Teichmüller spaces of invariant symmetric structures on the circle

Katsuhiko Matsuzaki
Department of Mathematics, Okayama University

§1. Past: Symmetric groups that are not the symmetric conjugates of Fuchsian groups

A *quasiconformal group* is a discrete group of quasiconformal automorphisms of the unit disk Δ whose maximal dilatations are uniformly bounded. A quasisymmetric automorphism of the unit circle S^1 is the boundary extension of a quasiconformal automorphism of Δ. A *quasisymmetric group* is a discrete group of quasisymmetric automorphisms of S^1 whose quasisymmetric constants are uniformly bounded. It is clear that the boundary extension of a quasiconformal group to S^1 is a quasisymmetric group. Due to Sullivan [16] and Tukia [17], every quasiconformal group is conjugate to a conformal group (Fuchsian group) by a quasiconformal homeomorphism $\Delta \to \Delta$.

On the other hand, since there is no canonical extension of the homeomorphisms $S^1 \to S^1$ to those of Δ preserving the group structure, it was difficult to see that every quasisymmetric group is conjugate to a Fuchsian group by a quasisymmetric homeomorphism $S^1 \to S^1$, or equivalently, every quasisymmetric group is the boundary extension of a quasiconformal group. Recently, this is shown to be true by Markovic [13], based on a famous result by Tukia [18], Gabai [9] and Casson and Jungreis [3] that a quasisymmetric group, which is a convergence group in the sense of Gehring and Martin [12], is conjugate to a Fuchsian group by a topological homeomorphism $S^1 \to S^1$.

An *asymptotically conformal group* is a quasiconformal group whose elements are asymptotically conformal automorphism of Δ. A symmetric automorphism is the extension of an asymptotically conformal automorphism to S^1, which was originally introduced by Gardiner and Sullivan [11]. A *symmetric group* is a quasisymmetric group whose elements are symmetric automorphisms of Δ. It is clear that the boundary extension of an asymptotically conformal group to S^1 is a symmetric group. However, we do not know whether the converse is true or not.

In this section, we consider an analogous problem to the above context; whether a symmetric group is conjugate to a Fuchsian group by a symmetric homeomorphism $S^1 \to \partial S^1$ or not. Our answer is negative: *To every infinite non-rigid Fuchsian group possibly with torsion, there exists a quasisymmetric conjugate symmetric group that is not conjugate to any Fuchsian group by a symmetric homeomorphism.* This assertion is a special case of the following theorem, which is the main result in this section.

Theorem 1.1. *Let $G \subset \mathrm{Conf}(R)$ be a group of conformal automorphisms of a hyperbolic Riemann surface R. Assume that G admits a non-trivial homogenous quasi-homomorphism $\phi : G \to \mathbb{R}$ and that the orbifold R/G is not $(0,3)$-type. Then there exists a quasiconformal homeomorphism $f : R \to R'$ such that $G_f = fGf^{-1}$ is a group of asymptotically conformal automorphisms of R' but there exists no asymptotically conformal homeomorphism $h : R' \to R''$ such that $hG_f h^{-1} \subset \mathrm{Conf}(R'')$.*

Here, a map $\phi : G \to \mathbb{R}$ is said to be a *quasi-homomorphism* if there exists a constant $D > 0$ such that $|\phi(g_1 g_2) - \phi(g_1) - \phi(g_2)| \leq D$ for any g_1 and g_2 in G. Moreover, it is *homogenous* if $\phi(g^n) = n\phi(g)$ for any $g \in G$ and any $n \in \mathbb{Z}$. A *hyperbolic group* is a finitely generated group that is a Gromov hyperbolic space with respect to the word metric. It is known that every cofinite area Fuchsian group is hyperbolic. The following fact is crucial in our arguments, which can be found in Epstein and Fujiwara [5].

Proposition 1.2. *For every infinite hyperbolic group G, there exists a non-trivial homogenous quasi-homomorphism $\phi : G \to \mathbb{R}$.*

The above theorem can be paraphrased as a statement on the existence of a fixed point of the isometric action of G on a closed subspace of the Teichmüller space, which is a fiber over the asymptotic Teichmüller space. See Section 3.

We can prove that every infinite Fuchsian group G (possibly infinitely generated and with torsion) admits a non-trivial homogenous quasi-homomorphism. Note that, the orbifold $R = \Delta/G$ often admits an infinite cyclic cover and hence there exists a surjective homomorphism $\phi : G \to \mathbb{Z}$. However, there remains the other possibility and we are concerned with such cases.

Lemma 1.3. *For every infinite Fuchsian group G, there exists a non-trivial homogenous quasi-homomorphism $\phi : G \to \mathbb{R}$.*

First we deal with a discrete model. For a countable group G in general, we define a topological linear space $L(G)$ of all real-valued functions $\xi : G \to \mathbb{R}$ endowed with the supremum norm $\|\xi\|_\infty = \sup_{g \in G} |\xi(g)|$. Also we consider a subspace $L_0(G)$ that consists of all elements $\xi \in L(G)$ vanishing at infinity. Namely, $\xi \in L(G)$ belongs to $L_0(G)$ if, for any $\varepsilon > 0$, there exists a finite subset $V \subset G$ such that $\sup_{g \in G - V} |\xi(g)| < \varepsilon$.

A canonical left-action of G on $L(G)$ is defined by $(\gamma \cdot \xi)(g) := \xi(\gamma g)$ $(g \in G)$ for any $\gamma \in G$ and for any $\xi \in L(G)$. This action is isometric with respect to the norm on $L(G)$.

Let S be an invertible generating system of G. We number the elements in S as $\{g_1^{\pm 1}, g_2^{\pm 1}, g_3^{\pm 1}, \dots\}$. For each generator $g_n^{\pm 1}$ $(n \in \mathbb{N})$, we give a weight n. The weighted word length $\ell(g)$ for an element $g \in G$ with respect to S is defined by the minimum of the sum of their weights when we represent g as a word of the generators. This is equal to the weighted path metric between $g \in G$ and the identity on the Cayley graph of G with respect to S. Here each edge assigned for $g_n^{\pm 1}$ has length n. The triangle inequality $\ell(g_1 g_2) \leq \ell(g_1) + \ell(g_2)$ is satisfied for any g_1 and g_2 in G. By this weighted word length ℓ, G is regarded as a locally compact (finite) topological space. The definition of vanishing at infinity of $\xi \in L(G)$ is equivalent to saying that $\xi(g)$ converges to 0 as g tends to the point at infinity of G.

