Altruistic Utility Functions

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Abstract

Utility functions that embody nonpaternalistic altruism, to be called altruistic utility functions, can be regarded as being formed through social interactions among altruistic individuals. As such, they possess socially formed characteristics, on which the present paper focuses. Assuming linear altruism, the paper shows that, as the degrees of altruism increase, different individuals' altruistic utility functions become more similar and converge to a single function in the limit if and only if everybody is altruistically connected to everybody else.

1. Introduction

This paper considers the role of social interactions in the formation of individuals' social preferences, namely preferences regarding social states. The importance of such a consideration in social welfare analysis was elaborated by Buchanan [1954] and Sen [1995]. The basic postulate of the present analysis is that the driving force of such social interactions is nonpaternalistic altruism, namely concern toward others that respects others' own preferences. Some results of analysis along this line are found in Bergstrom [1999] and Hori [2001].

Important preceding works that attempted to derive social preferences, instead of taking them as given, include Harsanyi [1955,1977] and Rawls [1971]. They derived social preferences or ethical criteria to which individuals would subscribe when they forced "a special impartial and impersonal attitude" upon themselves under the "veil of ignorance." Harsanyi further asserted that the social preferences that he derived would approach a unique utilitarian social welfare function if enough psychological laws governing personal preferences came to be known.

In contrast, in the present approach based on altruism, individuals are assumed to maintain a clear distinction between self and others in evaluating social situations. Therefore individuals subscribe to the social preferences, represented by their altruistic utility functions, in full awareness of their positions and without a special need to force impersonality upon themselves.

Although altruism is essentially a personal attitude toward others, an individual's utility function which is consistent with his nonpaternalistic altruism possesses important social characteristics because, due to the nonpaternalism, it is a result of altruistic social interactions with others. The present paper focuses on these socially formed characteristics. In particular, assuming linear altruism, the paper shows that, as individuals become more altruistic, their altruistic utility functions become more similar if every individual is altruistically connected to every other other individual, and converge to a single function in the limit.

The paper is organized as follows. Section 2 presents the model. Section 3 offers some semantic and methodological remarks. Section 4 analyzes the relationship between the degrees of altruism and the similarity of different individuals' utility functions. Section 5 analyzes the limiting situation for indecomposable altruism matrices, which represent the case where everybody is altruistically connected to everybody else. Section 6 discusses some properties of the function, to be called a consensus social utility function, which is obtained in the limiting situation. Section 7 analyzes the limiting situation for decomposable altruism matrices. Section 8 concludes.

2. The Model

The basic postulate is that individuals' concern toward others can be characterized by nonpaternalistic altruism. Let v^i denote the level of utility that individual i assigns to his personal position in a given social situation and call it *i's personal utility.* Also let U^i denote the level of utility that i assigns to the given social situation as a whole and call it *i's overall utility.* Assuming linearity, let a_{ij} denote the constant degree of i's altruism toward j, which is assumed to be nonnegative. Letting n denote the number of individuals in the society, nonpaternalistic altruism implies

$$U^i = v^i + \sum_{j \neq i} a_{ij} U^j, \ i = 1, 2, \cdots, n, \tag{1}$$

or, equivalently,

$$[I - A] U = v, \tag{2}$$

where $U = (U^1, U^2, \cdots, U^n)$, $v = (v^1, v^2, \cdots, v^n)$,

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} = 0,^1$$

and I is the n-dimensional identity matrix.

The system of linear equations (1) or (2) represents the altruistic interactions of the individuals in the society and has to be solvable for U in terms of v with nonnegaive coefficients, that is the inequality $[I - A]^{-1} \geq 0$ has to hold, in order for such a representation to make sense. We call the solutions *altruistic utility functions* if this condition is met. Since this is essentially a Leontief system (except that the diagonal elements of A are restricted to be zero), we can invoke the well-known Hawkins-Simon condition to state: *The system of linear equations (1) or (2) can be solved for altruistic utility functions if and only if all the principal minors of $I - A$ are positive.* See Hawkins and Simon [1949] or Nikaido [1968], Chapter II, Section 6.2, for the proof. We write $C = [c_{ij}] = [I - A]^{-1}$ so that the altruistic utility functions can be written

$$U^i = \sum_j c_{ij} v^j, \ i = 1, 2, \cdots, n, \tag{3}$$

or

$$U = Cv. \tag{4}$$

Note at this point that, although each individual's altruistic concern represented by (1) is personal, in the same sense as each individual's personal utility function v^i is personal, the resulting altruistic utility functions are not. This is because the altruistic interactions represented by the inversion of $I - A$ impart some social quality to the altruistic utility functions. In particular, the interactions

