id}] \in T$ coincides with $\text{Möb}(\mathbb{S})$. The condition that $g \in G$ fixes $[f] \in T$, that is, $g^*[f] = [f]$, can be written as $[fgf^{-1}] = [\text{id}]$, and this is equivalent to that $fgf^{-1} \in \text{Möb}(\mathbb{S})$.

Every element $\gamma \in \text{Möb}(\mathbb{S})$ acts on $B(\mathbb{D}^*)$ linear isometrically through the Bers embedding β. This means that, for $\beta([f]) = \varphi \in \beta(T)$, the image $\beta(\gamma^*[f])$ is represented by

$$(\gamma^*\varphi)(z) = \varphi(\gamma(z))\gamma'(z)^2,$$

where we regard γ is an element of $\text{Möb}(\mathbb{D}^*)$ and $\gamma^*\varphi$ is the pull-back of φ as a quadratic differential form. Clearly this action extends to $B(\mathbb{D}^*)$ and satisfies $\|\gamma^*\varphi\|_\infty = \|\varphi\|_\infty$. More generally, if $g \in \text{QS}$ has a fixed point $[f] \in T$, then the action of g on $T = \beta(T)$ is conjugate to the linear isometric action of fgf^{-1} on $\beta(T) \subset B(\mathbb{D})$ under the base point change automorphism $R_{[f]} : T \to T$ given by $[g] \mapsto [g \circ f^{-1}]$.

To prove Theorem 1, we need a characterization of symmetric automorphisms in Sym and the small universal Teichmüller space $T_0 = \text{Möb}(\mathbb{S})\backslash \text{Sym}$. We set a Banach subspace of $B(\mathbb{D}^*)$ consisting of the elements of vanishing at the boundary by

$$B_0(\mathbb{D}^*) = \{\varphi \in B(\mathbb{D}^*) \mid \lim_{|z|\to 1} \rho_{\mathbb{D}^*}^{-2}(z)|\varphi(z)| = 0\}.$$

The following result appeared in Gardiner and Sullivan [9], which were attributed to Fehlmann [8] and Becker and Pommerenke [4].

Lemma 3. *For a quasisymmetric automorphism $g \in \text{QS}$, the following conditions are equivalent:*

(1) *g belongs to* Sym*;*
(2) *there is $\mu \in \text{Bel}_0(\mathbb{D})$ such that $\pi(\mu) = [g] \in T$;*
(3) *$\beta([g]) \in \beta(T)$ is in $B_0(\mathbb{D}^*)$.*

Proof of Theorem 1. Set $\varphi = \beta([f])$, which belongs to the subspace $B_0(\mathbb{D}^*)$ by Lemma 3. The condition $f\Gamma f^{-1} \subset \text{Möb}(\mathbb{S})$ is equivalent to that $\gamma^*\varphi = \varphi$ for every $\gamma \in \Gamma$, where

$$(\gamma^*\varphi)(z) = \varphi(\gamma(z))\gamma'(z)^2 \quad (\gamma \in \text{Möb}(\mathbb{D}^*)).$$

Then

$$\rho_{\mathbb{D}^*}^{-2}(z)|\varphi(z)| = \rho_{\mathbb{D}^*}^{-2}(z)|\gamma^*\varphi(z)| = \rho_{\mathbb{D}^*}^{-2}(\gamma z)|\varphi(\gamma z)|,$$

which tends to 0 as $\gamma \in \Gamma$ runs over a hyperbolic cyclic subgroup because $\varphi \in B_0(\mathbb{D}^*)$. Thus $\varphi(z) \equiv 0$, which means that $[f] = [\text{id}]$, and equivalently $f \in \text{Möb}(\mathbb{S})$. □

The proof of Theorem 2 is given by modification and generalization of the arguments for Theorem 1. First we give a characterization of $\mathrm{Diff}^{1+\alpha}$ analogously to Lemma 3. Here are spaces we have to deal with in this situation:

$$\begin{aligned}&T_0^\alpha = \mathrm{M\ddot{o}b}(\mathbb{S})\backslash \mathrm{Diff}_+^{1+\alpha}(\mathbb{S});\\ &\mathrm{Bel}_0^\alpha(\mathbb{D}) = \{\mu \in \mathrm{Bel}_0(\mathbb{D}) \mid \|\mu\|_{\infty,\alpha} = \mathrm{ess.\,sup}_{z\in\mathbb{D}}\, \rho_{\mathbb{D}}^\alpha(z)|\mu(z)| < \infty\};\\ &B_0^\alpha(\mathbb{D}^*) = \{\varphi \in B_0(\mathbb{D}^*) \mid \|\varphi\|_{\infty,\alpha} = \sup_{z\in\mathbb{D}^*} \rho_{\mathbb{D}^*}^{-2+\alpha}(z)|\varphi(z)| < \infty\}.\end{aligned}$$