Let $\eta : \mathbb{R} \to [0,1]$ be a piecewise-linear continuous function defined by

$$\eta(x) = \begin{cases} 0 & (x \le -1) \\ (x+1)/2 & (-1 \le x \le 1) \\ 1 & (1 \le x) \end{cases}.$$

If x and x' satisfy either $|x - x'| < 2\varepsilon$, $\min\{x, x'\} \ge 1$ or $\max\{x, x'\} \le -1$ for an $\varepsilon > 0$, then $|\eta(x) - \eta(x')| < \varepsilon$.

Lemma 1.4. *Suppose that a countable group G admits a non-trivial homogenous quasi-homomorphism $\phi : G \to \mathbb{R}$ with a normalization condition $\phi(a) = 1$ for some $a \in G$. For an invertible generating system $S = \{g_1^{\pm 1}, g_2^{\pm 1}, g_3^{\pm 1}, \ldots\}$ of G with $g_1 = a$, let $\ell(g)$ be the weighted word length for $g \in G$ with respect to S. Let $\eta : \mathbb{R} \to [0,1]$ be the piecewise-linear function given above. Define a function $\xi : G \to [0,1]$ by*

$$\xi(g) = \eta\left(\frac{\phi(g)}{\ell(g)}\right) \qquad (g \in G).$$

Then the following properties are satisfied:

(1) *For every $\gamma \in G$, the function $\gamma \cdot \xi - \xi$ belongs to $L_0(G)$;*
(2) *For every $g \in G$, the values $(a^n \cdot \xi)(g)$ converge to 1 as $n \to \infty$ and converge to 0 as $n \to -\infty$.*

Proof. For property (1), we prove that, for any small ε $(0 < \varepsilon < 1/2)$, there exists ℓ_0 such that $|(\gamma \cdot \xi - \xi)(g)| < \varepsilon$ for all $g \in G$ with $\ell(g) \ge \ell_0$. Since ϕ is a quasi-homomorphism, there exists a constant $D \ge 0$ such that

$$|\phi(\gamma g) - \phi(g)| \le |\phi(\gamma)| + D$$

for any $g \in G$. Then we show that

$$\ell_0 := \frac{2\ell(\gamma) + |\phi(\gamma)| + D}{2\varepsilon}$$

is appropriate for proving the assertion above. Having

$$(\gamma \cdot \xi - \xi)(g) = \eta\left(\frac{\phi(\gamma g)}{\ell(\gamma g)}\right) - \eta\left(\frac{\phi(g)}{\ell(g)}\right),$$

we consider the function η for $x_g := \phi(\gamma g)/\ell(\gamma g)$ and $x'_g := \phi(g)/\ell(g)$. Then it suffices to see that either $|x_g - x'_g| < 2\varepsilon$, $\min\{x_g, x'_g\} \ge 1$ or $\max\{x_g, x'_g\} \le -1$ is satisfied for every $g \in G$ with $\ell(g) \ge \ell_0$.

Suppose that neither $\min\{x_g, x'_g\} \ge 1$ nor $\max\{x_g, x'_g\} \le -1$ is satisfied for some $g \in G$ with $\ell(g) \ge \ell_0$. Then this g holds either $|x_g| < 1$ or $|x'_g| < 1$. Indeed, if $x_g \ge 1$ and $-1 \ge x'_g$, then

$$\phi(\gamma g) - \phi(g) \ge \ell(\gamma g) + \ell(g) \ge \ell(g) \ge \ell_0.$$

However, the left-hand side is bounded by $|\phi(\gamma)| + D$, which contradicts the definition of ℓ_0. Similarly, we can rule out the case where $x'_g \geq 1$ and $-1 \geq x_g$.

In the case where $|x_g| = |\phi(\gamma g)|/\ell(\gamma g) < 1$ for $g \in G$ with $\ell(g) \geq \ell_0$, we have

$$\begin{aligned}|x_g - x'_g| &= \frac{|\phi(\gamma g)\ell(g) - \phi(g)\ell(\gamma g)|}{\ell(\gamma g)\ell(g)} \\ &\leq \frac{|\phi(\gamma g)| \cdot |\ell(g) - \ell(\gamma g)| + \ell(\gamma g) \cdot |\phi(\gamma g) - \phi(g)|}{\ell(\gamma g)\ell(g)} \\ &< \frac{|\ell(g) - \ell(\gamma g)| + |\phi(\gamma g) - \phi(g)|}{\ell(g)} \\ &\leq \frac{\ell(\gamma) + |\phi(\gamma)| + D}{\ell_0} \leq 2\varepsilon.\end{aligned}$$

Similar calculation can be applied to the case where $|x'_g| < 1$. Thus we complete the proof of (1).

Property (2) is shown as follows. Consider $(a^n \cdot \xi)(g) = \eta(\phi(a^n g)/\ell(a^n g))$. Since ϕ is a homogenous quasi-homomorphism and $\phi(a) = 1$, we see that

$$\begin{aligned}n + \phi(g) - D &= \phi(a^n) + \phi(g) - D \\ &\leq \phi(a^n g) \\ &\leq \phi(a^n) + \phi(g) + D = n + \phi(g) + D.\end{aligned}$$

Similarly, since $\ell(a^{\pm 1}) = 1$, we have $|n| - \ell(g) \leq \ell(a^n g) \leq |n| + \ell(g)$. And since η is a continuous increasing function, we conclude that

$$\begin{aligned}\lim_{n \to +\infty} (a^n \cdot \xi)(g) &\geq \lim_{n \to +\infty} \eta\left(\frac{n + \phi(g) - D}{n + \ell(g)}\right) = \eta(1) = 1; \\ \lim_{n \to -\infty} (a^n \cdot \xi)(g) &\leq \lim_{n \to -\infty} \eta\left(\frac{n + \phi(g) + D}{-n - \ell(g)}\right) = \eta(-1) = 0.\end{aligned}$$

This shows property (2). □ [Lemma 1.4]

Now we will move our arguments to the stage on Teichmüller spaces. Let $\mathrm{Belt}(R)$ be the Banach space of all Beltrami differentials μ on a Riemann surface R endowed with the supremum norm. Let $M(R)$ be the unit ball of $\mathrm{Belt}(R)$. The Teichmüller projection $\pi : M(R) \to T(R)$ is defined by $\pi(\mu) = [f_\mu]$ where $[f_\mu] \in T(R)$ is a Teichmüller class of the quasiconformal homeomorphism f_μ of R having the complex dilatation μ. The Teichmüller projection π is a holomorphic split submersion.