[1] For an arbitrary matrix $X = [x_{ij}]$, $X = 0$ means $x_{ij} \geq 0$ for all i, j, $X \geq 0$ means $X = 0$ and $X \neq 0$, and $X > 0$ means $x_{ij} > 0$ for all i, j.

restrict the class of utility functions that can serve as altruistic utility functions. For example, $U^1 = v^1 + v^2$ and $U^2 = 2v^1 + v^2$ are both legitimate value judgments regarding social welfare and represent social preferences of individuals 1 and 2. But these are not altruistic utility functions because, if one tries to uncover the altruistic concern that lies behind them, one comes up with $U^1 = -v^1 + U^2$ and $U^2 = -v^2 + 2U^1$, which are unreasonable.

Before inquiring into the nature of the restrictions imposed by the altruistic interactions on the utility functions, we offer some remarks in the next section.

3. Some Semantic and Methodological Remarks

First, we interpret *altruism* rather broadly as one's general positive concern toward others. It may be based on one's love of family members, relatives, and neighbors, the general fellow-feeling that one has, one's personal ethical beliefs, etc. What matters for the present analysis is that we regard each individual's degrees of altruism as given personal data in the same way as we regard each individual's personal utility functions v^i.

Second, nonpaternalistic altruism and altruistic utility functions are essentially of an ordinal nature and thus can be given an ordinal representation. See Hori [2002]. For technical reasons, however, this paper assumes linear altruism so that the utility functions treated here are cardinal. It is straightforward to show that linear transformations of utility functions do not affect the condition for the nonnegative invertibility of $I - A$.

Third, interpersonal utility comparisons made by each individual in the present context are personal and subjective; they do not have to satisfy interpersonal objectivity. What is required is that each individual have a utility function which is unique up to linear transformation and that every other individual also know it and apply some version of it in evaluating social situations. To see this, let U^j be the level of j's utility in a given social situation, measured in j's own scale, let $U^j(i)$ be the level of j's utility in the given social situation measured in i's version of j's cardinal utility, and let $a_{ij}(i)$ be the degree of i's altruism toward j applied to $U^j(i)$. Thus

$$U^i = v^i + \sum_{j \neq i} a_{ij}(i) U^j(i), i = 1, 2, \cdots, n. \quad (5)$$

Since $U^j(i)$ is a linear transformation of U^j, there is a constant $d_{ij} > 0$, independent of social situations, such that

$$U^j(i) = d_{ij} U^j, i, j = 1, 2, \cdots, n. \tag{6}$$

Letting $a_{ij} = d_{ij} a_{ij}(i)$, one obtains (1) from (5) and (6).

4. Nature of Altruistic Interactions

We now investigate the nature of altruistic interactions that impart a social quality to the utility functions. For this purpose, suppose that a_{ij} increases. This means that j's well-being, which depends on j's social welfare judgments, now means more to i. This should mean that j's social welfare judgments now have more influence on i's social welfare judgments. To phrase it differently, note the identity

$$\sum_k c_{ik} v^k = v^i + \sum_{l \neq i} a_{il} \left(\sum_k c_{lk} v^k \right),$$

which follows from (1) and (3). This identity shows that if a_{ij} increases, then $U^j = \sum_k c_{jk} v^k$ obtains more weight in the determination of the value of $U^i = \sum_k c_{ik} v^k$, and as a result U^i becomes more similar to U^j, provided that all the $c_{\cdot k}$'s in the right-hand side of the above equation remain fixed.

