On the volume conjecture of the Turaev-Viro invariant

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Geometry & physics of 3-dim. quantum gravity

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1. Volume Conjecture

For knots

R.Kashaev :
certain knot invariants $\longrightarrow$ hyperbolic volume

$\|$

H.Murakami-J.M. :
colored Jones Poly. J_N $\longrightarrow$ simplicial volume

H.M.-J.M.-M.Okamoto-T.Takata-Y.Yokota :
$J_N \longrightarrow$ hyperbolic volume $+\sqrt{-1}$ CS

For 3-manifolds

Witten-Reshetikhin-Turaev inv. of 3-mfd.
$\underset{\text{H.Murakami}}{\longrightarrow}$ hyperbolic volume $+\sqrt{-1}$ CS

$\downarrow$ absolute value^2 $\nearrow$ **?**
Turaev-Viro invariant
$\nwarrow$
simplicial decompositon

2. Turaev-Viro invariant

2.1. quantum $6j$-symbol

Notations: Fix $N \geq 3$ (integer)

$$I = \{0, \frac{1}{2}, 1, \frac{3}{2}, \cdots, \frac{N-3}{2}, \frac{N-2}{2}\}$$

q_0 : a $2N$-th root of unity, $q = {q_0}^2$

For $n \geq 1$, $[n] = \dfrac{{q_0}^n - {q_0}^{-n}}{q_0 - {q_0}^{-1}}$ (q-integer)

$[n]! = [n][n-1]\cdots[2][1]$ (q-factorial)

$(i,j,k) \in I^3$ is **addmissible** $\Leftrightarrow \begin{cases} i \leq j+k, \\ j \leq k+i, \\ k \leq i+j, \\ i+j+k \leq N-2 \end{cases}$

$$\Delta(i,j,k) = \sqrt{\frac{[i+j-k]![j+k-i]![k+i-j]!}{[i+j+k+1]!}}$$

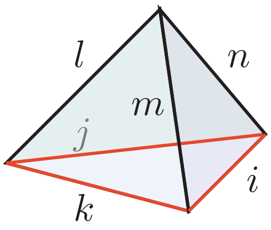
Rakah-Wigner symbol

$$\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW} =$$

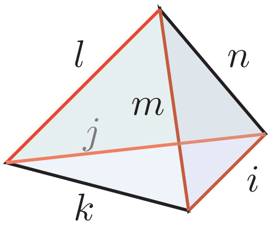
$$\Delta(i,j,k)\Delta(i,m,n)\Delta(j,l,n)\Delta(k,l,m) \times$$

$$\sum_r \frac{(-1)^r \, [z+1]!}{[r-a_1]! \cdots [r-a_4]! [b_1 - r]! \cdots [b_3 - r]!}$$

$$\begin{aligned} a_1 &= i+j+k \\ a_2 &= i+m+n \\ a_3 &= j+l+n \\ a_4 &= k+l+m \end{aligned}$$



$$\begin{aligned} b_1 &= i+j+l+m \\ b_2 &= i+k+l+n \\ b_3 &= j+k+m+n \end{aligned}$$



$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \sqrt{-1}^{-2(i+j+k+l+m+n)} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW}$$

$\longrightarrow$ **volume of T**

2.2 Turaev-Viro invariant

Notations:

M : 3-manifold,

$\mathcal{T}$: tetrahedral decompsition of M

a : number of vertices

b : number of edges

$\varphi : \{\text{edges of } M\} \longrightarrow I$ (**coloring**)

$$u_i = \left(\sqrt{-1}\right)^{2i} [2i+1]^{\frac{1}{2}}, \quad u = \frac{2r}{\left|q_0 - {q_0}^{-1}\right|^2}$$

$$|M|_\varphi = u \prod_{E\ :\ \text{edge}} u_{\varphi(E)} \prod_{T\ :\ \text{tetrahedron}} |\varphi(T)|$$

$$|M| = \sum_{\varphi\ :\ \text{coloring}} |M|_\varphi$$

Theorem (Turaev-Viro)

$|M|$ *is a topologial invariant of* M*.*

2.3 Volume conjecture for 3-mfd.

$\tau_N(M)$: Witten-Reshetikhin-Turaev invariant

Conjecture (H.Murakami, checked also by K.Ohnuki)

$$\underset{N\to\infty}{\text{o-lim}} \frac{2\pi \log \tau_N(M)}{N} = \text{Vol}(M) + \sqrt{-1}\,\text{CS}(M)$$

Known relation. $\left|\tau_N(M)\right|^2 = |M|$

Modified conjecture.

$$\underset{r\to\infty}{\text{o-lim}} \frac{\pi \log |M|}{N} = \text{Vol}(M)$$

Yokota's theory.