As usual $\rho_{\mathbb{D}}(z) = 2/(1-|z|^2)$ is the hyperbolic density on $\mathbb{D}$.

Lemma 4. *Let α be a constant with $0 < \alpha < 1$. For a quasisymmetric automorphism $g \in \mathrm{QS}$, the following conditions are equivalent:*

(1) *g belongs to $\mathrm{Diff}^{1+\alpha}(\mathbb{S})$;*
(2) *there is $\mu \in \mathrm{Bel}_0^\alpha(\mathbb{D})$ such that $\pi(\mu) = [g] \in T$;*
(3) *$\beta([g]) \in \beta(T)$ is in $B_0^\alpha(\mathbb{D}^*)$.*

A proof of this lemma will be given in [16], based on previous results on quasiconformal extension of univalent functions by Carleson [5], Pommerenke and Warschawski [19], Becker [3] and Anderson, Becker and Lesley [1]. In a similar flavor, we will also have the following lemma.

Lemma 5. *Let g belong to $\mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$. For every $f \in \mathrm{Sym}$ and for any $\alpha' < \alpha$,*

$$\beta([g\circ f]) \in \beta([f]) + B_0^{\alpha'}(\mathbb{D}^*).$$

Moreover, if $f \in \mathrm{Diff}_+^{1+\alpha'}(\mathbb{S})$, then

$$\beta([g\circ f]) \in \beta([f]) + B_0^{\alpha}(\mathbb{D}^*).$$

Proof of Theorem 2. Set $\varphi = \beta([f])$, which belongs to $B_0(\mathbb{D}^*)$. The condition $f\Gamma f^{-1} \subset \mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$ is equivalent to the existence of $g \in \mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$ with $[f\circ\gamma] = [g\circ f]$ for every $\gamma \in \Gamma$. Then Lemma 5 yields that $\gamma^*\varphi - \varphi \in B_0^{\alpha'}(\mathbb{D}^*)$.

Choose any hyperbolic element $\gamma \in \Gamma$ and set

$$\psi = \gamma^*\varphi - \varphi \in B_0^{\alpha'}(\mathbb{D}^*).$$

Then, by similar arguments for proving Theorem 1, we have:

Claim. $\varphi = -\sum_{i=0}^\infty (\gamma^*)^i\psi = \sum_{i=1}^\infty (\gamma^*)^{-i}\psi$.

Using this representation of φ in terms of $\psi \in B_0^{\alpha'}(\mathbb{D}^*)$, we see that $\varphi = \beta([f])$ also belongs to $B_0^{\alpha'}(\mathbb{D}^*)$ by the following lemma:

Lemma 6. *If $\varphi = -\sum_{i=0}^\infty (\gamma^*)^i\psi = \sum_{i=1}^\infty (\gamma^*)^{-i}\psi$ for a hyperbolic element $\gamma \in \mathrm{M\ddot{o}b}(\mathbb{D}^*)$ and if $\psi \in B_0^\alpha(\mathbb{D}^*)$, then $\varphi \in B_0^\alpha(\mathbb{D}^*)$.*

The condition $\varphi \in B_0^{\alpha'}(\mathbb{D}^*)$ implies that $f \in \mathrm{Diff}_+^{1+\alpha'}(\mathbb{S})$ by Lemma 4. Having this, we repeat the same argument as above from the beginning. At first, Lemma 5 yields that $\gamma^*\varphi - \varphi \in B_0^{\alpha}(\mathbb{D}^*)$ in this turn. Then, we see that $\varphi \in B_0^{\alpha}(\mathbb{D}^*)$, which means that $f \in \mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$ by Lemma 4. □

3. Conjugation of a group of circle diffeomorphisms to a Möbius group

As an application of our rigidity theorem, we prove a certain conjugation problem of a group of circle diffeomorphisms. We note that there is a related work by Navas [18].