We say that a quasiconformal homeomorphism f_μ of R is asymptotically conformal if the Beltrami differential μ vanishes at infinity. The asymptotic Teichmüller space $AT(R)$ is the set of all asymptotic equivalence classes of quasiconformal homeomorphisms of R and the asymptotic equivalence is defined similarly to the Teichmüller equivalence by using asymptotically conformal homeomorphisms. This is a quotient space of the Teichmüller space $T(R)$ and the quotient map $\alpha : T(R) \to AT(R)$ is holomorphic with respect to the complex structure endowed with $AT(R)$. The asymptotic Teichmüller projection $\hat{\pi} : M(R) \to AT(R)$ is defined by $\hat{\pi} = \alpha \circ \pi$.

Suppose that R admits a group G of conformal automorphisms. Let $\mathrm{Belt}(R,G)$ denote the subspace of $\mathrm{Belt}(R)$ consisting of all Beltrami differentials invariant under G and $M(R,G)$ its unit ball. The closed submanifold $\pi(M(R,G))$ in $T(R)$ can be identified with the Teichmüller space $T(R/G)$ for the orbifold R/G and it is denoted by this notation.

Let $\check{\mu}$ be an arbitrary Beltrami differential on the orbifold $\check{R} = R/G$. Take a disk $\check{W}$ in $\check{R}$ avoiding cone singularities and consider the restriction $\check{\mu}|_{\check{W}}$ of $\check{\mu}$ to $\check{W}$. We lift $\check{\mu}|_{\check{W}}$ to R and obtain a Beltrami differential μ_0 belonging to $\mathrm{Belt}(R,G)$. We also consider the lifts of $\check{W}$ to R, which can be represented by the disjoint union $\sqcup_{g\in G} W_g$, where $W_g = g(W_1)$ for some lift W_1 biholomorphically equivalent to $\check{W}$.

Using the function $\xi \in L(G)$ given in Lemma 1.4, we define a Beltrami differential μ on R by

$$\mu(z)\frac{d\bar{z}}{dz} = \sum_{g\in G} \xi(g) 1_{W_g}(z)\mu_0(z)\frac{d\bar{z}}{dz},$$

where $1_W(z)$ is the characteristic function of W on R.

For all $t \in \mathbb{R}$ with $\|t\mu_0\| < 1$, we consider the Beltrami differential $t\mu$ and define a curve $p(t)$ in the Teichmüller space $T(R)$, where $p(t) = \pi(t\mu)$ is the Teichmüller class determined by the quasiconformal homeomorphism $f_{t\mu} : R \to R_{p(t)}$ having the complex dilatation $t\mu$. Also define $G_{t\mu} = f_{t\mu} G f_{t\mu}^{-1}$, which is a group of quasiconformal automorphisms of the Riemann surface $R_{p(t)}$.

Lemma 1.5. *Every quasiconformal automorphism in the group $G_{t\mu}$ is asymptotically conformal.*

Proof. Set $f = f_{t\mu}$ in brief. For every $\gamma \in G$, we consider the complex dilatation $\mu_{f\gamma f^{-1}}$ of $f\gamma f^{-1} \in G_{t\mu}$. It satisfies

$$|\mu_{f\gamma f^{-1}}(f(z))| = \frac{|\mu_{f\gamma}(z) - \mu_f(z)|}{|1 - \overline{\mu_{f\gamma}(z)}\mu_f(z)|} \leq \frac{|\mu_{f\gamma}(z) - \mu_f(z)|}{1 - \|t\mu_0\|_\infty^2}.$$

Here we have

$$\begin{aligned}
\mu_f(z) &= \sum_{g\in G} \xi(g) 1_{W_g}(z) t\mu_0(z); \\
\mu_{f\gamma}(z) &= \mu_f(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} \\
&= \sum_{g\in G} \xi(g) 1_{W_g}(\gamma(z)) t\mu_0(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} \\
&= \sum_{g\in G} (\gamma\cdot\xi)(g) 1_{W_g}(z) t\mu_0(z).
\end{aligned}$$

Hence the numerator of the right side term in the above inequality is estimated as

$$|\mu_{f\gamma}(z) - \mu_f(z)| \leq \sum_{g\in G} |(\gamma\cdot\xi - \xi)(g)| 1_{W_g}(z) \|t\mu_0\|_\infty.$$

When $f(z)$ tends to the point at infinity of $R_{p(t)}$, so does z. If $z \in W_g$ and z tends to the point at infinity, then such $g \in G$ tends to the point at infinity of G. Since $\gamma \cdot \xi - \xi$ vanishes at infinity by property (1) in Lemma 1.4, this implies that $\mu_{f\gamma f^{-1}}$ also vanishes at infinity. Therefore $f\gamma f^{-1}$ is asymptotically conformal. $\square$ [Lemma 1.5]

For a Riemann surface R, let $N(R)$ denote the subspace of $\mathrm{Belt}(R)$ consisting of all infinitesimally trivial Beltrami differentials. To define this space precisely, let $Q(R)$ be the Banach space of all integrable holomorphic quadratic differentials φ on R with the norm $\|\varphi\|_1 = \int_R |\varphi|$. Then the tangent space of $T(R)$ at the base point o is identified with the dual space $Q(R)^*$. Each element $\mu \in \mathrm{Belt}(R)$ induces a bounded linear functional $v_\mu \in Q(R)^*$ by $v_\mu(\varphi) = \int \mu\varphi$. We say that $\mu \in \mathrm{Belt}(R)$ is *infinitesimally trivial* if $v_\mu = 0$, that is, $\int \mu\varphi = 0$ for every $\varphi \in Q(R)$. For the Teichmüller projection $\pi : M(R) \to T(R)$, the kernel of the derivative $d\pi_0$ at the origin is coincident with $N(R)$.

