Scissors Congruence: The Birth of Hyperbolic Volume

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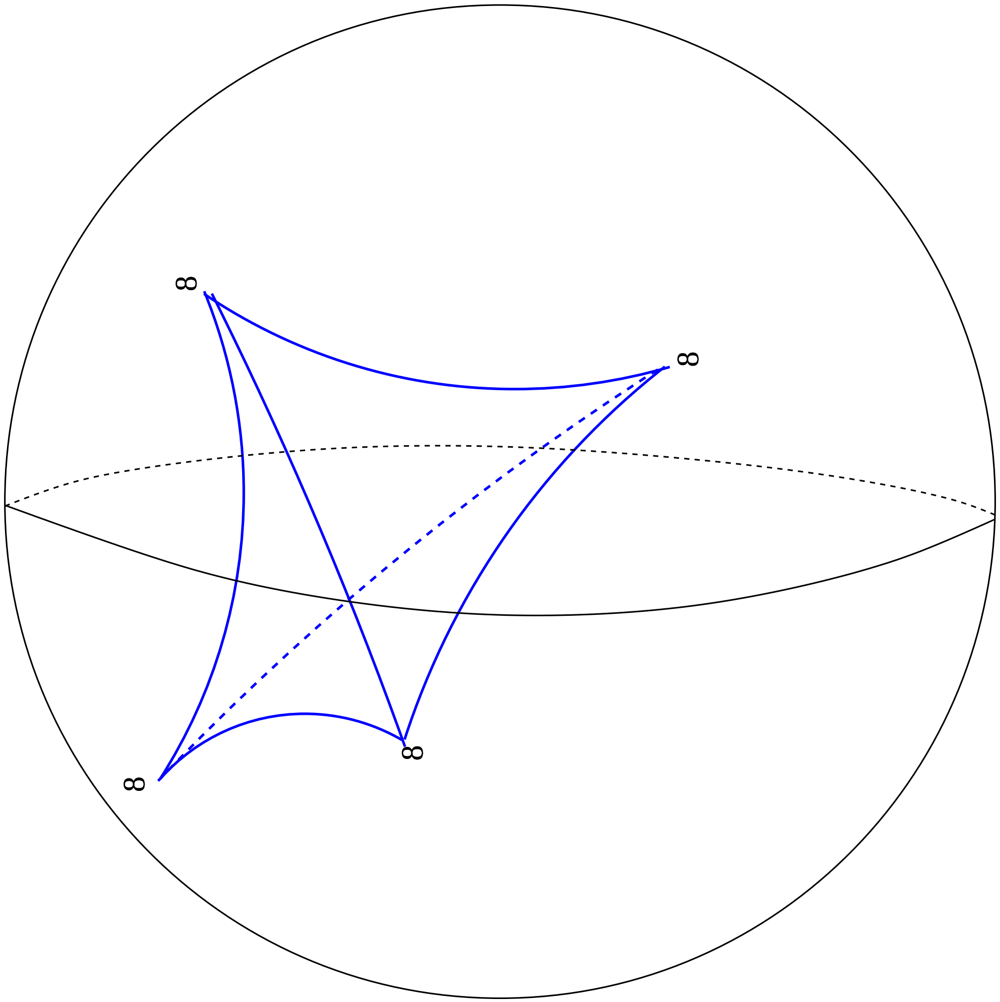
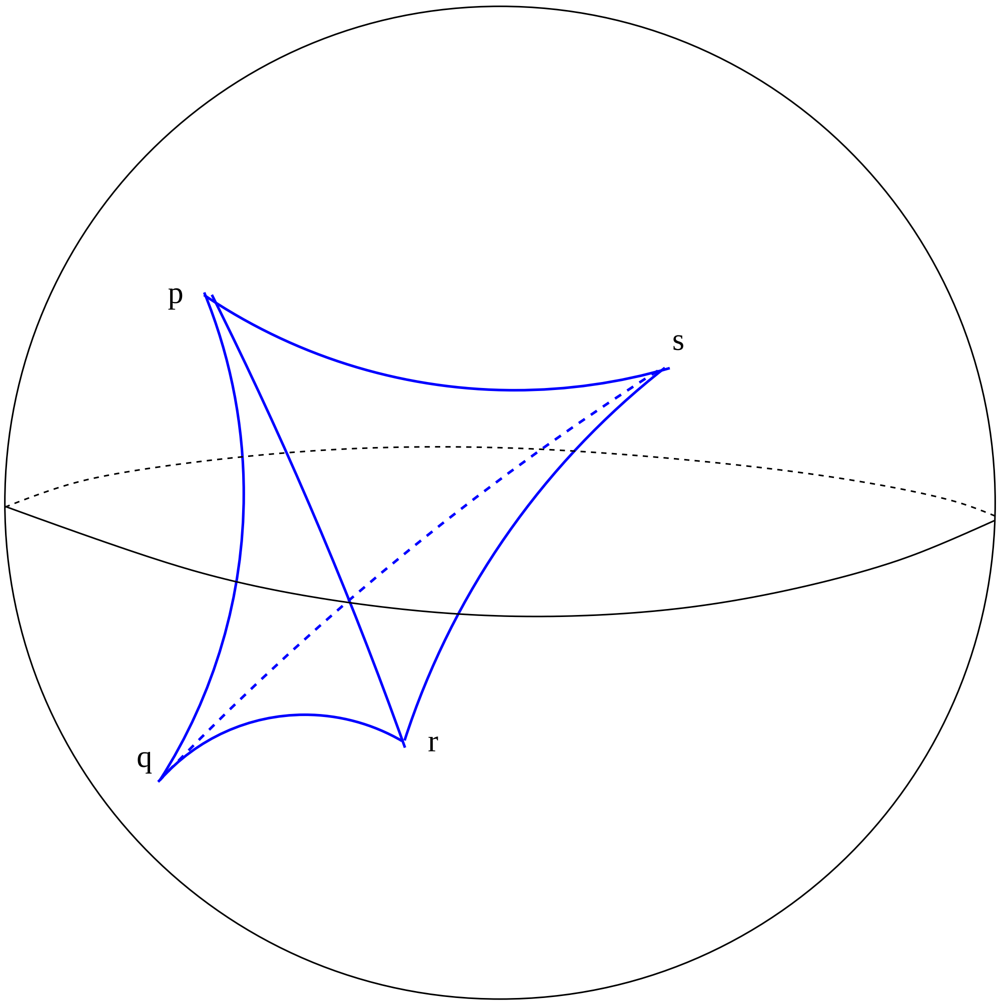
The Ideal Tetrahedron

Here we see the oriented convex hull of four ideal points, an *ideal tetrahedron*.



