Finite-type invariants detecting the mutant knots

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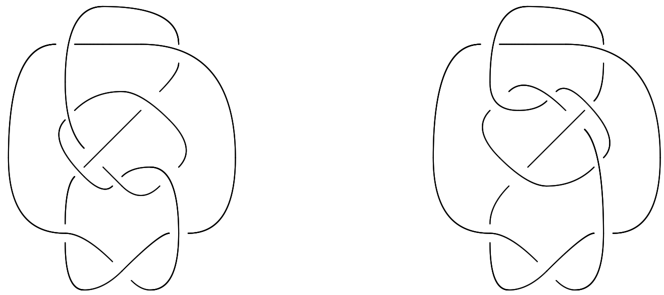
Introduction

The notion of finite-type invariants was introduced by Gusarov [2] and then appeared in the work of Vassiliev [8] concerning to certain 0-homology of the space of knots approimated by polynomial mappings. Kontsevich [3] gives a universal description for all the finite-type invariants by using chord diagrams and iterated integral.

In this paper, we show that there is no finite-type invariant of degree less than 11 which distinguishes mutant knots. A finite-type invariant of degree 11 which can distinguish some mutant knots is constructed as follows. Let $Q_K(q)$ be the quantum invariant of a knot K corresponding to the representation of the partition (2,1) of the quantum enveloping algebra $\mathcal{U}_q(sl_4)$. Let K_C be the Conway's 11-crossing knot, K_{KT} be the Kinoshita-Terasaka knot, and $\bigcirc$ be the trivial knot. Then, by using the computer software "KnotTheoryByComputer" by M. Ochiai [7], we have

$$\frac{Q_{K_C}(q) - Q_{K_{KT}}(q)}{Q_{\bigcirc}(q)} = q^8 \left(1+q^4\right) \left(-1+q^2\right)^6 \left(-1+q^6\right)^2 \left(-1+q^{12}\right) \left(-1+q^{28}\right)^2.$$

This is divisible by $(q-1)^{11}$ and is not divisible by $(q-1)^{12}$. This means that there is a finite-type invariant of degree actually 11 which has different values for K_C and K_{KT}.



Kinoshita-Terasaka knot K_{KT} Conway's 11 crossing knot K_C

FIGURE 1. Kinoshita-Terasaka knot and Conway's 11 crossing knot

The aim of this paper is to show that all the finite-type invariants of degree less than or equal to 10 cannot distinguish mutant knots. This fact is shown by using the Kontsevich invariant and by investigating the symmetry of web diagrams corresponding to 2-tangles.

1. Finite-type invariants of 2-tangles

1.1. **Web diagrams.** Let I be an oriented 1-manifold. A web diagram W on I is a uni-trivalent graph whose univalent vertices are attached to I and edges attached to a trivalent vertex are cyclically ordered as in Figure 2. In the diagram, I is denoted by solid lines and the graph is denoted by dashed lines. The total degree of a web diagram W is

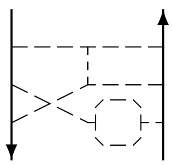


FIGURE 2. An example of a web diagram

the number of all vertices of W, which is an even number. The inner degree of W is the number of the trivalent vertices of W. Let $\mathcal{W}(I)$ be the spaces of web diagrams on I subjected to the STU relation and the IHX relation as in Figure 3.