A *degenerating sequence* is a sequence $\{\varphi_n\} \subset Q(R)$ such that $\|\varphi_n\|_1 = 1$ and φ_n converge locally uniformly to zero. We say that $\mu \in \mathrm{Belt}(R)$ is *infinitesimally asymptotically trivial* if $\lim_{n\to\infty} |v_\mu(\varphi_n)| = 0$ for every degenerating sequence $\{\varphi_n\} \subset Q(R)$. Let $\hat{N}(R)$ denote the subspace of $\mathrm{Belt}(R)$ consisting of all infinitesimally asymptotically trivial Beltrami differentials. For the asymptotic Teichmüller projection $\hat{\pi} : M(R) \to AT(R)$, the kernel of the derivative $d\hat{\pi}_0$ at the origin is coincident with the Banach subspace $\hat{N}(R) \subset \mathrm{Belt}(R)$. This is implicitly given in [6] and [10].

It is clear that $N(R)$ is contained in $\hat{N}(R)$. Actually, for the Banach subspace $\mathrm{Belt}_0(R) \subset \mathrm{Belt}(R)$ consisting of all Beltrami differentials vanishing at infinity, we know that $\hat{N}(R) = N(R) + \mathrm{Belt}_0(R)$. Here we say that $\mu \in \mathrm{Belt}(R)$ vanishes at infinity if, for every $\varepsilon > 0$, there exists a compact subset $V \subset R$ such that $\operatorname{ess.sup}_{z\in R-V} |\mu(z)| < \varepsilon$.

Consider a Riemann surface R having a group G of conformal automorphisms. Assume that the orbifold $\check{R} = R/G$ is not a sphere with three singular points (punctures or cone points). This is equivalent to assuming that there exists a non-trivial bounded integrable holomorphic quadratic differential ψ on $\check{R}$. For the hyperbolic density $\rho_{\check{R}}$ on $\check{R}$, the harmonic Beltrami differential for ψ is given by $\rho_{\check{R}}^{-2}\overline{\psi}$. For a disk $\check{W}$ in $\check{R}$ avoiding cone singularities, the Beltrami differential $\check{\mu}$ is defined by the restriction of $\rho_{\check{R}}^{-2}\overline{\psi}$ to $\check{W}$. We lift $\check{\mu}$ to R and obtain the Beltrami differential $\mu_0 \in \mathrm{Belt}(R, G)$. This does not belong to $N(R)$ because $\int_{\check{R}} \check{\mu}\psi > 0$. Then we define a Beltrami differential

$$\mu(z)\frac{d\bar{z}}{dz} = \sum_{g\in G} \xi(g) 1_{W_g}(z)\mu_0(z)\frac{d\bar{z}}{dz}$$

on R as before.

Lemma 1.6. *The Beltrami differential* $\mu \in \mathrm{Belt}(R)$ *does not belong to the direct sum* $\mathrm{Belt}(R, G) + \hat{N}(R)$.

Proof. Suppose to the contrary that we can write $\mu = \nu + \lambda$ where $\nu \in \mathrm{Belt}(R, G)$ and $\lambda \in \hat{N}(R)$. Let $a \in G$ be the element of G chosen in Lemma 1.4. Take any

$\varphi \in Q(R)$ with $\|\varphi\|_1 = 1$ and set $\varphi_n = a_*^n \varphi$ for every $n \in \mathbb{N}$. Then $\{\varphi_n\}$ is a degenerating sequence.

Using the facts that a^n acts isomorphically and that ν is G-invariant, we have

$$\int_R (a^n)^* \mu \cdot \varphi = \int_R \mu \varphi_n = \int_R \nu \varphi + \int_R \lambda \varphi_n.$$

Here, $(a^n)^* \mu(z) = \sum_{g \in G} (a^n \cdot \xi)(g) 1_{W_g}(z) \mu_0(z)$. Since $(a^n \cdot \xi)(g) \to 1$ as $n \to \infty$ by property (2) in Lemma 1.4, we see that $(a^n)^* \mu(z) \to \mu_0(z)$ pointwise. Hence the left-hand side of the above equality converges to $\int_R \mu_0 \varphi$ by the dominated convergence theorem. On the other hand, $\int_R \lambda \varphi_n$ converges to 0 because $\lambda \in \hat{N}(R)$ and $\{\varphi_n\}$ is a degenerating sequence. Hence $\int_R \mu_0 \varphi = \int_R \nu \varphi$ for every $\varphi \in Q(R)$, which implies that $\mu_0 - \nu \in N(R) \subset \hat{N}(R)$. From this and $\mu = \nu + \lambda$, we have $\mu_0 - \mu = \lambda'$ for another $\lambda' \in \hat{N}(R)$.

Next, we set $\varphi_{-n} = a_*^{-n} \varphi$ for every $n \in \mathbb{N}$ and take another degenerating sequence $\{\varphi_{-n}\}$. Similar to the above and by the fact that μ_0 is G-invariant, it holds

$$\int_R (\mu_0 - (a^{-n})^* \mu) \varphi = \int_R (\mu_0 - \mu) \varphi_{-n} = \int_R \lambda' \varphi_{-n}.$$

This time, $(a^{-n})^* \mu(z) = \sum_{g \in G} (a^{-n} \cdot \xi)(g) 1_{W_g}(z) \mu_0(z)$. Since $a^{-n} \cdot \xi(g) \to 0$ as $n \to \infty$ again by property (2) in Lemma 1.4, we see that $(a^{-n})^* \mu(z) \to 0$ pointwise. The left-hand side of the above equality converges to $\int_R \mu_0 \varphi$ and the right-hand side converges to 0 as $n \to \infty$. Hence $\int_R \mu_0 \varphi = 0$ for every $\varphi \in Q(R)$, which implies that $\mu_0 \in N(R)$. However, this contradicts the fact that μ_0 is chosen so that $\mu_0 \notin N(R)$. □ [Lemma 1.6]

Summing up all the above arguments, we have the proof of Theorem 1.1.