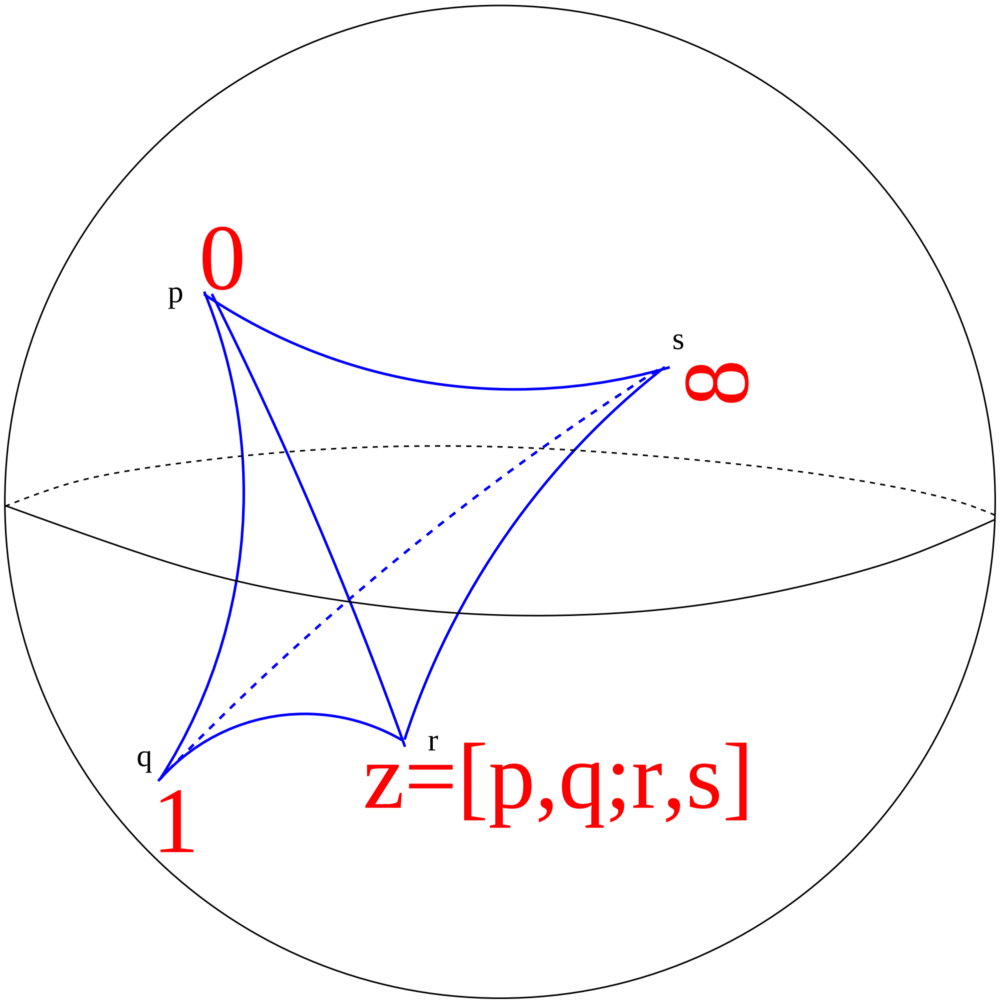
Ideal Tetrahedron

The boundary at infinity is the Riemann sphere with hyperbolic isometries corresponding to conformal mappings. Hence we label the points...



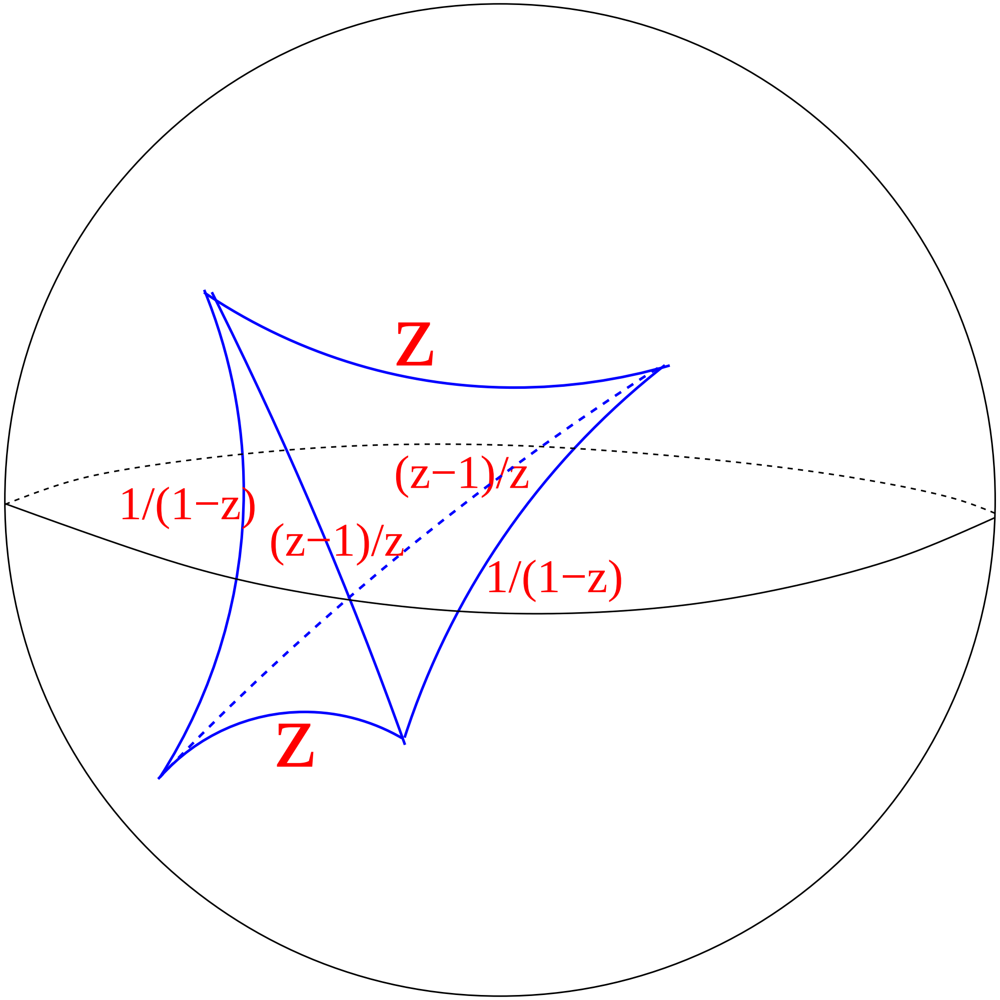
Ideal Tetrahedron

..and compute the cross ratio z. This cross ration parameterizes these labeled oriented ideal tetrahedra.



Ideal Tetrahedron

It is easy to see that this cross ration depends really only on a choice of orientation and a choice of a pair of opposite edges. Hence, the complex coordinate parameterize the space of ideal oriented tetrahedra with a specified pair of opposite edges.



Another Big Free Groups

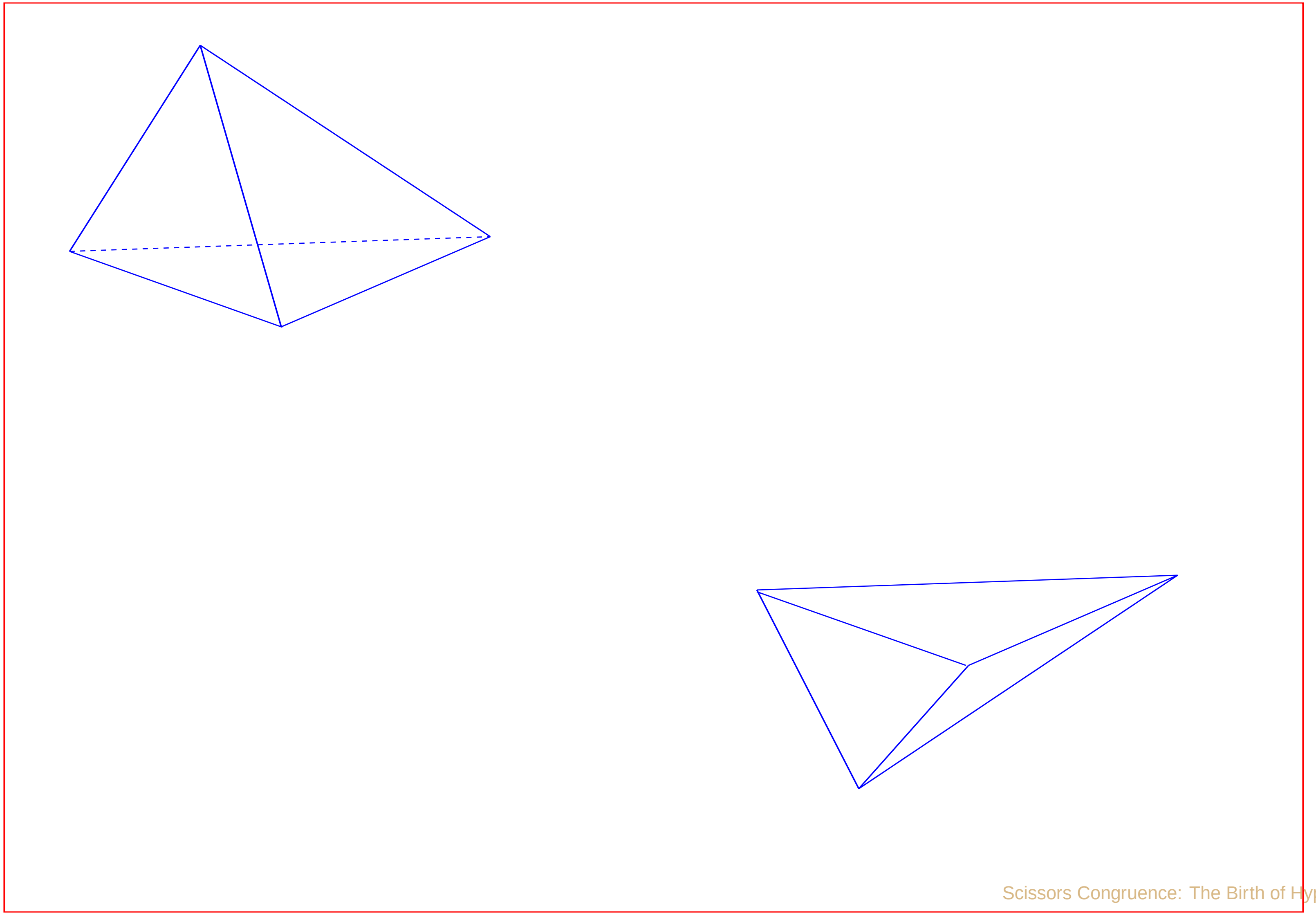
Let

$$< C >$$

denote the free Abelian group generated by all complex numbers, i.e. all ideal tetrahedra.