$$\mathcal{W}(I) = \mathbf{C}\{\text{web diagram on } I\}/\text{STU, IHX}.$$

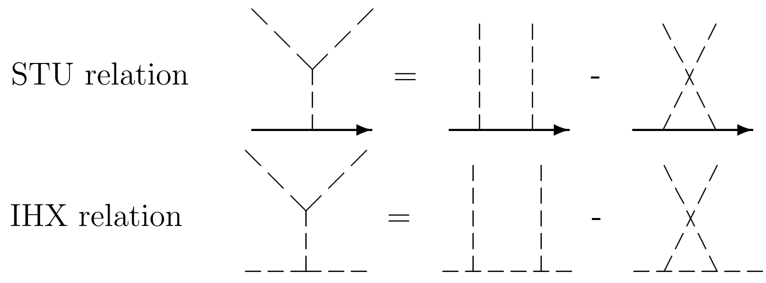


FIGURE 3. STU and IHX relation

These relations are homogeneous with respect to the total degree and so $\mathcal{W}(I)$ is graded by the total degree. The STU relation is not homogeneous with respect to the inner degree and there is a filtration of $\mathcal{W}(I)$ coming from the inner degree. Let $\mathcal{W}^{(t)}(I)$ denote the subspace of $\mathcal{W}(I)$ spanned by the web diagrams of the total degree t, and $\mathcal{W}^{(t,i)}(I)$ denote the subspace of $\mathcal{W}(I)$ spanned by the web diagrams of the total degree t and the inner degree greater than or equal to i.

1.2. **Symmetrized web diagrams.** Let I be an oriented 1-manifold consist of connected components $I_1, I_2, \cdots, I_k$. Let W be a web diagram on I and let $p_1^{(l)}, p_2^{(l)}, \cdots, p_{q_l}^{(l)}$ be the univalent vertices of W attached to the l-th component I_l. Let S_n denote the symmetric group of n letters. For $\sigma \in S_{q_1} \times \cdots \times S_{q_k}$, let σW be the web diagram on I whose univalent vertices are exchanged according to σ. Let

$$W^S = \frac{1}{q_1!\, q_2! \cdots q_k!} \sum_{\sigma \in S_{q_1} \times \cdots \times S_{q_k}} \sigma W,$$

and we call W^S the symmetrized web diagram of W.

Let

$$\mathcal{W}^S(I) = \{W^S \mid W \in \mathcal{W}(I)\},$$

and we call $\mathcal{W}^S(I)$ the space of symmetrized web diagrams. We represent W^S by the graph of W whose univalent vertex is labeled accoeding to the attached component of I. For example, let W be the web diagram given in Figure 2, then W^S is expressed as in FIgure 4.

Proposition. $\mathcal{W}^S(I) = \mathcal{W}(I)$.

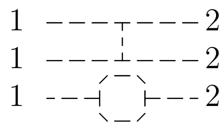


FIGURE 4. An example of a symmetrized web diagram

The proof of this proposition is similar to the proof of Theorem 8 in [1], and we omit it.

1.3. **Primitive diagrams.** The coproduct Δ of $\mathcal{W}(I)$ is a linear mapping from $\mathcal{W}(I)$ to $\mathcal{W}(I) \otimes \mathcal{W}(I)$ distributing the connected component of the graph of a web diagram as in [1]. An element x of $\mathcal{W}(I)$ is called *primitive* if

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

Let W be a web diagram on I which is primitive in $\mathcal{W}(I)$, then the graph of W is connected and the symmetrized diagram W^S is also primitive.

Let $I^{(n)}$ be a union of intervals and we define a multiplication of elements of $\mathcal{W}(I^{(n)})$ by connecting corresponding intervals. For two web diabrams W_1, W_2 in $\mathcal{W}(I^{(n)})$, we define symmetrized product $W_1 * W_2$ by

$$W_1 * W_2 = (W_1 \, W_2)^S.$$

This definition implies that

$${W_1}^S * {W_2}^S = (W_1 \, W_2)^S.$$

This symmetrized product is commutative and so $\mathcal{W}(I^{(n)})$ has a commutative algebra structure with this product. As in the case of web diagrams on S^1 discussed in [1], we have the following.

Proposition. *$\mathcal{W}(I^{(n)})$ is isomorphic to the symmetric tensor algebra of $\mathcal{W}(I^{(n)})^{prim}$, where $\mathcal{W}(I^{(n)})^{prim}$ is the subspace of $\mathcal{W}(I^{(n)})$ spanned by the primitive elements.*

Moreover, we have the following.