For a hyperbolic knot complement,

o-lim $\longleftrightarrow$ geometrization condition

saddle point hyperbolicity equation

What about a closed 3-manifold ?

2.4 Optimistic limit

Step 1. *Operation R:* Replace

$$[n]! \longrightarrow \exp \frac{N}{2\pi\sqrt{-1}}\, g(z_n)$$

$$g(z_n) \underset{\text{def}}{=} \frac{\pi\sqrt{-1}}{2} \log z_n - \frac{(\log z_n)^2}{4} - \mathrm{Li}_2(z_n)$$

$$\left(z_n = q^n = \exp \frac{2\pi n\sqrt{-1}}{N}, \quad \mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-x)}{x} \right)$$

$$n = \frac{N}{2\pi\sqrt{-1}} \log z_n,$$

$$\begin{aligned}
\log([n]!) &= -n \log\left(-q_0 + {q_0}^{-1}\right) + \sum_{k=1}^{n} \log(-{q_0}^k + {q_0}^{-k}) \\
&= -n \log\left(-q_0 + {q_0}^{-1}\right) - \frac{n(n+1)}{2} \log q_0 + \sum_{k=1}^{n} \log(1 - q^k) \\
&\sim \frac{N}{4} \log z_n - \frac{N}{8\pi\sqrt{-1}} (\log z_n)^2 + \int_0^n \log\left(1 - e^{2\pi t\sqrt{-1}/N}\right) dt \\
&\sim \frac{N}{2\pi\sqrt{-1}} \left(\frac{\pi\sqrt{-1}}{2} \log z_n - \frac{(\log z_n)^2}{4} + \int_1^{z_n} \frac{\log(1-x)}{x} dx \right) \\
&= \frac{N}{2\pi\sqrt{-1}} \left(\frac{\pi\sqrt{-1}}{2} \log z_n - \frac{(\log z_n)^2}{4} - \mathrm{Li}_2(z_n) \right)
\end{aligned}$$

Step 2. *Operation S:*

Apply **Saddle point method:**

$$f(z_1, \cdots, z_n) : \text{holomorphic function}$$

$$\left| \int_{z_1, \cdots, z_n} \exp \sqrt{-1} N f(z_1, \cdots, z_n) \right| \underset{N \to \infty}{\sim} \exp N \left| \sqrt{-1} f(z_1^0, \cdots, z_n^0) \right|,$$

where z_1^0, $\cdots$, z_n^0 satisfy

$$\frac{\partial}{\partial z_i} f(z_1^0, \cdots, z_n^0) = 0 \qquad (i = 1, \cdots, n)$$

+ condition about the integral path
(e.g. steepest decent)

In the volume conjedture for 3-manifolds, such condition is **ignored**. ⟶ **Oprimistic limit**

Remark.
There may be several oprimistic limits.

Example: Optimistic limit of $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$

($z_i = q^i$, $z_j = q^j$, $\cdots$, $z_n = q^n$ are fixed.)

Recall

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \sqrt{-1}^{-2(i+j+k+l+m+n)} \begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW},$$

$$\begin{Bmatrix} i & j & k \\ l & m & n \end{Bmatrix}^{RW} =$$

$$\Delta(i,j,k)\Delta(i,m,n)\Delta(j,l,n)\Delta(k,l,m) \times \sum_r \frac{(-1)^r\,[z+1]!}{[r-a_1]!\cdots[r-a_4]![b_1-r]!\cdots[b_3-r]!}.$$

Operation R

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \sim \exp\frac{\sqrt{-1}N}{2\pi}U(\mathbf{z}), \qquad (\mathbf{z} = (z_i, \cdots, z_n))$$

$$U(\mathbf{z}) \;=\; \pi\sqrt{-1}\,(\log z_i + \cdots + \log z_n) + V(\mathbf{z}),$$

$$V(\mathbf{z}) = \delta(z_i, z_j, z_k) + \cdots + \delta(z_k, z_l, z_m) + W(\mathbf{z}),$$

$$\delta(x,y,z) = \frac{1}{2}\left(g(\frac{xy}{z}) + g(\frac{xz}{y}) + g(\frac{yz}{x}) - g(xyz) \right).$$