Although many $c_{\cdot k}$'s in the right-hand side are affected by changes in a_{ij}, the main result holds true. To state and prove the result, however, it is convenient to work with the adjoint matrix $B = [b_{ij}]$ of $I - A$ instead of $C = [I - A]^{-1}$, where b_{ij} is the cofactor of the (j, i) element of $I - A$. The adjoint matrix B satisfies a useful identity

$$B[I - A] = [I - A]B = |I - A| I. \tag{7}$$

Needless to say, $[I - A]^{-1} = |I - A|^{-1} B$ if $|I - A| \neq 0$. The reason for working with B instead of C is that the scale factor $|I - A|^{-1}$ is immaterial for the utility functions as long as $|I - A| > 0$ and that B remains bounded and well-defined while C explodes as A approaches its limit for the nonnegative invertibility of $I - A$. Thus, instead of (4), we write the altruistic utility functions as

$$U^i = \sum_j b_{ij} v^j, \ i = 1, 2, \cdots, n, \tag{8}$$

or

$$U = Bv. \tag{9}$$

Assume throughout this section that $I-A$ is nonnegatively invertible, let $b^k = (b_{k1}, b_{k2}, \cdots, b_{kn})$, and measure the distance between utility functions $U^k = b^k v$ and $U^l = b^l v$ by

$$M(b^k, b^l) \equiv \sum_p \left(\frac{b_{kp}}{\sum_q b_{kq}} - \frac{b_{lp}}{\sum_q b_{lq}} \right)^2, \tag{10}$$

where the right-hand side is well-defined because B is nonnegative and nonsingular. Then it holds that *the distance between $U^i = b^i v$ and $U^j = b^j v$ decreases as a_{ij} increases.* Furthermore, say that *k is altruistically connected to l* if either $a_{kl} > 0$ or there is a sequence of individuals indexed $k_1, k_2, \cdots, k_t$ such that $a_{kk_1} > 0, a_{k_1k_2} > 0, \cdots$, and $a_{k_tl} > 0$. Then it also holds that *the distance between $U^k = b^k v$ and $U^j = b^j v$ decreases as a_{ij} increases if and only if k is altruistically connected to i.*

The rest of this section is devoted to the proof of this assertion. We will utilize the following useful identity which connects the minors of a nonsingular square matrix X to the minors of its inverse Y. Let x_{ij} and y_{ij} denote the (i,j) element of X and Y respectively, let

$$X\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} = \begin{vmatrix} x_{i_1j_i} & x_{i_1j_2} & \cdots & x_{i_1j_k} \\ x_{i_2j_1} & x_{i_2j_2} & \cdots & x_{i_2j_k} \\ \cdot & \cdot & \cdots & \cdot \\ x_{i_kj_1} & x_{i_kj_2} & \cdots & x_{i_kj_k} \end{vmatrix},$$

and define $Y\begin{pmatrix} \cdots \\ \cdots \end{pmatrix}$ similarly. Then, letting n be the dimension of X and Y, it holds that

$$\begin{aligned} & Y\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} \\ = & \frac{(-1)^{\sum_{\nu=1}^k i_\nu + \sum_{\nu=1}^k j_\nu}}{|X|} \times X\begin{pmatrix} j'_1 & j'_2 & \cdots & j'_{n-k} \\ i'_1 & i'_2 & \cdots & i'_{n-k} \end{pmatrix}, \end{aligned} \tag{11}$$

where $i_1 < i_2 < \cdots < i_k$ and $i'_1 < i'_2 < \cdots < i'_{n-k}$ form a complete system of indices $1, 2, \cdots, n$, as do $j_1 < j_2 < \cdots < j_k$ and $j'_1 < j'_2 < \cdots < j'_{n-k}$. See Gantmacher [1959], vol.I, Chapter I, Section 4, for the proof.