Proposition. *Let $\hat{Z}_f(T)$ be the Kontsevich invariant of a framed n-tangle T [5]. Then $\hat{Z}_f(T)$ is a group-like element i.e.*

$$\Delta(\hat{Z}_f(T)) = \hat{Z}_f(T) \otimes \hat{Z}_f(T),$$

and there is $x \in \mathcal{W}(I^{(n)})^{prim}$ such that

$$\hat{Z}_f(T) = \exp(x) = 1 + x + \frac{x * x}{2!} + \frac{x * x * x}{3!} + \cdots .$$

1.4. **Generators.** We give a generator set of $\mathcal{W}^{(2t)}(I^{(2)})^{\text{prim}}$. Here we use that $\mathcal{W}^{(2t)}(I^{(2)})^{\text{prim}}$ is spanned by symmetrized diagrams.

$t = 0$ The space $\mathcal{W}^{(0)}(I^{(2)})^{\text{prim}}$ is spanned by the empty graph ϕ.

$t = 1$ The space $\mathcal{W}^{(2)}(I^{(2)})^{\text{prim}}$ is spanned by 3 elements. These are all consists of a single line and the endpoints of the line are labeled by (1,1), (1,2), (2,2) respectively.

1 ---- 1   1 ---- 2   2 ---- 2

FIGURE 5. Generators of $\mathcal{W}^{(1)}(I^{(2)})^{\text{prim}}$

$t = 2$ The space $\mathcal{W}^{(4)}(I^{(2)})^{\text{prim}}$ is spanned by the 3 elements in Figure 6.

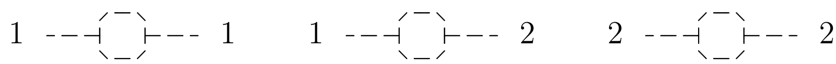


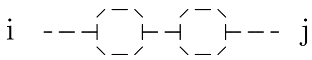
FIGURE 6. Generators of $\mathcal{W}^{(2)}(I^{(2)})^{\text{prim}}$

$t = 3$ The space $\mathcal{W}^{(6)}(I^{(2)})^{\text{prim}}$ is spanned by the 4 elements in Figure 7.

$t = 4$ The space $\mathcal{W}^{(8)}(I^{(2)})^{\text{prim}}$ is spanned by the 9 elements in Figure 8.

$t = 5$ The space $\mathcal{W}^{(10)}(I^{(2)})^{\text{prim}}$ is spanned by the 10 elements in Figure 9.

We can show the above by using the following relations coming from the STU and the IHX relations.



$(i,j) = (1,1), (1,2), (2,2),$

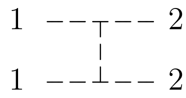
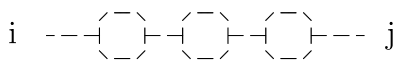
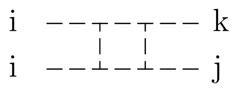


FIGURE 7. Generators of $\mathcal{W}^{(6)}(I^{(2)})^{\text{prim}}$



$(i,j) = (1,1), (1,2), (2,2),$



$(i,j,k) = (1,1,1), (1,1,2), (1,2,2), (2,1,2), (2,2,2),$

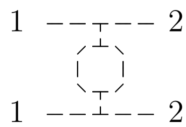


FIGURE 8. Generators of $\mathcal{W}^{(8)}(I^{(2)})^{\text{prim}}$

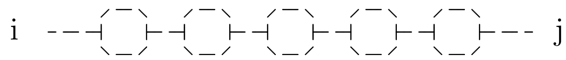
Lemma. *In $\mathcal{W}(I)$, symmetrized diagrams satisfy the relations in Figure 10.*

2. Symmetry of web diagrams

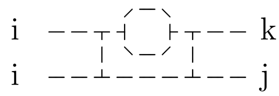
For a web diagram W in $\mathcal{W}(I^{(2)})$, let $f(W)$ denote the flipped diagram of W exchanging the two lines of $I^{(2)}$. For example, $f(P_1) = P_2$.