Operation R (continued)

$$W(\mathbf{z}) = \int_x w(\mathbf{z}; x)\ dx,$$

$$\begin{aligned}
&w(\mathbf{z}; x) = \\
&\pi\sqrt{-1}\,\mathsf{log}\,x + g(x) \\
&\quad - g(\frac{x}{z_i z_j z_k}) - g(\frac{x}{z_i z_j z_k}) - g(\frac{x}{z_i z_j z_k}) - g(\frac{x}{z_i z_j z_k}) \\
&\qquad - g(\frac{z_i z_j z_l z_m}{x}) - g(\frac{z_i z_k z_l z_n}{x}) - g(\frac{z_j z_k z_m z_n}{x}).
\end{aligned}$$

$$\left(\begin{aligned}
&U(\mathbf{z}) = \\
&\quad \pi\sqrt{-1}(\mathsf{log}\,z_i + \cdots + \mathsf{log}\,z_n) + \\
&\quad \underbrace{\delta(z_i, z_j, z_k) + \cdots + \delta(z_k, z_l, z_m) + \overbrace{\int w(\mathbf{z} : x)\,dx}^{W(\mathbf{z})}}_{V(\mathbf{z})}
\end{aligned}\right)$$

Operation S

Apply saddle point method to $\int_x w(\mathbf{z};x)\,dx$.

Solve $\dfrac{\partial}{\partial x} w(\mathbf{z};x) = 0,$

i.e.

$$\frac{(1-x)(1-\frac{x}{z_iz_jz_lz_m})(1-\frac{x}{z_iz_kz_lz_n})(1-\frac{x}{z_jz_kz_mz_n})}{(1-\frac{x}{z_iz_jz_k})(1-\frac{x}{z_iz_mz_n})(1-\frac{x}{z_jz_lz_n})(1-\frac{x}{z_kz_lz_m})} = 1.$$

(Essentially a quadratic equation of x.)

Answers $x = 0,\ x_1,\ x_2.$

Substitute x_1, x_2 to W, V, U.

Then we get the values of the saddle point of $\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}$ at the saddle point w.r.t. x.

Let

$$\begin{aligned} W_i(\mathbf{z}) &\underset{\text{def}}{=} w(\mathbf{z}; x_i)\,, \\ V_i(\mathbf{z}) &\underset{\text{def}}{=} \delta(z_i, z_j, z_k) + \cdots + \delta(z_k, z_l, z_m) + W_i(\mathbf{z})\,, \\ U_i(\mathbf{z}) &\underset{\text{def}}{=} \pi\sqrt{-1}\,(\log z_i + \cdots + \log z_n) + V_i(\mathbf{z})\,. \end{aligned}$$

Then

U_1, U_2 : **optimistic limits** of $\begin{vmatrix} z_i & z_j & z_k \\ z_l & z_m & z_n \end{vmatrix}$.

Speculation:

Turaev-Viro $|M|$ $\underset{\text{o-lim}}{\longrightarrow}$ Volume of M (Conjecture)

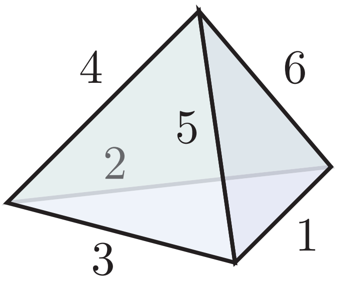
U_1, U_2 $\longrightarrow$ **Volume of tetrahedron ?**

Remark.

$$\begin{aligned} U_1(\mathbf{z}) = -U_2(\mathbf{z}) &= \frac{W_1(\mathbf{z}) - W_2(\mathbf{z})}{2} \\ &= \frac{w(\mathbf{z}; x_1) - w(\mathbf{z}; x_2)}{2}. \end{aligned}$$

3. Volume of tetrahedron

3.1. Observations



Obserbation I: (hyperbolic case)

For a hyperbolic tetrahedron,

let $z_p = \exp l_p$,

$$\sqrt{-1}\theta_p^{(i)} = \frac{\partial}{\partial l_p} U_i(\mathbf{z}) \qquad (\mathbf{z} = (z_1, \cdots, z_6))$$

$$(p = 1, \cdots, 6,\ i = 1, 2),$$

$$\widetilde{U}_i(\mathbf{z}) = U_i(\mathbf{z}) - \sqrt{-1} \sum_{p=1}^{6} \theta_p^{(i)} l_p.$$

Conjecture.

$$2\sqrt{-1}\,\mathrm{Vol}(T) = \pm\widetilde{U}_1(\mathbf{z}) = \mp\widetilde{U}_2(\mathbf{z})\,.$$

Checked for milions of examples.