Proof of Theorem 1.1. Consider the arc $p(t) = \pi(t\mu)$ in $T(R)$ and its projection $\hat{\pi}(t\mu)$ on $AT(R)$. By Lemma 1.5, $G_{t\mu}$ is a group of asymptotically conformal automorphisms of $R_{p(t)}$. On the other hand, since $\mu \notin \mathrm{Belt}(R, G) + \hat{N}(R)$ by Lemma 1.6, the tangent vector $d\hat{\pi}_0(\mu)$ of the arc $\hat{\pi}(t\mu)$ at $\alpha(o) \in AT(R)$ does not along the submanifold $\alpha(T(R/G))$ in $AT(R)$. Hence, for some $p = \pi(t\mu)$ with sufficiently small $t > 0$, $\alpha(p) = \hat{\pi}(t\mu)$ does not belong to $\alpha(T(R/G))$. This means that $G_{t\mu}$ is not conjugate to a group of conformal automorphisms by an asymptotically conformal homeomorphism of $R_{p(t)}$. □ [Theorem 1.1]

§2. Present: Teichmüller spaces of symmetric structures

The space of the symmetric structures on the circle S^1 introduced by Gardiner and Sullivan [11] has been generalized to the asymptotic Teichmüller spaces of arbitrary Riemann surfaces by Earle, Gardiner and Lakic [6], [7]. In this present work, we seek another direction for the generalization of the original study of the symmetric structures, namely, we consider Teichmüller spaces of group invariant symmetric structures on S^1.

Let $QC(\Delta)$ denote the space of all quasiconformal automorphisms of the unit disk and $QC_0(\Delta)$ its subspace consisting of all asymptotically conformal automorphisms of Δ. Here $\tilde{f}$ is said to be asymptotically conformal if the complex dilatation $\mu_{\tilde{f}}(z)$ vanishes at S^1, that is, it is arbitrarily close to 0 in a neighborhood of S^1.

An orientation-preserving homeomorphism f of S^1 is called *quasisymmetric* if it extends to a quasiconformal automorphism $\tilde{f} \in QC(\Delta)$. Moreover, f is called *symmetric* if it extends to an asymptotically conformal automorphism $\tilde{f} \in QC_0(\Delta)$. Let us denote the space of all quasisymmetric homeomorphisms by QS and the subspace of all symmetric homeomorphisms by Sym. Topology on QS is endowed by that on $QC(\Delta)$. Although QS itself does not constitute a topological group, Sym is the largest topological subgroup in QS.

Each element of the quotient $\mathrm{Sym} \backslash QS$ is called a *symmetric structure* on S^1 and the space of all symmetric structures on S^1 is denoted by AT and called the *universal asymptotic Teichmüller space*. In this fashion, the ordinary universal Teichmüller space T can be defined by $T = \mathrm{Mob} \backslash QS$, where Mob is the group of all Möbius transformations of S^1. Since Mob is contained in Sym, there exists a natural projection $\alpha : T \to AT$. By using the complex dilatations of the quasiconformal automorphisms in $QC(\Delta)$, these spaces endowed with complex Banach manifold structures as well as Finsler metric structures. The projection α is holomorphic with respect these complex structures.

For a discrete subgroup G of Mob (which we also call a Fuchsian group), we denote the space of all (G, Mob)-compatible quasisymmetric homeomorphisms by

$$QS(G) = \{f \in QS \mid fGf^{-1} \subset \mathrm{Mob}\}.$$

Then $T(G) = \mathrm{Mob} \backslash QS(G)$ is a subspace of T, which is the Teichmüller space for the Fuchsian group G, or equivalently, the Teichmüller space $T(R)$ of the Riemann surface $R = \Delta/G$ uniformized by G, where G is extended to a discrete subgroup of the Möbius transformations of Δ.

In contrast to the definitions above, we consider the space of all (G, Sym)-compatible quasisymmetric homeomorphisms, which is defined by

$$QS(G, \mathrm{Sym}) = \{f \in QS \mid fGf^{-1} \subset \mathrm{Sym}\}.$$

Then $AT(G) = \mathrm{Sym} \backslash QS(G, \mathrm{Sym})$ is a subspace of AT, which is the space of the *G-invariant symmetric structures* on S^1. Remark that this is a different object from the asymptotic Teichmüller space $AT(R)$ for the Riemann surface R.

For the projection $\alpha : T \to AT$, the inverse image $\alpha^{-1}(q)$ of any point $q \in AT$ is denoted by $T_q \subset T$. In particular, $T_o = \alpha^{-1}(o)$ for the basepoint $o \in AT$ is the space of the asymptotically conformal Teichmüller classes. This is a closed submanifold of T satisfying the second countability axiom. The inverse image $\alpha^{-1}(AT(G))$ of $AT(G)$ is denoted by $\widetilde{AT}(G) \subset T$. On the other hand, the image $\alpha(T(G))$ of $T(G)$ is denoted by $\alpha T(G) \subset AT$. We will investigate fundamental properties of the space $AT(G)$ by using $\widetilde{AT}(G)$ and T_o.

Theorem 2.1. *For any Fuchsian group G, the space $AT(G)$ of the G-invariant symmetric structures on S^1 is contractible.*

Theorem 2.2 modulo a certain claim. *For any Fuchsian group G, the space $AT(G)$ is infinite dimensional. In fact, $AT(G)$ is not separable.*

An outline of the proof of Theorem 2.1. For a proof of the fact that the Teichmüller space $T(G)$ is contractible, the conformally barycentric extension $E : QS \to QC(\Delta)$ due to Douady and Earle [4] played an important role. This is a continuous global section of the boundary extension $\Phi : QC(\Delta) \to QS$ of a quasiconformal automorphism of Δ to its boundary value on S^1. The barycenter of $f \in QS$ viewed at $z \in \Delta$ is defined to be $w \in \Delta$ where the Poisson integral

$$\frac{1}{2\pi}\int_{S^1} P_w(f(\zeta))|P_z'(\zeta)||d\zeta|$$

takes 0 for $P_z(\zeta) = (\zeta - z)(1 - \bar{z}\zeta)^{-1}$. The barycenter uniquely exists and the correspondence $z \mapsto w$ defines a quasiconformal automorphism $E(f)$ of Δ. The barycentric extension satisfies the following conformal naturality:

$$E(\psi_1 \circ f \circ \psi_2) = E(\psi_1) \circ E(f) \circ E(\psi_2)$$

for any $\psi_1, \psi_2 \in \mathrm{Mob}$ and for any $f \in QS$.