Definition. A Beltrami coefficient $\mu \in \mathrm{Bel}(\mathbb{D})$ is p-integrable if

$$\|\mu\|_p^p = \int_{\mathbb{D}} |\mu(z)|^p \rho_{\mathbb{D}}^2(z)\,dxdy < \infty.$$

The space of all p-integrable Beltrami coefficients on $\mathbb{D}$ is denoted by $\mathrm{Ael}^p(\mathbb{D})$.

Theorem 7. *Let G be a non-abelian, infinite subgroup of $\mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$ with $\alpha > 1/2$. Then $f^{-1}Gf \subset \mathrm{M\ddot{o}b}(\mathbb{S})$ for some $f \in \mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$ if and only if G satisfies both of the following uniform boundedness conditions for all $g \in G$:*

(1) $k_2(g) = \inf_{q(\tilde g)=g} \|\mu_{\tilde g}\|_2 \leq\ ^\exists C < \infty$;
(2) $k_\infty(g) = \inf_{q(\tilde g)=g} \|\mu_{\tilde g}\|_\infty \leq\ ^\exists c < 1$.

Remark. We can replace the above infimum taken over all quasiconformal extension $\tilde g : \mathbb{D} \to \mathbb{D}$ of g with the complex dilatation $\mu_{\tilde g}$ of the conformal baricentric extension $\tilde g$ of g introduced by Douady and Earle [7].

We will find a subspace of T where $\mathrm{Diff}_+^{1+\alpha}(\mathbb{S}) \subset \mathrm{QS}$ acts.

Definition. A quasisymmetric automorphism $g : \mathbb{S} \to \mathbb{S}$ belongs to Sym^p for $p \geq 2$ if g has a quasiconformal extension $\tilde g : \mathbb{D} \to \mathbb{D}$ whose complex dilatation $\mu_{\tilde g}$ belongs to $\mathrm{Ael}^p(\mathbb{D})$. The p-integrable Teichmülcer space is defined by

$$T^p = \mathrm{M\ddot{o}b}(\mathbb{S}) \backslash \mathrm{Sym}^p \subset T.$$

We also consider the space of all p-integrable holomorphic functions:

$$A^p(\mathbb{D}^*) = \{\varphi \in \mathrm{Hol}(\mathbb{D}^*) \mid \|\varphi\|_p^p = \int_{\mathbb{D}^*} \rho_{\mathbb{D}^*}^{2-2p}(z)|\varphi(z)|^p\,dxdy < \infty\}.$$

The inclusion relation $A^p(\mathbb{D}^*) \subset B_0(\mathbb{D}^*)$ is known. Cui [6] and Guo [12] proved that the Bers embedding of T^p satisfies $\beta(T^p) = \beta(T) \cap A^p(\mathbb{D}^*)$. This in particular implies that

$$\mathrm{Sym}^p \subset \mathrm{Sym}; \quad T^p \subset T_0.$$

Moreover, if $p\alpha > 1$, then $B_0^{\alpha}(\mathbb{D}^*) \subset A^p(\mathbb{D}^*)$. This also shows that $\mathrm{Diff}_+^{1+\alpha}(\mathbb{S}) \subset \mathrm{Sym}^p$ in this case.

For every point $\tau = [f] \in T$, the base point change automorphism $R_\tau : T \to T$ is defined by $[g] \mapsto [g \circ f^{-1}]$ as before. It can be proved that if $\tau \in T^p$ then R_τ preserves T^p. The canonical coordinate of T^p at each $\tau \in T^p$ is given by

$$\beta_\tau = \beta \circ R_\tau : T^p \to \beta(T) \cap A^p(\mathbb{D}^*).$$

Definition. The p-Weil-Petersson metric d^p_{WP} on T^p is the norm $\|\cdot\|_p$ on the tangent space $T_\tau(T^p)$ at each $\tau \in T^p$, which is identified with $A^p(\mathbb{D}^*)$ by the canonical coordinate β_τ.

It is clear from definition that $\mathrm{Sym}^p \subset \mathrm{QS}$ acts on (T^p, d^p_{WP}) isometrically.

For $p = 2$, Cui [6] proved that (T^2, d^2_{WP}) is complete and contractible. Takhtajan and Teo [20] proved later that (T^2, d^2_{WP}) is negatively curved. From these properties, we see that (T^2, d^2_{WP}) is a CAT(0) space.