But not proved yet.

Observation II: (Elliptic case)

For a tetrahedron in a 3-sphere S^3,

let $z_p = \exp \sqrt{-1}\, l_p$,

$$\theta_p^{(i)} = \frac{\partial}{\partial l_p} U_i(\mathbf{z}) \qquad (p = 1, \cdots, 6,\ \ i = 1, 2),$$

$$\widetilde{U}_i(\mathbf{z}) = U_i(\mathbf{z}) + \sum_{p=1}^{6} \theta_p^{(i)} l_p.$$

Conjecture.

$$\boxed{2\,\mathrm{Vol}(T) = \pm \widetilde{U}_1(\mathbf{z}) = \mp \widetilde{U}_2(\mathbf{z})\,.}$$

Corollary

$$4\ \mathrm{Vol}(T) = \left| w(\mathbf{z};x_1) - w(\mathbf{z};x_2) + \varepsilon \sum_{p=1}^{6} l_p \frac{\partial(w(\mathbf{z};x_1) - w(\mathbf{z};x_2))}{\partial l_p} \right|,$$

where $\varepsilon = 1$(elliptic), -1(hyperbolic)

Recall that

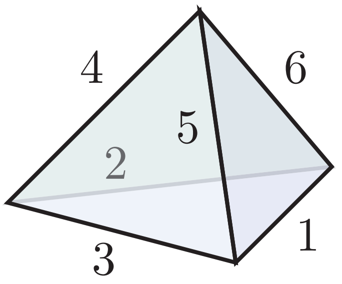
$$\begin{aligned} w(\mathbf{z};x) = {} & \pi\sqrt{-1}\log x + g(x) \\ & - g\left(\frac{x}{z_i z_j z_k}\right) - g\left(\frac{x}{z_i z_j z_k}\right) - g\left(\frac{x}{z_i z_j z_k}\right) - g\left(\frac{x}{z_i z_j z_k}\right) \\ & - g\left(\frac{z_i z_j z_l z_m}{x}\right) - g\left(\frac{z_i z_k z_l z_n}{x}\right) - g\left(\frac{z_j z_k z_m z_n}{x}\right), \end{aligned}$$

$$g(z_n) = \frac{\pi\sqrt{-1}}{2}\log z_n - \frac{(\log z_n)^2}{4} - \mathrm{Li}_2(z_n),$$

$$\frac{\partial}{\partial x} w(\mathbf{z};x_i) = 0, \qquad (i = 1, 2,\ x_i \neq 0)$$

3.2. Volume from angles

Y.Cho and H.Kim
Discrete Comput. Geom.
22 (1999), 347
(from ideal tetrahedron)



J.Murakami and M. Yano
to appear in
Commun. in Analysis and Geometry
(from quantum 6j-symbol as below)

Let $z_p = -\exp\sqrt{-1}\,\theta_p$, then

Theorem. $\bar{\mathbf{z}} = (z_{\mathbf{4}}, z_{\mathbf{5}}, z_{\mathbf{6}}, z_{\mathbf{1}}, z_{\mathbf{2}}, z_{\mathbf{3}})$
(corresponding to opposite edges)

$$\mathrm{Vol}(T) = \frac{\sigma}{4}\,(W_1 - W_2)\,(\bar{\mathbf{z}})\,.$$

where $\sigma = 1$ (elliptic), $\sqrt{-1}$ (hyperbolic).

4. Geometric structure

4.1. o-lim of Turaev-Viro inv.

Recall **notations:**

M : 3-manifold,

$\mathcal{T}$: tetrahedral decompsition of M

- a : number of vertices
- b : number of edges

$\varphi : \{\text{edges of } M\} \longrightarrow I$ (**coloring**)

$$u_i = \left(\sqrt{-1}\right)^{2i} [2i+1]^{\frac{1}{2}}, \quad u = \frac{2r}{\left|q_0 - q_0{}^{-1}\right|^2}$$

$$|M|_\varphi = u \prod_{E\,:\ \text{edge}} u_{\varphi(E)} \prod_{T\,:\ \text{tetrahedron}} |\varphi(T)|$$

$$|M| = \sum_{\varphi\,:\ \text{coloring}} |M|_\varphi \quad \text{(Turaev-Viro Invariant)}$$

Modified conjecture.