Lemma. *For the symmetrized web diagram W^S, we have*

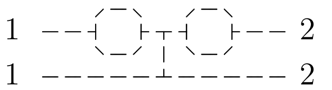
$$f(W^S) = f(W)^S$$



$$(i,j) = (1,1), (1,2), (2,2),$$



$$(i,j,k) = (1,1,1), (1,1,2), (1,2,2), (2,1,2), (2,2,2),$$



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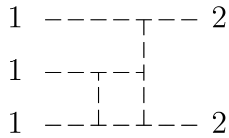
$P_1 =$ 

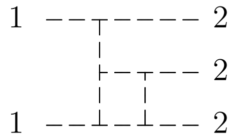
, $P_2 =$ 

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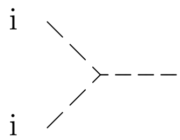
FIGURE 9. Generators of $\mathcal{W}^{(10)}(I^{(2)})^{\text{prim}}$



$= 0,$

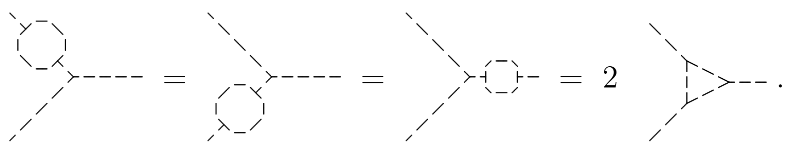


FIGURE 10. Relations among web diagrams

Proposition. *Let x be one of generators of $\mathcal{W}^{(2t)}(I^{(2)})$ given in the previous section, which is not equal to P_1 nor P_2. Then x satisfies*

$$f(x) = x.$$

The generator P_1 and P_2 satisfies

$$f(P_1 + P_2) = P_1 + P_2, \qquad f(P_1 - P_2) = -(P_1 - P_2).$$

Proof. Let x be one of the generators other than P_1, P_2, $x = f(x)$ because the shape of x and the lemma in the previous section. For P_1 and P_2, we have $f(P_1) = P_2$ and $f(P_2) = P_1$. □

3. Mutation

Let K be a knot consisting of two 2-tangles T_1 and T_2. Then the Kontsevich invariant $\hat{Z}_f(K)$ of K is a composition of the Kontsevich invariants $\hat{Z}_f(T_1)$ and $\hat{Z}_f(T_2)$ of T_1 and T_2 as the knot K is constructed from T_1 and T_2 as in Figure 11. We denote this composition

$\hat{Z}_f(T_1) \natural \hat{Z}_f(T_2) =$

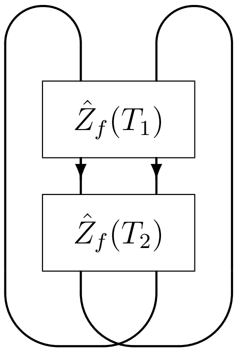


FIGURE 11. Composition of the Kontsevich invariant of tangles

by $\hat{Z}_f(T_1) \natural \hat{Z}_f(T_2)$ and so

$$\hat{Z}_f(K) = \hat{Z}_f(T_1) \natural \hat{Z}_f(T_2).$$

Let $f(T_1)$ denote the flipped tangle as in Figure 12 and let K' be the knot constructed by $f(T_1)$ and T_2. These two knots K and K' are called *mutant*.

Figure 11 implies the following.

Lemma. *We have*

$$\begin{aligned}\hat{Z}_f(K) - \hat{Z}_f(K') &= \left(\hat{Z}_f(T_1) - f(\hat{Z}_f(T_1))\right) \natural \hat{Z}_f(T_2) \\ &= \hat{Z}_f(T_1) \natural \left(\hat{Z}_f(T_2) - f(\hat{Z}_f(T_2))\right).\end{aligned}$$