Let us see how the elements of B are affected by a change in a_{ij}. Since b_{kp} is the cofactor of $\delta_{pk} - a_{pk}$ in $|I-A|$, where δ_{pk} is Kronecker's delta and $a_{kk} = 0 \; \forall k$, one can see by expanding $|I-A|$ along the ith row or the jth column that b_{kp} is independent of a_{ij} if $p=i$ or $k=j$. Thus

$$\frac{\partial b_{kp}}{\partial a_{ij}} = 0 \text{ if } k=j \text{ or } p=i. \tag{12}$$

For $k \neq j$ and $p \neq i$, let $A^{(pk)}$ denote the minor of $|I-A|$ that is obtained by deleting the pth row and the kth column from the latter so that $b_{kp} = (-1)^{k+p} A^{(pk)}$. Expand $A^{(pk)}$ along the row containing $-a_{ij}$ to obtain $A^{(pk)} = \sum_{q \neq k} (\delta_{iq} - a_{iq}) A_{iq}^{(pk)}$, where $A_{iq}^{(pk)}$ is the cofactor of $\delta_{iq} - a_{iq}$ in $A^{(pk)}$. Since $A_{iq}^{(pk)}$ does not contain a_{ij} for any q, we have $\partial A^{(pk)}/\partial a_{ij} = -A_{ij}^{(pk)}$. Thus $\partial b_{kp}/\partial a_{ij} = (-1)^{k+p+1} A_{ij}^{(pk)}$. Now apply formula (11) to $A_{ij}^{(pk)}$ to obtain

$$\frac{\partial b_{kp}}{\partial a_{ij}} = |I-A|^{-1} (b_{ki} b_{jp} - b_{kp} b_{ji}) \text{ if } k \neq j \text{ and } p \neq i. \tag{13}$$

Next differentiate $M(b_k, b_j)$ with respect to a_{ij} and use (12) and (13) to obtain

$$\frac{\partial M(b_k, b_j)}{\partial a_{ij}} = -\frac{2 b_{ki}}{|I-A|} \frac{\sum_q b_{jq}}{\sum_q b_{iq}} \sum_p \left(\frac{b_{kp}}{\sum_q b_{kq}} - \frac{b_{jp}}{\sum_q b_{jq}} \right)^2, \tag{14}$$

where $|I-A| > 0$ and $\left(\sum_q b_{jq} / \sum_q b_{iq}\right) \sum_p \left(b_{kp}/\sum_q b_{kq} - b_{jp}/\sum_q b_{jq}\right)^2 > 0$ for all $k \neq j$ because B is nonnegative and nonsingular.

For $k=i$, b_{ii} is positive due to the Hawkins-Simon condition. Thus (14) shows that the first part of the assertion is true.

For $k \neq i$, it remains to show that $b_{ki} > 0$ if and only if k is altruistically connected to i. Without loss of generality, let $i=1$, let $2, \cdots, p$ be the indices of the individuals who are altruistically connected to 1, and let

$$A = \begin{bmatrix} A_1 & A_{12} \\ O & A_2 \end{bmatrix}, \tag{15}$$

where A_1 and A_2 are square matrices of order p and $n-p$ respectively while A_{12} is a $p \times (n-p)$ matrix. The adjoint matrix of A is of the form

$$B = \begin{bmatrix} B_1 & B_{12} \\ O & B_2 \end{bmatrix},$$

where B_1 and B_2 are of the same order as A_1 and A_2 respectively. It immediately follows from this that $b_{k1} = 0$ if $k > p$, namely if k is not altruistically connected to 1.

Next suppose that k is altruistically connected to 1 so that $1 < k \leq p$ and assume without loss of generality that $a_{k,k-1} > 0, a_{k-1,k-2} > 0, \cdots,$ and $a_{21} > 0$. Now note that $B_1 = |I - A| \sum_{q=0}^{\infty} A_1^q$. Let $\left(A_1^{k-1}\right)_{k1}$ denote the $(k, 1)$-element of A_1^{k-1}. Then, since A_1 is nonnegative, we have

$$\left(A_1^{k-1}\right)_{k1} = a_{k,k-1}a_{k-1,k-2} \cdots a_{21} + \cdots \geq a_{k,k-1}a_{k-1,k-2} \cdots a_{21},$$

the extreme right-hand side of which is positive by assumption. Since

$$b_{k1} = |I - A| \sum_{q=0}^{\infty} (A_1^q)_{k1} \geq \left(A_1^{k-1}\right)_{k1},$$

it follows that $b_{k1} > 0$. Thus the second part of the assertion is also true.