$$\underset{r\to\infty}{\text{o-lim}} \frac{\pi \log |M|}{N} = \mathrm{Vol}(M)$$

Operation R, S:

$z(E)$: parameter corresponding to E

$$|M| \xrightarrow[\text{Operation R}]{} \exp\left(\frac{\sqrt{-1}N}{2\pi}\times\right.$$

$$\underbrace{\left.\underset{E:\text{edge}}{\int_{z_E}} - \sum_{E:\text{edge}} 2\pi\sqrt{-1}\log z_E + \sum_{T:\text{tetrahedron}} U(T)\right).}_{R}$$

$$\text{Operation S} \quad \downarrow \quad \underset{z_E^0}{\text{value at saddle point}}$$

$$z_E^0 : \text{solution of} \quad \frac{\partial R}{\partial z_E} = 0 \quad (\text{for all edges } E),$$

$$\text{i.e.} \quad \frac{2\pi\sqrt{-1}}{z_E} = \sum \frac{\partial U}{\partial z_E} = \sum_{T\ni E} \frac{\partial U(T)}{\partial z_E}$$

$$\underset{\textbf{observation}}{=} \sum_{T\ni E} \sqrt{-1}\,\frac{\theta_E(T)}{z_E}.$$

Suppose that $\log z_E^0$ is real and is the length of E. Then

$$R_0 = \sum_E 2\pi\sqrt{-1}\log z_E^0 + \sum_{T:\text{tetrahedron}} U_i(T)\big|_{z_E=z_E^0}$$

$$= \sum_{T:\text{tetrahedron}} \widetilde{U}_i(T)\Big|_{z_E=z_E^0}$$

$$\underset{\textbf{conjecture}}{=} \boxed{\sum_{T:\text{tetrahedron}} 2\sqrt{-1}\,\text{Vol}_0(T)}\,.$$

If $l_E^0 (= \log z_E^0)$ is a **positive real** number for each edge E and satisfy the trigonometric inequarity, these parameters determin a hyperbolic structhre of M.

(Assuming that the conjecture is true.)

4.2. Casson's algorithm

Schönflies' formula T : tetrahedron,

$$l_E = \frac{\varepsilon}{2}\frac{\partial \mathrm{Vol}(P)}{\partial \theta_E} \quad \left(\varepsilon = \begin{cases} 1 \text{ elliptic (spherical)}, \\ -1 \text{ hyperbolic.} \end{cases} \right)$$

M : hyperbolic 3-manifold,
$\mathcal{T}$: tetrahedral decompsition of M,
$\varphi : \{\text{edges}\} \to \mathbf{R}$ (length).

$$P_\varphi(M) = \sum_{T\,:\,\text{tetrahedron}} \mathrm{Vol}(T_\varphi) + \frac{1}{2}\sum_{E\,:\,\text{edge}} \theta_E\,\varphi_E$$

φ^0 : solution of $\dfrac{\partial P_\varphi}{\partial \varphi_E} = \pi$.

Argorithm. (Casson)
φ^0 often determines the hyperbolic structure of M.

Remark.

$P_\varphi(M)$ corresponds to the previous $\sum_T U(T)$.

$$|M| \xrightarrow[\text{Operation R}]{} \underbrace{\exp\left(\frac{\sqrt{-1}N}{2\pi}\times \int_{\substack{z_E \\ E:\text{edge}}} - \sum_{E:\text{edge}} 2\pi\sqrt{-1}\log z_E + \sum_{T:\text{tetrahedron}} U(T)\right)}_{R}.$$

$$\begin{aligned}\frac{\partial P_\varphi}{\partial \varphi_E} &= \sum_k \frac{\partial \operatorname{Vol}(T^k)}{\partial \theta_E^k}\frac{\partial \theta_E^k}{\partial \varphi_E} + \sum_k \frac{1}{2}\frac{\partial \theta_E^k}{\partial \varphi_E}\varphi_E + \sum_k \frac{1}{2}\theta_E^k \\ &= \sum_k \frac{1}{2}\theta_E^k \left(= \pi \text{ at } \varphi^0\right)\end{aligned}$$

Saddle point of R corresponds to φ^0, a solution of $\dfrac{\partial P_\varphi}{\partial \varphi_E} = \pi$.

Conclusion

If quantum $6j$-symbol determines the volume of tetrahedron as conjectured, then the Turaev-Viro invariant may determine the hyperbolic structure of M along with Casson's argorithm.

Ploblem.

What about 3-manifold with **general** geometric structure?