By a property of CAT(0) space, if the orbit of G in T^2 is bounded with respect to d^2_{WP}, then G has a fixed point $[f]$ in T^2 (see Ballmann [2]). This implies that there exists a symmetric automorphism $f \in \mathrm{Sym}^2 \subset \mathrm{Sym}$ such that $f^{-1}Gf \subset \text{Möb}(\mathbb{S})$.

Assumptions (1) and (2) of Theorem 7 implies that G has a bounded orbit. This is seen from the following lemma.

Lemma 8. *For any $g \in \mathrm{Sym}^p$, the p-Weil-Petersson distance satisfies*

$$d^p_{WP}([\mathrm{id}], [g]) \leq c\, k_p(g),$$

where c is a constant depending only on $k_\infty(g)$.

Proof of Theorem 7. We have obtained $f \in \mathrm{Sym}$ satisfying $f^{-1}Gf \subset \text{Möb}(\mathbb{S})$. Then, by applying Theorem 2, we see that $f \in \mathrm{Diff}^{1+\alpha}_+(\mathbb{S})$. Conversely, estimates of k_2 and k_∞ under the composition yield that if $f^{-1}Gf \subset \text{Möb}(\mathbb{S})$ for $f \in \mathrm{Diff}^{1+\alpha}_+(\mathbb{S})$ then G satisfies (1) and (2). Actually the estimate for k_∞ is well-known. The estimate for k_2 can be found in [6] and [20]. □

Theorem 7 can be generalized to some extend for an arbitrary $\alpha > 0$:

Theorem 9. *If a non-abelian, infinite subgroup $G \subset \mathrm{Diff}^{1+\alpha}_+(\mathbb{S})$ for $\alpha > 0$ satisfies $k_p(g) \leq \varepsilon_p$ for all $g \in G$ and for a sufficiently small constant $\varepsilon_p > 0$ depending only on p with $p\alpha > 1$, then there exists $f \in \mathrm{Diff}^{1+\alpha}_+(\mathbb{S})$ such that $f^{-1}Gf \subset \text{Möb}(\mathbb{S})$.*

Note that $\varepsilon_p > 0$ can be chosen so that $k_p(g) \leq \varepsilon_p$ implies $k_\infty(g) \leq c$ for all $g \in G$ and for some $c < 1$. Such a group G satisfying condition (2) of Theorem 7 is called uniformly quasisymmetric.

Markovic [14] proved that a uniformly quasisymmetric group $G \subset \mathrm{QS}$ is conjugate into $\text{Möb}(\mathbb{S})$ by a quasisymmetric automorphism $f_0 \in \mathrm{QS}$. This theorem guarantees the existence of a fixed point $\tau_0 = [f_0]$ of G in T. Then G acts on the Banach space $B(\mathbb{D}^*)$ linear isometrically through the Bers embedding $\beta_{\tau_0} = \beta \circ R_{\tau_0}$.

Proof of Theorem 9. The condition that $k_p(g)$ are uniformly bounded implies that the linear isometric action of G on $B(\mathbb{D}^*)$ preserves the affine subspace $\varphi_0 + A^p(\mathbb{D}^*)$ invariant and the orbit of $\varphi_0 = \beta_{\tau_0}(o)$ is bounded with respect to the norm of $A^p(\mathbb{D}^*)$. This gives a fixed point of G in $\varphi_0 + A^p(\mathbb{D}^*)$ since the Banach space $A^p(\mathbb{D}^*)$ is uniformly convex.

Moreover, by the assumption $k_p(g) \leq \varepsilon_p$, the diameter of the orbit of φ_0 is sufficiently small and hence the fixed point of G is in

$$\beta_{\tau_0}(T^p) \subset \beta(T) \cap (\varphi_0 + A^p(\mathbb{D}^*)).$$

Having the new fixed point $\tau = [f] \in T^p$, we perform the same argument as in the case where $\tau \in T^2$ before. Then Theorem 2 shows that $f \in \mathrm{Diff}_+^{1+\alpha}(\mathbb{S})$, which conjugates G into $\mathrm{M\ddot{o}b}(\mathbb{S})$. □

Remark. If the metric space (T^p, d^p_{WP}) has a property that every isometry group with a bounded orbit has a fixed point, then the assumption of Theorem 9 can be improved.

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