5. The Limit of Assimilation: The Indecomposable Case

The result in the preceding section shows that if $A \geq A'$, then the altruistic utility functions generated by A are more similar to each other than those generated by A'. Let us now consider how far such an assimilation proceeds. As can be seen from the previous section, this depends crucially on the nature of the network of altruistic interactions. Assume in this section that everybody is altruistically connected to everybody else so that the altruism matrix A is indecomposable.[2].

Call a nonnegative matrix A *maximal* if its Frobenius root is unity so that $|I - A| = 0$ and $|I - \mu A| > 0$ for all $\mu \in [0, 1)$ .In this case $[I - \mu A]$ is nonnegatively invertible for all $\mu \in [0, 1)$. If A is indecomposable and maximal, then (i) A has a positive Frobenius vector which is unique up to scalar multiplication and (ii) any proper principal minor of $I - A$ is positive. See Gantmacher [1959], vol.II, Chapter XIII, Section 2, or Nikaido [1968], Chapter II, Section 7.

Now, if A is maximal, then in view of (7), the adjoint matrix B of $I - A$ satisfies

$$B[I - A] = O, \tag{16}$$

[2]A nonnegative $n \times n$ matrix A is called decomposable (or reducible) if there is a nonempty proper subset J of $\{1, 2, \cdots, n\}$ such that $a_{ij} = 0$ for all $i \in J$ and $j \notin J$. Otherwise, A is called indecomposable (or irreducible). It can be shown that A is indecomposable if and only if everybody is altruistic to everybody else either directly or indirectly. See Nikaido [1968], Lemma 8.1.

where O is the matrix whose elements are all zero. Since $b_{ii} > 0$ by (ii) above, it follows from (16) that any row of B is a Frobenius vector. Moreover, by (i), all the rows of B are proportional. Thus we can conclude that, *if the altruism matrix A is indecomposable and maximal, all the individuals' altruistic utility functions coincide.* Let w denote a row Frobenius vector of $I - A$ and call $W = wv$ a *consensus social welfare function.* Note that $w > 0$ also by (i).

6. Consensus Social Welfare Function

A few remarks are in order concerning the consensus social welfare function.

[I] The vector w clearly depends on the altruism matrix A, and any ethical principle that w may represent must be derivable from the properties of A. Consider utilitarianism, for example, which can be represented by $w_1 = w_2 = \cdots = w_n > 0$. *A sufficient condition for this is cyclicity of the altruism matrix in the sense that, after a suitable simultaneous permutation of rows and columns, we have*

$$A = \left| \begin{array}{cccccc} 0 & -a_1 & -a_2 & \cdots & -a_{n-2} & a_{n-1} \\ -a_{n-1} & 0 & -a_1 & \cdots & -a_{n-3} & -a_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & 0 \end{array} \right| . \tag{17}$$

Note that this includes a subcase where $a_1 = a_2 = \cdots = a_{n-1}$, namely where everybody is equally altruistic to everybody else, as well as a subcase where $a_1 > 0$ and $a_j = 0 \; \forall j \neq 1$, namely where everybody is altruistic only to his/her next of kin.

The proof of this assertion follows. Since all the rows of B are proportional to each other, it suffices to show that $b_{11} = b_{12} = \cdots = b_{1n}$ if A satisfies (17).

First note that the maximality of A implies

$$\sum_{j=1}^{n-1} a_j = 1, \tag{18}$$

because, in this case, the row sum is equal to $\sum_{j=1}^{n-1} a_j$ for every row of A and A's Frobenius root is unity. Also note that the cyclicity and $\sum_{j=1}^{n-1} a_j > 0$ imply indecomposability of A.

Now, by (17), we have

$$b_{11} = \begin{vmatrix} 1 & -a_1 & -a_2 & \cdots & -a_{n-2} \\ -a_{n-1} & 1 & -a_1 & \cdots & -a_{n-3} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -a_2 & -a_3 & -a_4 & \cdots & 1 \end{vmatrix},$$

and for $j = 2, 3, \cdots, n$,

$$b_{1j} = (-1)^{1+j} \begin{vmatrix} -a_{n-1} & 1 & -a_1 & \cdot\cdot & -a_{j-3} & -a_{j-1} & \cdot\cdot & -a_{n-2} \\ -a_{n-2} & -a_{n-1} & 1 & \cdot\cdot & -a_{j-4} & -a_{j-2} & \cdot\cdot & -a_{n-3} \\ \cdot & \cdot & \cdot & \cdot\cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ -a_1 & -a_2 & -a_3 & \cdot\cdot & -a_{j-1} & -a_{j+1} & \cdot\cdot & 1 \end{vmatrix}.$$