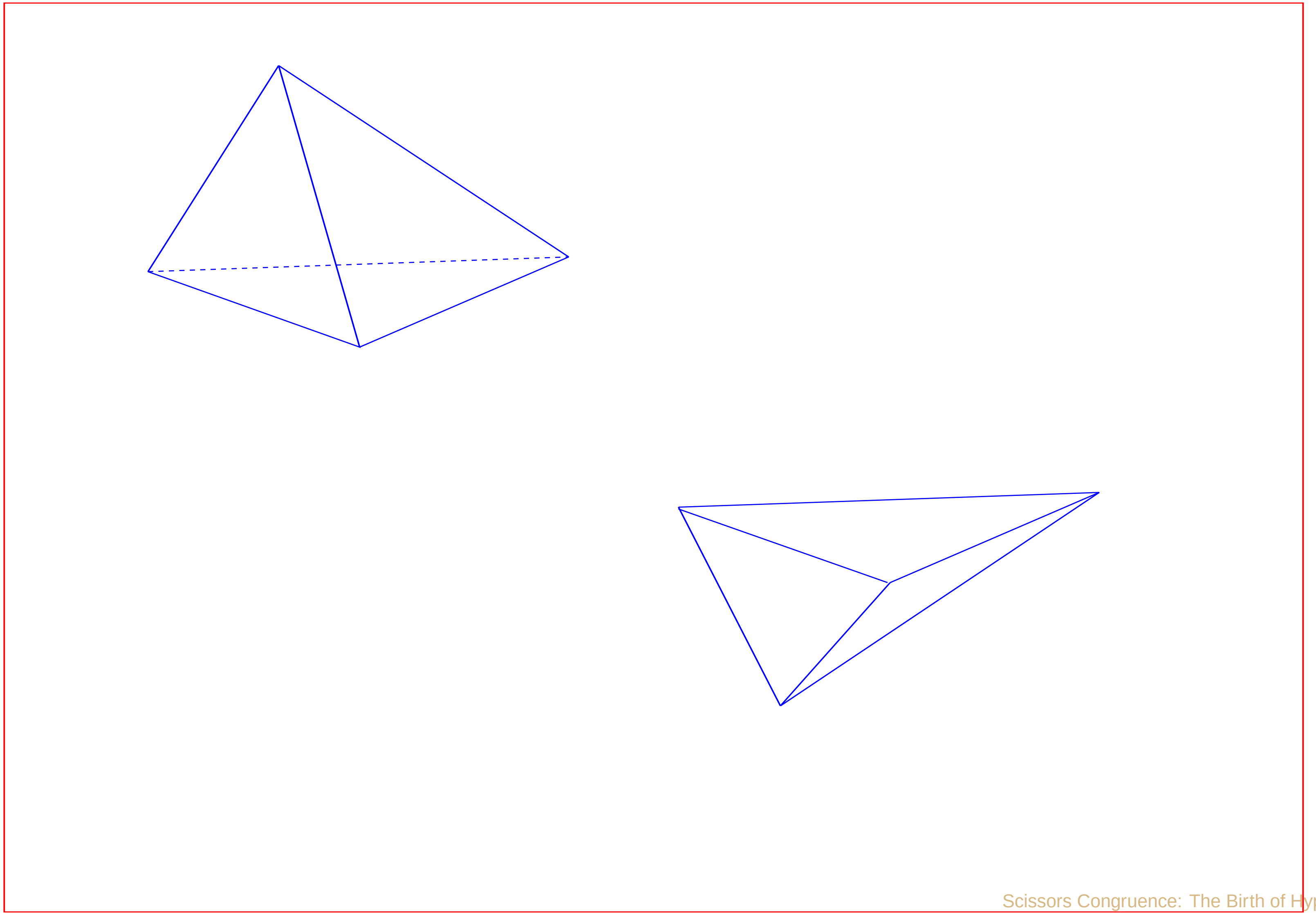
Key Relations

Two understand the need relations, take a pair of ideal tetrahedra and...



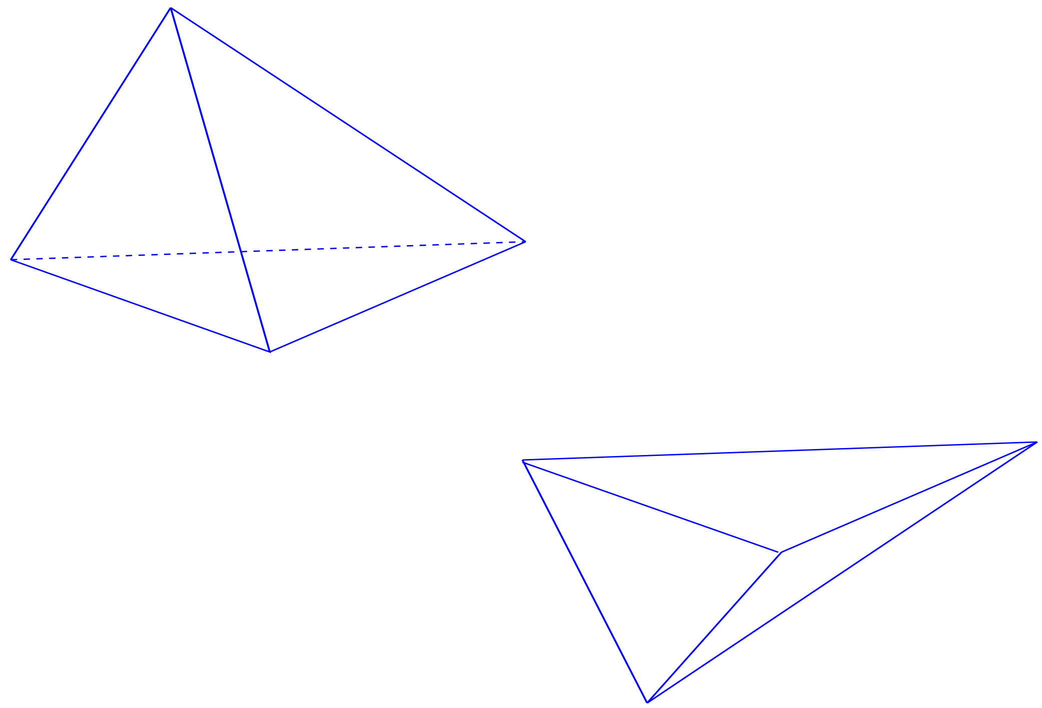
Key Relations

and...