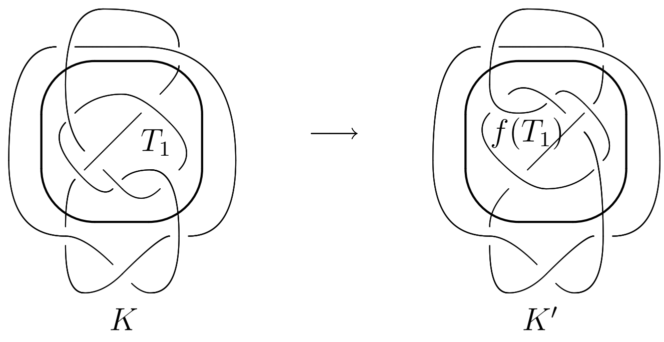


FIGURE 12. Flip the tangle T_1

Let $\hat{Z}_f^{(d)}$ denote the degree d part of the Kontsevich invariant. Then the above lemma and the previous proposition imply the following.

Lemma. *Let T be a 2-tangle. Then, for $d \leq 4$,*

$$\hat{Z}_f^{(d)}(T) - f(\hat{Z}_f^{(d)}(T)) = 0.$$

If $d \leq 9$, then each term of $\hat{Z}_f^{(d)}(K) - \hat{Z}_f^{(d)}(K')$ is a composition of terms of $\hat{Z}_f^{(d_1)}(T_1)$ or $f(\hat{Z}_f^{(d_1)}(T_1))$, and, $\hat{Z}_f^{(d_2)}(T_2)$ or $f(\hat{Z}_f^{(d_2)}(T_2))$, where one of d_1 and d_2 is less than or equal to 4. Therefore, the above lemma and the previous proposition imply the following.

Proposition. $\hat{Z}_f^{(d)}(K) - \hat{Z}_f^{(d)}(K') = 0$ *if* $d \leq 9$.

Next, we consider about the degree 10 part. The web diagram we have to consider is the non-symmetric generators P_1 and P_2. From Figure 11, we have

$$P_1 \natural P_1 = P_1 \natural P_2 = P_2 \natural P_2 = P_2 \natural P_1.$$

Let

$$\hat{Z}_f^{(5)}(T_1) = c_1\,(P_1 - P_2) + \text{symmetric part},$$
$$\hat{Z}_f^{(5)}(T_2) = c_2\,(P_1 - P_2) + \text{symmetric part}.$$

Then, we have

$$\hat{Z}_f^{(10)}(K) - \hat{Z}_f^{(10)}(K') = 2\,c_1\,c_2\,(P_1 - P_2) \natural (P_1 - P_2) = 0.$$

Hence the following theorem holds.

Theorem. *For $d \leq 10$,*

$$\hat{Z}_f^{(d)}(K) - \hat{Z}_f^{(d)}(K') = 0.$$

References

[1] D. Bar-Natan: *On the Vassiliev knot invariants*, Topology **34** (1995), 423–472.

[2] M. Gusarov: *On n-equivalence of knots and invariants of finite degree*, Advances in Soviet Math. **18** (1994), 173–192.

[3] M. Kontsevich: *Vassiliev's knot invariants*, in 'Collection: I. M. Gelfand Seminar', Adv. Soviet Math. **16**, Part 2, Amer. Math. Soc., Providence, RI, 1993, pp. 137–150.

[4] T. Q. T. Le and J. Murakami: *Kontsevich's integral for the HOMFLY polynomial and relations between values of multiple zeta functions*, Topology Appl. **62** (1995), 193-206.

[5] T. Q. T. Le and J. Murakami: *Representations of the category of tangles by Kontsevich's iterated integral*, Comm. Math. Phys. **168** (1995), 535–562.

[6] T. Q. T. Le, J. Murakami and T. Ohtsuki: *On the universal perturbative invariant of 3-manifolds*, Topology **37** (1998), 539–574.

[7] M. Ochiai: *Knottheory by Computer*, ftp://ftp.ics.nara-wu.ac.jp/pub/ochiai/

[8] V. A. Vassiliev: *Cohomology of knot spaces*, Theory of singularities and its Applications (Providence)(V. I. Arnold, ed.), Amer. Math. Soc., Providence (1990).