By the standard operations on determinants, we obtain

$$\begin{aligned} b_{1j} &= (-1) \begin{vmatrix} 1 & -a_1 & \cdot\cdot & -a_{j-3} & -a_{n-1} & -a_{j-1} & \cdot\cdot & -a_{n-2} \\ -a_{n-1} & 1 & \cdot\cdot & -a_{j-4} & -a_{n-2} & -a_{j-2} & \cdot\cdot & -a_{n-3} \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ -a_2 & -a_3 & \cdot\cdot & -a_{j-1} & -a_1 & -a_{j+1} & \cdot\cdot & 1 \end{vmatrix} \\ &= (-1) \begin{vmatrix} 1 & -a_1 & \cdot\cdot & -a_{j-3} & 1 - \sum_{k \neq j-2} a_k & -a_{j-1} & \cdot\cdot & -a_{n-2} \\ -a_{n-1} & 1 & \cdot\cdot & -a_{j-4} & 1 - \sum_{k \neq j-3} a_k & -a_{j-2} & \cdot\cdot & -a_{n-3} \\ \cdot & \cdot & \cdot\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ -a_2 & -a_3 & \cdot\cdot & -a_{j-1} & 1 - \sum_{k \neq j} a_k & -a_{j+1} & \cdot\cdot & 1 \end{vmatrix}. \end{aligned}$$

Using (18) in the extreme right-hand side of the above equation, we obtain $b_{1j} = b_{11}$ for all $j = 2, 3, \cdots, n$.

[II] Needless to say, we should not expect a consensus social welfare function to materialize spontaneously; a maximal altruism matrix constitutes a bifurcation point between existence and non-existence of altruistic utility functions. The basic importance of a consensus social welfare function lies in that, if the altruism matrix is sufficiently close to being maximal, then every individual's altruistic utility function is close to it. A society with such an altruism matrix will not agree to a social welfare function which is too different from the consensus social welfare function, whatever ethical principle it embodies. Also in this case, the range of individuals' possible altruistic utility functions is rather limited, and the universal domain condition of Arrow [1963] for rational social choice may be too stringent.

[III] If individuals are altruistic, two versions of Bergson-Samuelson individualistic social welfare function (see Bergson [1938] and Samuelson [1947]) are conceivable; the one which takes personal utility levels as arguments and the one which takes overall utility levels as arguments. The consensus social welfare function serves as a legitimate first version because the coefficient vector w in $W = wv$ is positive, but is degenerate as a second version because $U^i = U^j$ for all i, j.

[IV] A consensus social welfare function has the following interesting property. Suppose that a society's altruism matrix is given by μA with $0 < \mu < 1$, where A is indecomposable and maximal, and let $W = wv$ denote the consensus social welfare function for A.

Let $B(\mu)$ be the adjoint matrix of $I - \mu A$ so that $U = B(\mu) v$ is the vector of altruistic utility functions for the society. Since $B(\mu)$ is nonsingular, a linear combination of these utility functions can yield $W = wv$. Let z denote the coefficient vector to be placed on U so that $W = zU$. Then $w = zB(\mu)$ so that

$$z = w [B(\mu)]^{-1} = |I - \mu A|^{-1} w [I - \mu A] = |I - \mu A|^{-1} (1 - \mu) w,$$

where the last equality is due to $w = wA$. Thus we obtain

$$W = wv = |I - \mu A|^{-1} (1 - \mu) wU.$$

Since the scalar $|I - \mu A|^{-1} (1 - \mu)$ is immaterial and w is strictly positive, we can conclude that (1) *the consensus social welfare function for A can serve both as a first and a second version of individualistic social welfare function for a society whose altruism matrix is μA*, $0 < \mu < 1$, and that (2) *the same vector w can be used as weights both on v and U.*