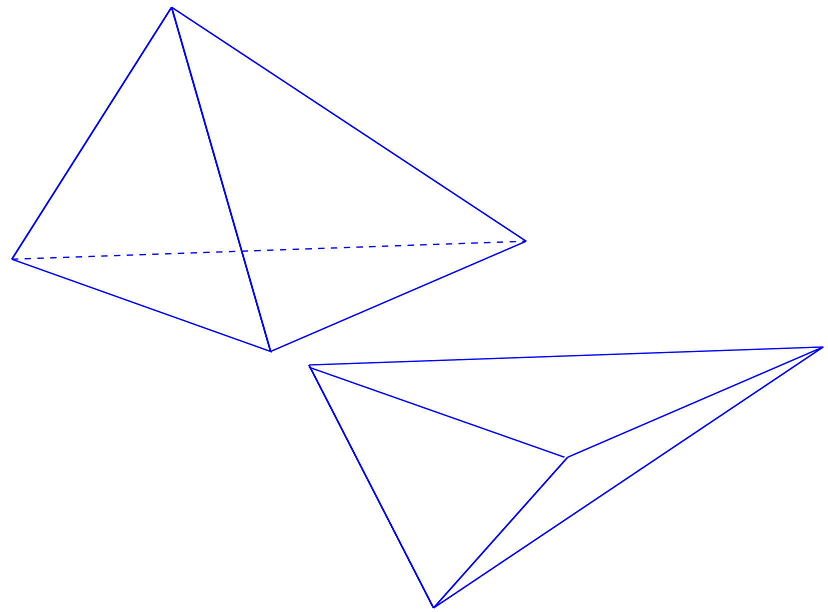
Key Relations

and...



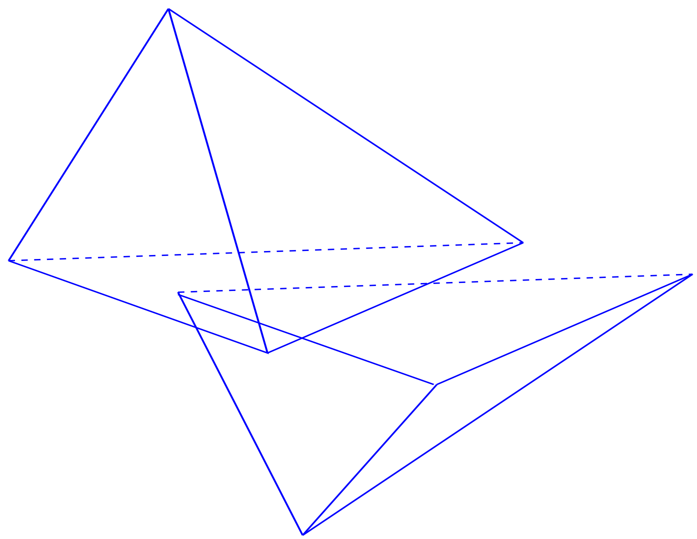
Key Relations

and...



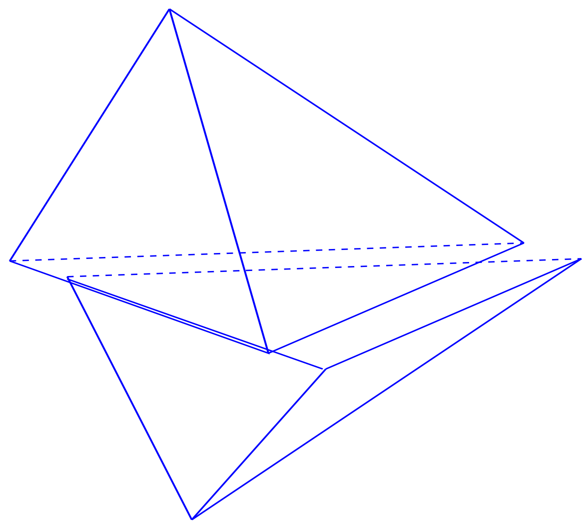
Key Relations

and...



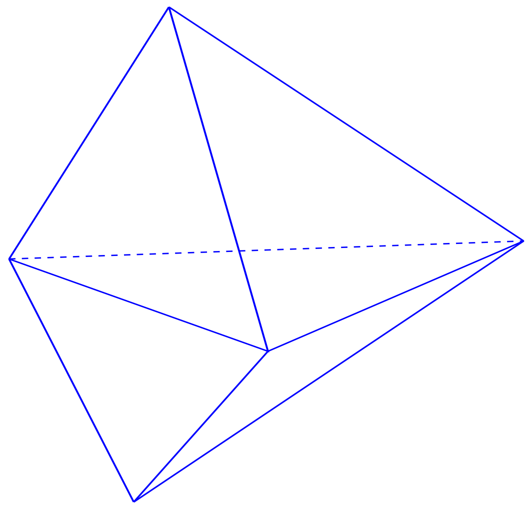
Key Relations

and...



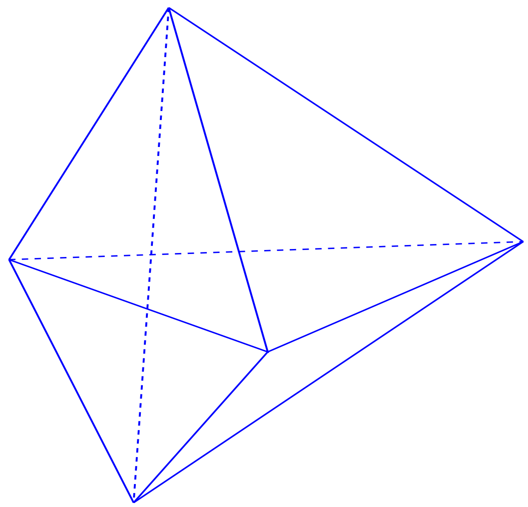
Key Relations

glue them together.



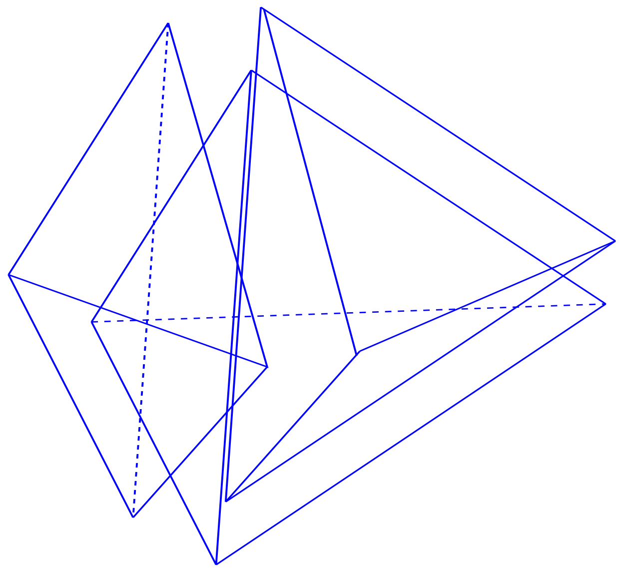
Key Relations

Now "firepole" this pair and...



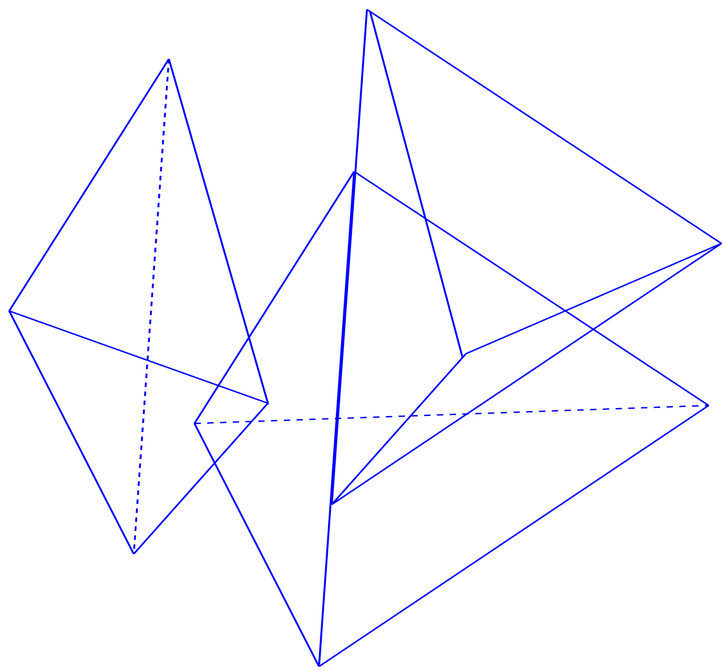
Key Relations

and...