We restrict the boundary extension D to the space of the asymptotically conformal automorphisms $QC_0(\Delta)$ and consider the map $\Pi_0 : QC_0(\Delta) \to \mathrm{Sym}$. Then Earle, Markovic and Saric [8] proved that the conformally barycentric extension E_0 restricted to Sym also gives a continuous global section for Π_0.

Let M be the unit ball of the Banach space $\mathrm{Belt}(\Delta)$ of all measurable functions on Δ with L^∞-norm. The measurable Riemann mapping theorem asserts that, for any $\mu \in M$, there is a quasiconformal automorphism $\tilde{f} \in QC(\Delta)$ with $\mu_{\tilde{f}} = \mu$ uniquely up to the post composition of a Möbius transformation. Hence M is identified with $\mathrm{Mob}\backslash QC(\Delta)$. Then the map $\Pi : QC(\Delta) \to QS$ induces a map $\pi : M \to T$, which is known to be a holomorphic submersion. Since the barycentric extension $E : QS \to QC(\Delta)$ is a section for Π and it has the conformal naturality, this induces a map $e : T \to M$ which is a continuous (not holomorphic) global section for $\pi : M \to T$. In particular, the contractibility of M leads that of T.

Let M_0 be the unit ball of the Banach subspace $\mathrm{Belt}_0(\Delta)$ of $\mathrm{Belt}(\Delta)$ consisting of all elements vanishing at S^1. In other words, we set $M_0 = \mathrm{Mob}\backslash QC_0(\Delta)$ under the identification $M = \mathrm{Mob}\backslash QC(\Delta)$. Let AM be the unit ball of the quotient Banach space $AM = \mathrm{Belt}_0(\Delta)\backslash \mathrm{Belt}(\Delta)$. Since the holomorphic submersion $\pi : M \to T$ maps M_0 onto T_0, it induces a well-defined holomorphic submersion

$$\pi_\alpha : AM = M_0\backslash M \to AT = \mathrm{Sym}\backslash QS$$

Since $E_0 : \mathrm{Sym} \to QC_0$ is the section for $\Pi_0 : QC_0 \to \mathrm{Sym}$, we have a continuous global section $e_\alpha : AT \to AM$. Again the contractibility of AM leads that of AT. These results have been given in [6] and [8].

Now we take Γ-invariance into account. For a Fuchsian group G, let $AM(G)$ be the G-invariant subspace of AM, that is,

$$AM(G) = \{[\mu] \in AM \mid [g^*\mu] = [\mu] \text{ for } \forall g \in G\}.$$

Here the condition $[g^*\mu] = [\mu]$ is equivalent to saying that $\tilde{f}_\mu g \tilde{f}_\mu^{-1}$ belongs to $QC_0(\Delta)$ for all $g \in G$. We also consider its image $\pi_\alpha(AM(G))$ by $\pi_\alpha : AM \to AT$. Since the above condition becomes $fgf^{-1} \in \mathrm{Sym}$ on S^1 for the boundary value f of $\tilde{f}_\mu$, we see that $\pi_\alpha(AM(G))$ is a subspace of $AT(G)$. However, in virtue of the conformally barycentric extension, we can prove the following.

Lemma 2.3. *The map $\pi_\alpha : AM(G) \to AT(G)$ is a holomorphic submersion and, in particular, $AT(G) = \pi_\alpha(AM(G))$.*

Also the restriction of the section $e_\alpha : AT \to AM$ to $AT(G)$ induces a continuous global section $e_\alpha : AT(G) \to AM(G)$ for π_α. Since $AM(G)$ is contractible, we see that $AT(G)$ is contractible as well. $\square$ [Theorem 2.1]

For the projection $\alpha : T \to AT$, the space $\alpha T(G) = \alpha(T(G))$ is a closed submanifold of $AT(G)$ in AT. In Section 1, we have seen that $AT(G) - \alpha T(G)$ is not empty for any infinite Fuchsian group G except in the case that G is rigid. Note that, for any infinite Fuchsian group G, the projection α is injective on $T(G)$. This implies that, if $T(G)$ is a non-separable infinite dimensional space, then so is $\alpha T(G)$. Hence, if G is of analytically infinite type, then $AT(G)$ is a non-separable infinite dimensional space. Hereafter, we only consider the case that G is of analytically finite type and prove that $AT(G)$ is a non-separable infinite dimensional space.

An outline of the proof of Theorem 2.2. The proof consists of two major ingredients. The first one is the following claim, which says that our problem can be passed to a normal subgroup of finite index. In particular, this is convenient for dealing with a rigid Fuchsian group. As we have seen, such a Fuchsian group should be excluded from our previous argument. However, by passing to a non-rigid subgroup of finite index, we see that the result is valid even for any rigid Fuchsian group.

Claim 2.4. *Let G be a Fuchsian group of analytically finite type and Γ a normal subgroup of G of finite index. If $AT(\Gamma)$ is non-separable, then so is $AT(G)$.*

An idea of the proof of Claim 2.4. We use the asymptotic Bers embedding of AT. Let B be the Banach space of all bounded holomorphic quadratic differentials φ on Δ and B_0 a Banach subspace consisting of the elements φ that vanishes at S^1, that is, $\rho^{-2}(z)|\varphi(z)|$ is arbitrarily close to 0 in a neighborhood of S^1. The quotient Banach space is denoted by $AB = B_0 \backslash B$. The Bers projection is a holomorphic map $P : M \to B$ which is defined by using the Schwarzian derivative. The (Teichmüller) projection $\pi : M \to T$ divides P into an injection $\beta : T \to B$. This is a holomorphic embedding of T onto a bounded contractible domain in B, which is called the Bers embedding. The Bers projection P maps M_0 into B_0. Hence the asymptotic Bers projection $P_\alpha : AM \to AB$ is well-defined. The asymptotic (Teichmüller) projection $\pi_\alpha : AM \to AT$ also divides P_α into an injection $\beta_\alpha : AT \to AB$, which is a holomorphic embedding called the asymptotic Bers embedding. See [6] and [8]. Hereafter we identify T with the image $\beta(T)$ in B, AT with the image $\beta_\alpha(AT)$ in AB and the projection $\alpha : T \to AT$ with the quotient map $B \to AB = B_0 \backslash B$.