7. The Limit of Assimilation: The Decomposable Case

We next consider the case where there exists an individual who is not altruistically connected to some other individual so that A is decomposable. In this case, we can apply a simultaneous renumbering of rows and columns to transform A to a form like (15), where the diagonal blocks A_1 and A_2 are square matrices. If any one of the diagonal blocks is decomposable, we can repeatedly apply the same operation and arrive finally at a form in which all the diagonal blocks are indecomposable. For the present purpose, we can assume that A is already of the form (15) and that A_1 and A_2 are indecomposable. Then the network of altruism is of the following nature: First, every individual is altruistically connected to every other individual within each group; second, an individual in group 1 may or may not be altruistically connected to individuals in group 2; and third, an individual in group 2 is not altruistically connected to any individuals in group 1. Let the set of indices be $J_1 = \{1, 2, \cdots, n_1\}$ for A_1 and $J_2 = \{n_1 + 1, n_1 + 2, \cdots, n\}$ for A_2.

Supposing that A is maximal, consider altruistic utility functions generated by μA, $0 < \mu < 1$. By (15), we have

$$|I - \mu A| = |I_1 - \mu A_1| \times |I_2 - \mu A_2|, \tag{19}$$

where I_1 and I_2 are identity matrices of dimensions n_1 and $n - n_1$ respectively. Let $B(\mu)$ and $B_k(\mu)$ be the adjoint matrix of $I - \mu A$ and $I_k - \mu A_k$, $k = 1, 2$. Then

$$B(\mu) = \begin{bmatrix} |I_2 - \mu A_2| \times B_1(\mu) & \mu B_1(\mu) A_{12} B_2(\mu) \\ O & |I_1 - \mu A_1| \times B_2(\mu) \end{bmatrix}. \tag{20}$$

Let r_k be the Frobenius root of A_k and let $\mu_k = 1/r_k$. Then $1 = \min(\mu_1, \mu_2)$ due to the maximality of A and (19). μ_k is finite unless A_k is of order one.

First suppose that $A_{12} = O$ so that there are no altruistic interactions between the two groups. In this case the society as a whole cannot have a consensus social welfare function because each group is indifferent to the well-being of the members of the other. The society is naturally split into two groups and we can consider each group separately. Let $U^{(1)} = (U^1, \cdots, U^{n_1}), U^{(2)} = (U^{n_1+1}, \cdots, U^n), v^{(1)} = (v^1, \cdots, v^{n_1})$, and $v^{(2)} = (v^{n_1+1}, \cdots, v^n)$. Then the altruistic utility functions are given by $U^{(k)} = B_k(\mu) v^{(k)}$ for $\mu < \mu_k$, $k = 1, 2$, which converge to the respective consensus social welfare functions as $\mu \to \mu_k$.

Next suppose that $A_{12} \geq O$ so that group 1 is altruistic to group 2 but group 2 is not altruistic to group 1. Even in this case, group 2 is in the same situation as in the case of $A_{12} = O$. Members of this group are indifferent to the well-being of group 1 and the altruistic utility functions are given by $U^{(2)} = B_2(\mu) v^{(2)}$, which converge to a consensus social welfare function as $\mu \to \mu_2$. The situation for group 1 is different. It can be shown that $\mu B_1(\mu) A_{12} B_2(\mu) > 0$ for all $\mu \in (0, \min\{\mu_1, \mu_2\}]$ and that all its rows become proportional to each other as $\mu \to \min\{\mu_1, \mu_2\}$. Noting this fact, one can see that, although the altruistic utility functions for group 1 also converge to a consensus social welfare function as $\mu \to \min\{\mu_1, \mu_2\}$, it is of the form $W = \sum_{j=n_1+1}^{n} w_j v^j$ if $\mu_2 \leq \mu_1$. Thus the consensus among the members of group 1 places no value on the group's own consumption utilities.

8. Concluding Remarks

This paper has argued that, in a society whose altruism matrix is indecomposable, individuals' altruistic utility functions become more and more similar as the degrees of altruism increase, approaching some consensus social welfare function. Although a consensus social welfare function in its literal sense cannot be expected to materialize spontaneously, the point is that altruism contributes to assimilating individuals' views about the well-being of a society. A society in which people hold similar criteria concerning the society's well-being should be more likely to agree to a proper social welfare function or social decision.

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