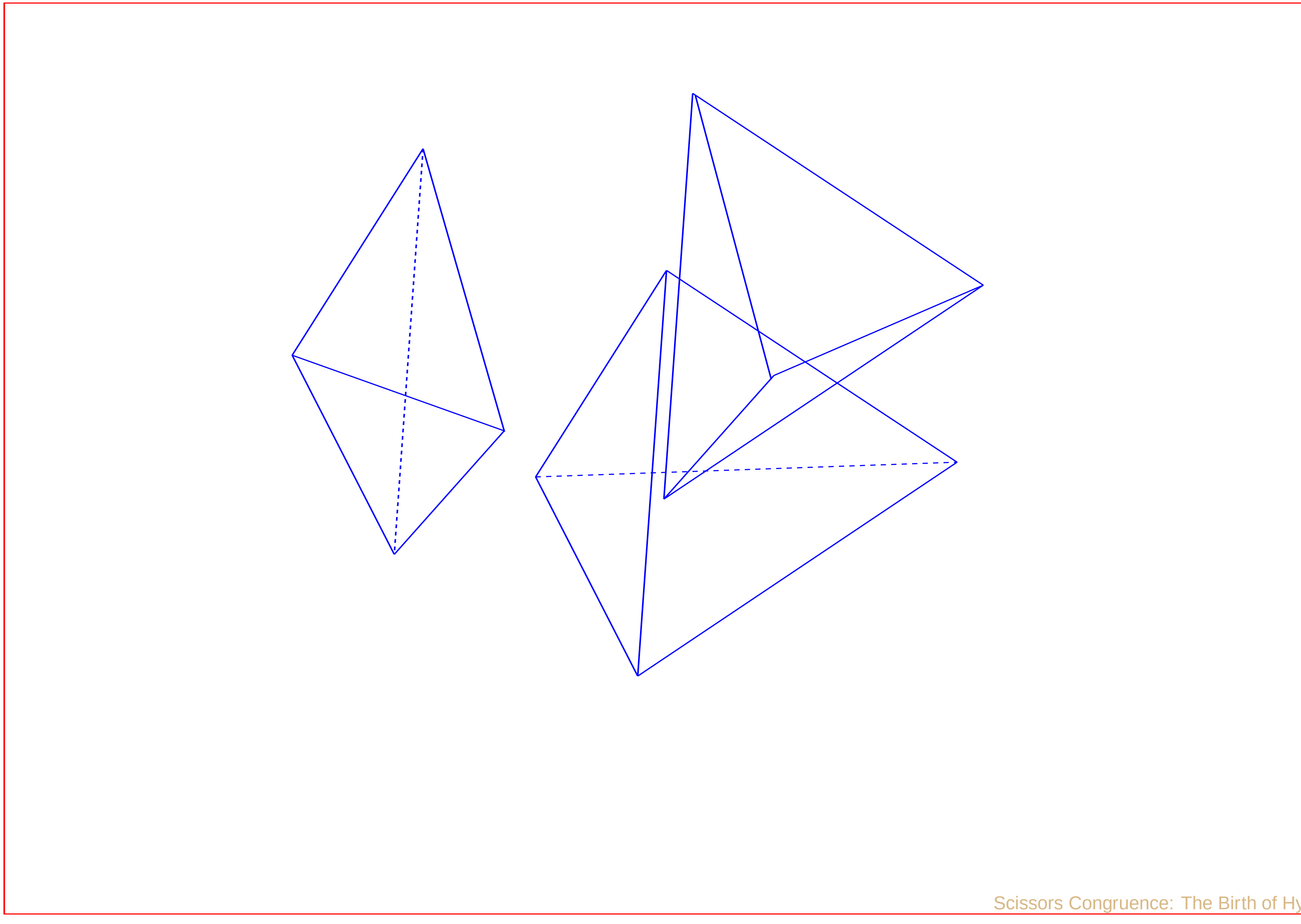
Key Relations

and...



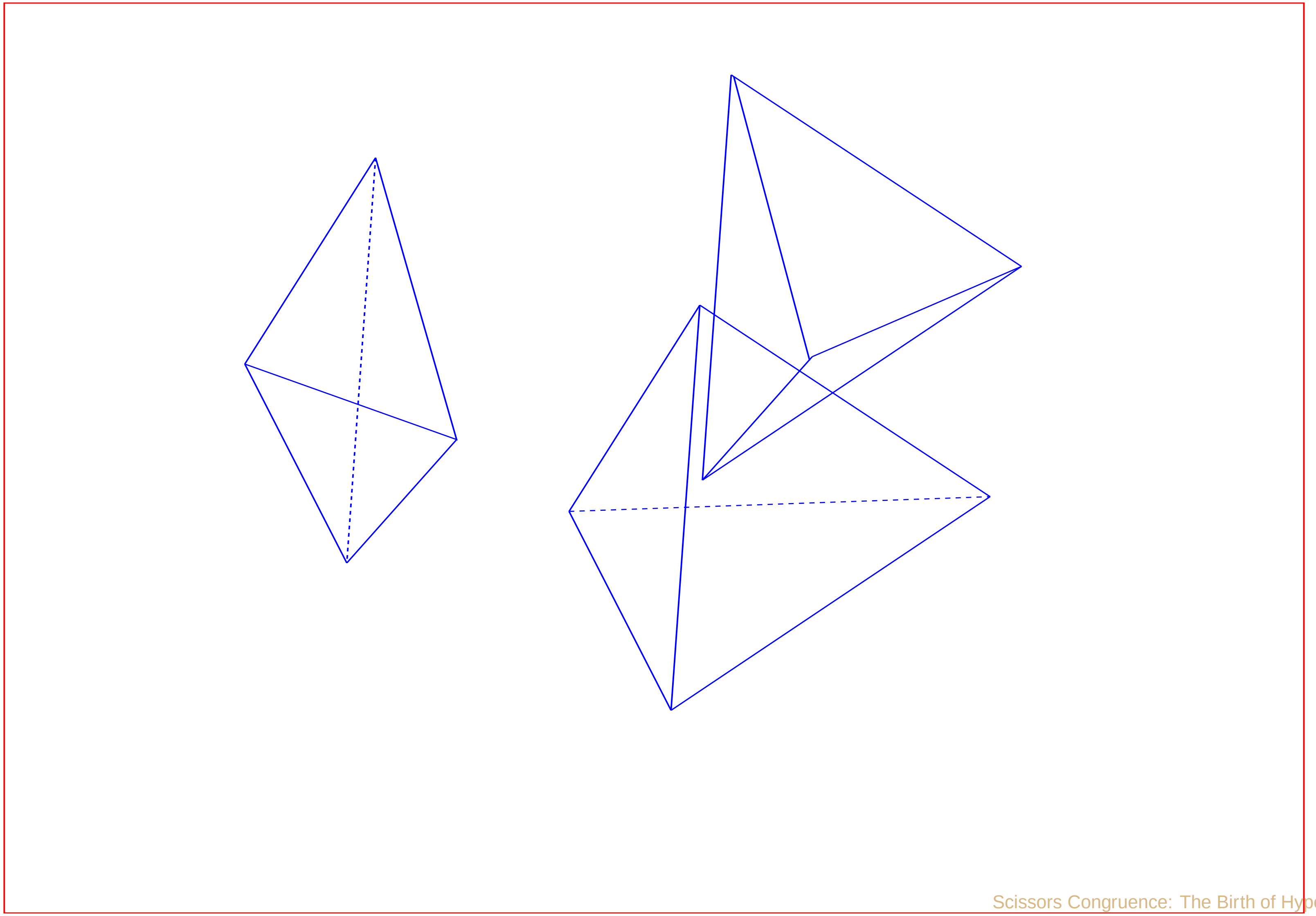
Key Relations

and...