Each element g of Mob acts on B and AB as $\varphi \mapsto g^*\varphi$ and $[\varphi] \mapsto [g^*\varphi]$ as bounded linear operators. For a Fuchsian group G, we set

$$B(G) = \{\varphi \in B \mid g^*\varphi = \varphi \ \text{ for } \forall g \in G\};$$
$$AB(G) = \{[\varphi] \in AB \mid [g^*\varphi] = [\varphi] \ \text{ for } \forall g \in G\};$$
$$\widetilde{AB}(G) = \{\varphi \in B \mid [g^*\varphi] = [\varphi] \ \text{ for } \forall g \in G\},$$

which contain $T(G)$, $AT(G)$ and $\widetilde{AT}(G)$ respectively as bounded contractible domains.

Let Γ be a normal subgroup of G of finite index and let $\{g_1, g_2, \ldots, g_k\}$ be a system of the representatives of the cosets G/Γ, where we choose $g_1 = id$. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a generating system of Γ. For each γ_i $(i = 1, 2, \ldots, n)$, we consider a bounded linear operator $\tilde{L}_i$ on B defined by

$$\tilde{L}_i(\varphi) = \gamma_i^*\varphi + \sum_{j=2}^{k} g_j^*\varphi.$$

If we restrict $\tilde{L}_i$ to $\widetilde{AB}(\Gamma)$, then we see that the image $\tilde{L}_i(\widetilde{AB}(\Gamma))$ is contained in $\widetilde{AB}(G)$. We consider the operator $L_i : \widetilde{AB}(\Gamma) \to \widetilde{AB}(G)$. Each $\tilde{L}_i$ has $B(G)$ as a fixed point set. Concerning the kernel, we have the following.

Lemma 2.5. *The intersection* $\bigcap_{i=1}^{n} \operatorname{Ker} \tilde{L}_i$ *of the kernels* $\operatorname{Ker} \tilde{L}_i$ *is contained in* $B(\Gamma)$.

Consider a decreasing sequence of the intersections of the kernels as follows:

$$\operatorname{Ker} \tilde{L}_1 \subset \operatorname{Ker} \tilde{L}_1 \cap \operatorname{Ker} \tilde{L}_2 \subset \cdots \subset \operatorname{Ker} \tilde{L}_1 \cap \operatorname{Ker} \tilde{L}_2 \cap \cdots \cap \operatorname{Ker} \tilde{L}_n.$$

We may assume that each inclusion above is strict, for otherwise, we just skip taking the intersection in that turn. Set $K_0 = \widetilde{AB}(\Gamma)$ and $K_1 = \operatorname{Ker} \tilde{L}_1$. We restrict $\tilde{L}_2$ to K_1 and set $K_2 = \operatorname{Ker} \tilde{L}_2|_{K_1}$, which is coincident with $\operatorname{Ker} \tilde{L}_1 \cap \operatorname{Ker} \tilde{L}_2$. Inductively, when K_{j-1} has been defined, we set $K_j = \operatorname{Ker} \tilde{L}_j|_{K_{j-1}}$, which is coincident with $\bigcap_{i=1}^{j} \operatorname{Ker} \tilde{L}_i$.

By assumption, K_0 is non-separable, whereas K_n is separable by Lemma. Hence there is some j such that K_{j-1} is non-separable but K_j is separable. Then we define $K := K_{j-1}$ and $\tilde{L} := \tilde{L}_j|_{K_{j-1}}$. Further, we take the composition $\tilde{L}$ with $\alpha : \widetilde{AB}(G) \to AB(G)$, which is denoted by

$$L : K(\subset \widetilde{AB}(\Gamma)) \to AB(G).$$

The kernel of L is $\operatorname{Ker} L = \tilde{L}^{-1}(B_0)$. The quotient Banach space $\tilde{L}^{-1}(B_0)/\operatorname{Ker} \tilde{L}$ is (topologically) isomorphic to the image $\tilde{L}(\tilde{L}^{-1}(B_0))$, which is contained in the separable Banach space B_0. Also $\operatorname{Ker} \tilde{L}(= K_j)$ is separable by construction. From these facts, we see that $\operatorname{Ker} L = \tilde{L}^{-1}(B_0)$ is separable. Note that, for the above argument, we have to verify that $\tilde{L}(\tilde{L}^{-1}(B_0)) = \tilde{L}(K) \cap B_0$ is closed.

We will see that $AB(G)$ is non-separable. From this fact, non-separability of $AT(G)$ follows. The image $L(K)$ is contained in $AB(G)$. Differently from the above situation, we can easily find that $L(K)$ is closed in this case. Indeed, since $\widetilde{AB}(G)$ is a fixed point set of $\tilde{L}_j$, we see that $L(K)$ is coincident with $\alpha(\widetilde{AB}(G) \cap K)(\subset AB(G))$. This is closed because α preserves closed sets. Hence the quotient Banach space $K/\operatorname{Ker} L$ is (topologically) isomorphic to $L(K)$ by the open mapping theorem. Since K is non-separable whereas $\operatorname{Ker} L$ is separable as is seen in the previous paragraph, we conclude that $L(K)$ and hence $AT(G)$ is non-separable. □ [Claim 2.4]

The second ingredient for the proof of Theorem 2.2 is to construct a Beltrami differential μ that is "asymptotically invariant" under the Fuchsian group Γ (which is a normal subgroup of G of finite index). This has been done in Section 1, however, in order to show that $AT(\Gamma)$ is non-separable, we have to make a lot of such Beltrami differentials. Our construction is based on homogenous quasi-homomorphisms $\phi : \Gamma \to \mathbb{R}$. The vector space of all such maps is identified with the second bounded cohomology $H_b^2(\Gamma, \mathbb{R})$ of Γ.