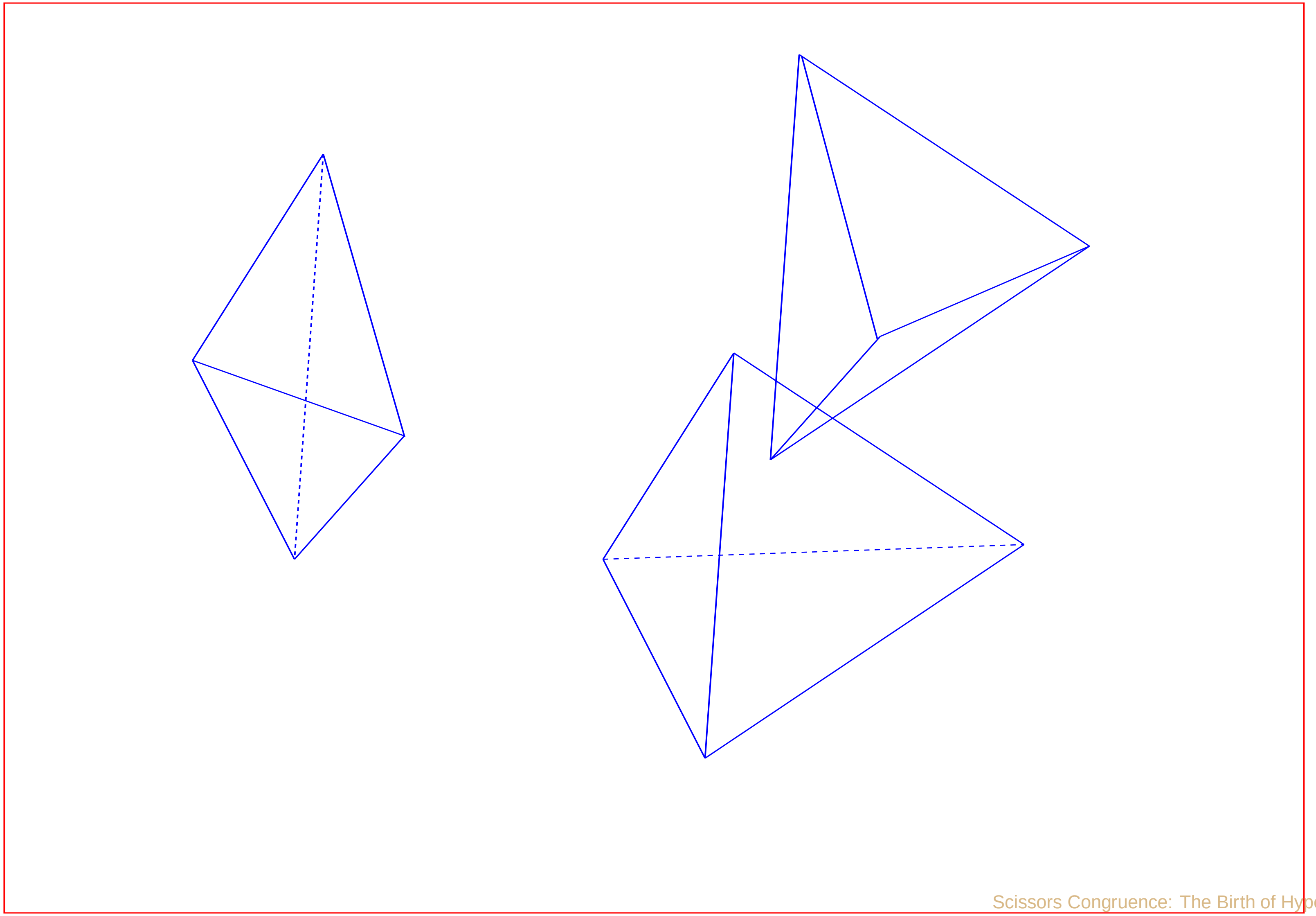
Key Relations

and...



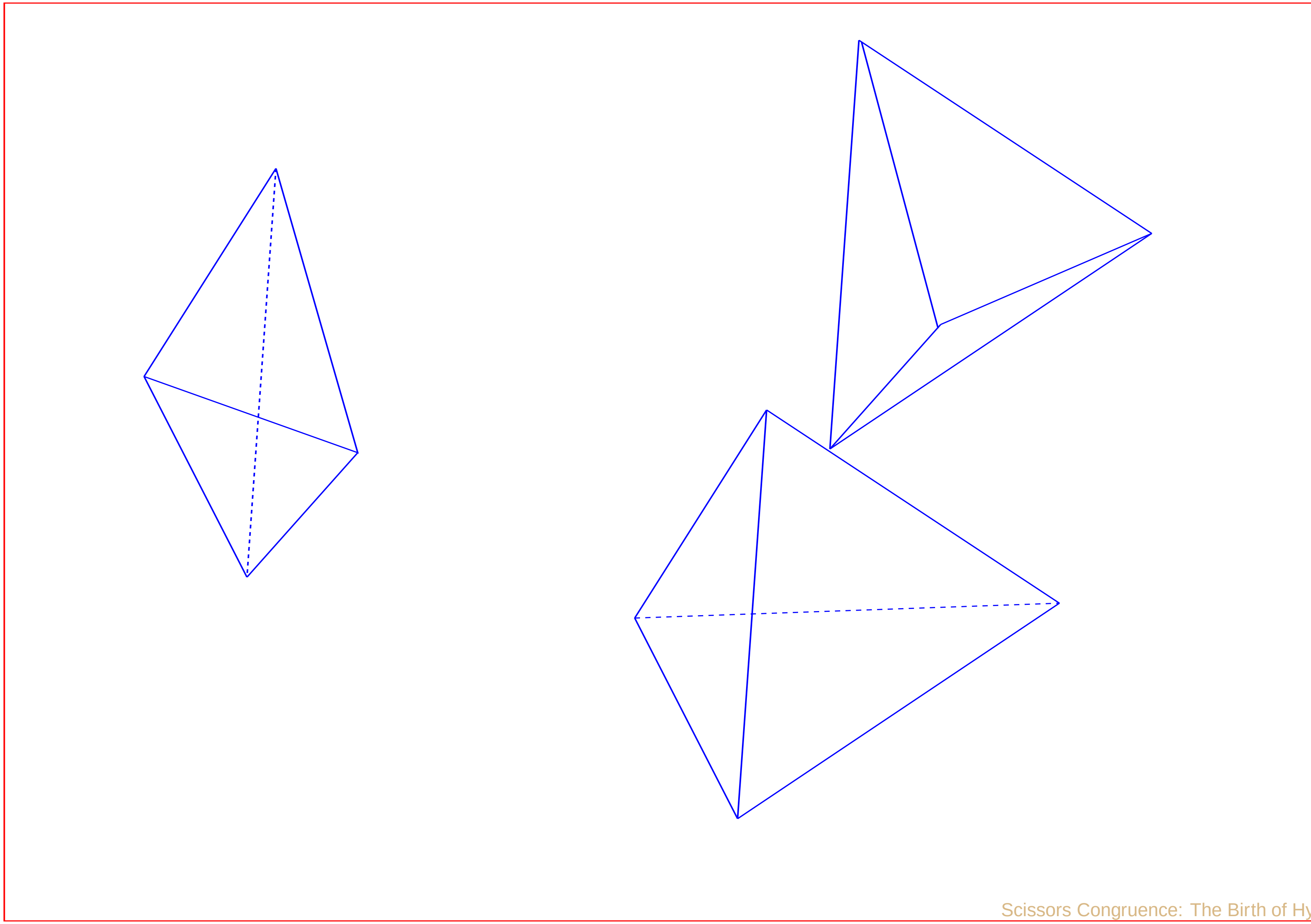
Key Relations

and...



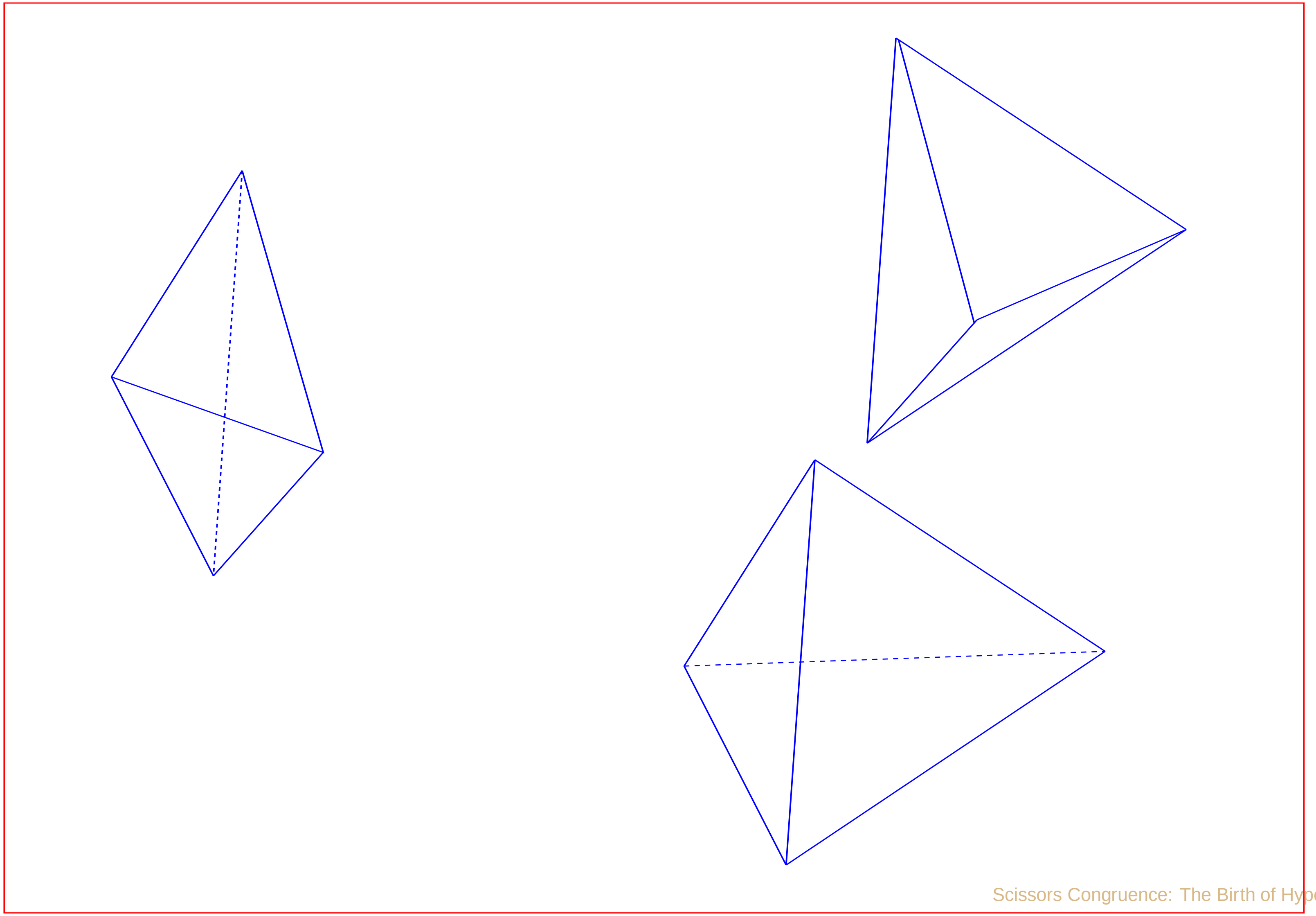
Key Relations

and...



Key Relations

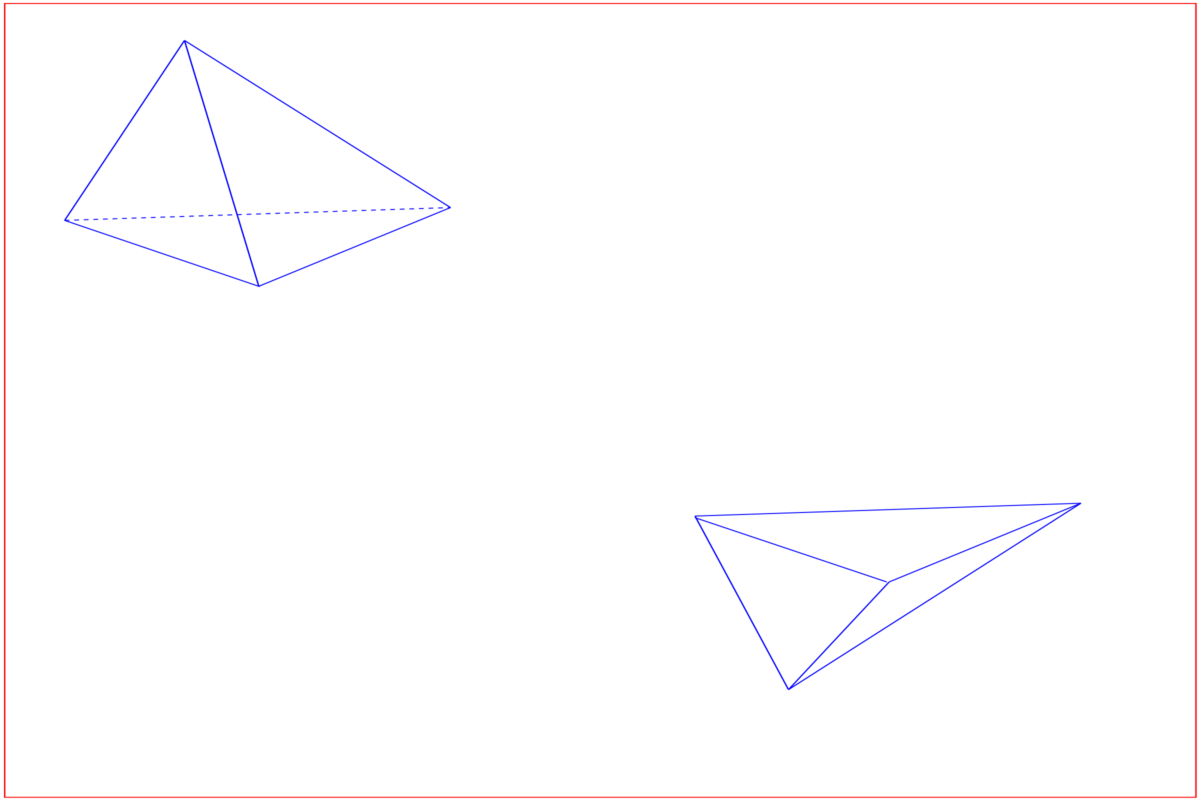
we can re-express this pair as three ideal tetrahedra. This is called a *2-3 move*.



Key Relations

In terms of the z coordinates we have

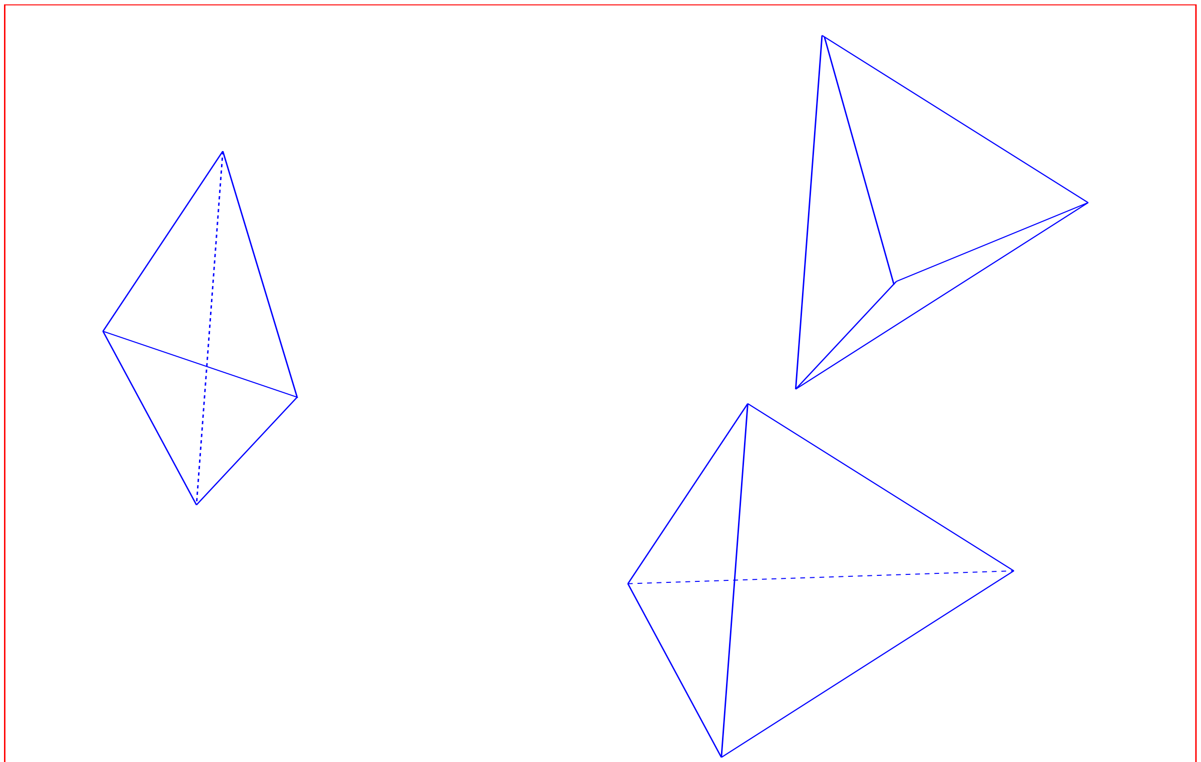
$$[z] + [w]$$



Key Relations

equals

$$[zw] + \left[\frac{z - zw}{1 - zw}\right] + \left[\frac{w - zw}{1 - zw}\right]$$