Since Γ is of analytically finite type, it is a free group or a surface group as an abstract group. For such a special group, the structure of $H_b^2(\Gamma, \mathbb{R})$ has been investigated by Brooks and Series [1], [2] and Mitsumatsu [15]. In particular, it is known that $H_b^2(\Gamma, \mathbb{R})$ is of uncountably infinite dimension as a vector space. However, for our purpose, we need a finer result than this; we will show that $H_b^2(\Gamma, \mathbb{R})$ is non-separable with respect to the topology on $H_b^2(\Gamma, \mathbb{R})$ induced by L^∞-norm. This can be done by using so called a Brooks cocycle determined by counting a certain word appearing in the canonical representation of each element of Γ.

Remark that, to prove the non-separability of $AT(\Gamma)$ from the non-separability of $H_b^2(\Gamma, \mathbb{R})$, we need a certain technical detail. This requires that Γ has a large fundamental region in Δ. By passing to a smaller normal subgroup Γ of a given G but still of finite index, this requirement does not make trouble for the proof of Theorem 2.2 in virtue of Claim 2.4 again. □ [Theorem 2.2]

§3. Future: Invariant symmetric structures on Riemann surfaces and a fixed point problem on Teichmüller spaces

In the previous section, we consider the space of invariant symmetric structures in the universal asymptotic Teichmüller space AT. The asymptotic Teichmüller spaces is generalized for an arbitrary Riemann surface R and the projection $\alpha_R : T(R) \to AT(R)$ is defined similarly as in Section 1. In this situation, we take a subgroup G of the conformal automorphism group $\mathrm{Conf}(R)$ of the Riemann surface R, and consider the subspace $AT(R, G)$ of $AT(R)$ invariant under the action of G. The problems in Section 2 can be also formulated for this $AT(R, G)$. The dimension of $AT(R, G)$ will be determined by both the group structure of G and the topological structure of R. Since an arbitrary countable group G can be a subgroup of $\mathrm{Conf}(R)$ for some Riemann surface R, the variety of the phenomena will become much larger than the universal Teichmüller case.

The quasiconformal mapping class group $\mathrm{MCG}(R)$ of a Riemann surface acts on the Teichmüller space $T(R)$ as well as on the asymptotic Teichmüller space AT. Since $\mathrm{Conf}(R)$ is regarded as a subgroup of $\mathrm{MCG}(R)$, a group G of conformal transformations of R acts on G fixing the base point of $T(R)$ and $AT(R)$. We assume that G fixes another point $q \in AT(R)$ and hence G keeps the fiber $T_q \subset T(R)$ invariant. The result in Section 1 says that, when G admits a non-trivial homogenous quasi-homomorphism, it may not have a fixed point in T_q. As a next problem, we consider an infinite group G without non-trivial homogenous quasi-homomorphisms and ask whether it always has a fixed point in T_q or not. A lattice G of a semi-simple Lie group of higher rank and an infinite group G all of whose elements are of finite order are examples of such groups. This can be regarded as a fixed point problem of new kind for groups acting on Teichmüller spaces.

References

[1] R. Brooks, *Some remarks on bounded cohomology*, Riemann surfaces and related topics, Ann. Math. Studies, vol. 97, Princeton Univ. Press, 1980, pp. 53–63.

[2] R. Brooks and C. Series, *Bounded cohomology for surface groups*, Topology **23** (1984), 29–36.

[3] A. Casson and D. Jungreis, *Convergence groups and Seifert fibered 3-manifolds*, Invent. Math. **118** (1994), 441–456.

[4] A. Douady and C. Earle, *Conformally natural extension of homeomorphisms of the circle*, Acta Math. **157** (1986), 23–48.

[5] D. Epstein and K. Fujiwara, *The second bounded cohomology of word-hyperbolic groups*, Topology **36** (1997), 1275–1289.

[6] C. Earle, F. Gardiner and N. Lakic, *Asymptotic Teichmüller space, Part I: the complex structure*, In the tradition of Ahlfors and Bers, Contemp. Math., vol. 256, AMS, 2000, pp. 17–38.

[7] C. Earle, F. Gardiner and N. Lakic, *Asymptotic Teichmüller space, Part II: the metric structure*, In the tradition of Ahlfors and Bers III, Contemp. Math., vol. 355, AMS, 2004, pp. 187–219.

[8] C. Earle, V. Markovic and D. Saric, *Barycentric extension and the Bers embedding for asymptotic Teichmüller space*, Complex manifolds and hyperbolic geometry, Contemp. Math., vol. 311, AMS, 2002, pp. 87–105.

[9] D. Gabai, *Convergence groups are Fuchsian groups*, Ann. of Math. **136** (1992), 447–510.

[10] F. Gardiner and N. Lakic, *Quasiconformal Teichmüller theory*, SURV 76, American Mathematical Society, 2000.

[11] F. Gardiner and D. Sullivan, *Symmetric structures on a closed curve*, Amer. J. Math. **114** (1992), 683–736.

[12] F. Gehring and G. Martin, *Discrete quasiconformal groups*, Proc. London Math. Soc. **55** (1987), 331–358.

[13] V. Markovic, *Quasisymmetric groups*, J. Amer. Math. Soc. **19** (2006), 673–715.

[14] K. Matsuzaki, *Elliptic modular transformations of infinite dimensional Teichmüller spaces*, The proceedings of the international workshop on Teichmüller theory and moduli problems, (preprint).

[15] Y. Mitsumatsu, *Bounded cohomology and ℓ^1-homology of surfaces*, Topology **23** (1984), 465–471.

[16] D. Sullivan, *On the ergodic theory at infinity of an arbitrary discrete group of hyperbolic motions*, Riemann surfaces and related topics, Ann. Math. Studies, vol. 97, Princeton Univ. Press, 1981, pp. 465–496.

[17] P. Tukia, *On two-dimensional quasiconformal groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. **5** (1980), 73–78.

[18] P. Tukia, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math. **391** (1988), 1–54.