The Relations

Let T the subgroup of $< C >$ generated by all elements in the form

$$[z] + [w] - [zw] - \left[\frac{z - zw}{1 - zw}\right] - \left[\frac{w - zw}{1 - zw}\right],$$

where z and w are complex numbers, together with all elements in the form $[z] + [\bar{z}]$.

The Dupont and Sah Theorem

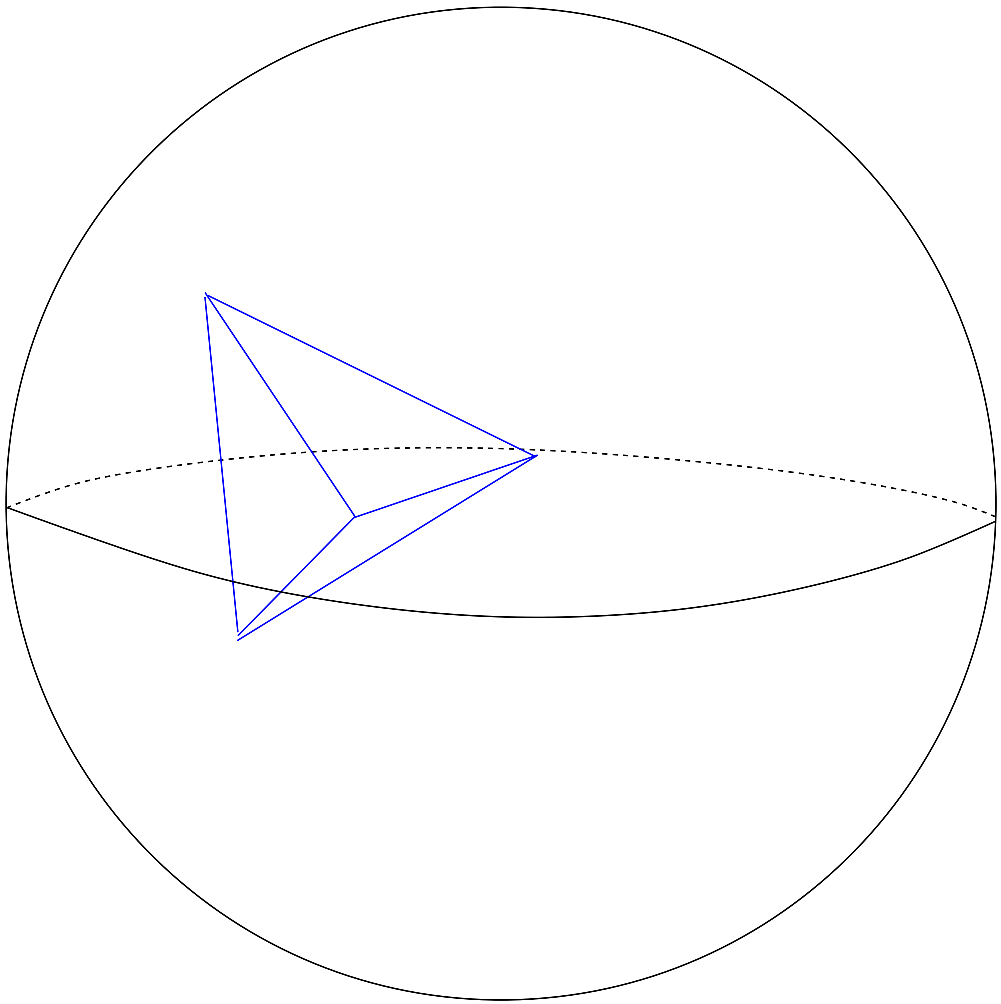
Wonderfully enough these are all the relations we need. **Theorem:**

(Dupont, Sah)

$$Sis(H^n) \cong \frac{< C >}{T},$$

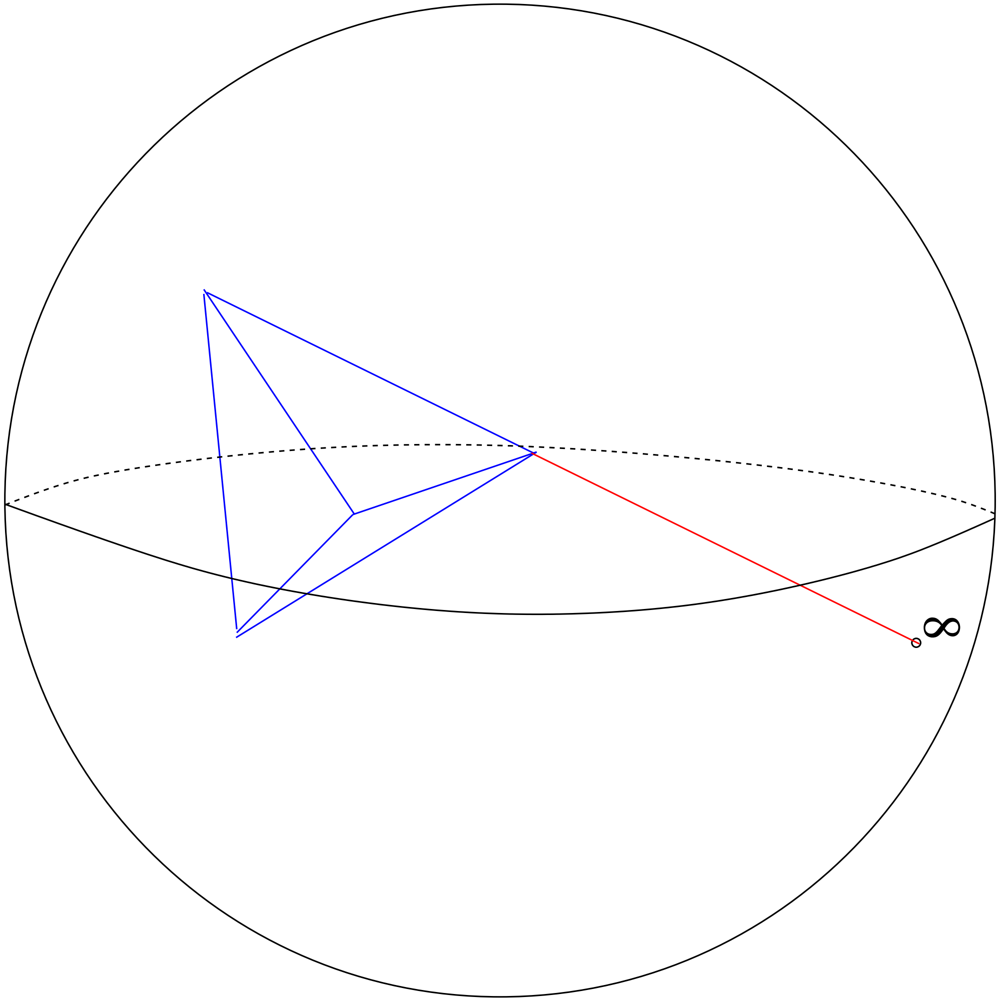
The Proof

Recall $Sis_\infty(H^n) \cong Sis(H^n)$. A key step in the proof is showing we can express a finite tetrahedron using ideal tetrahedra.



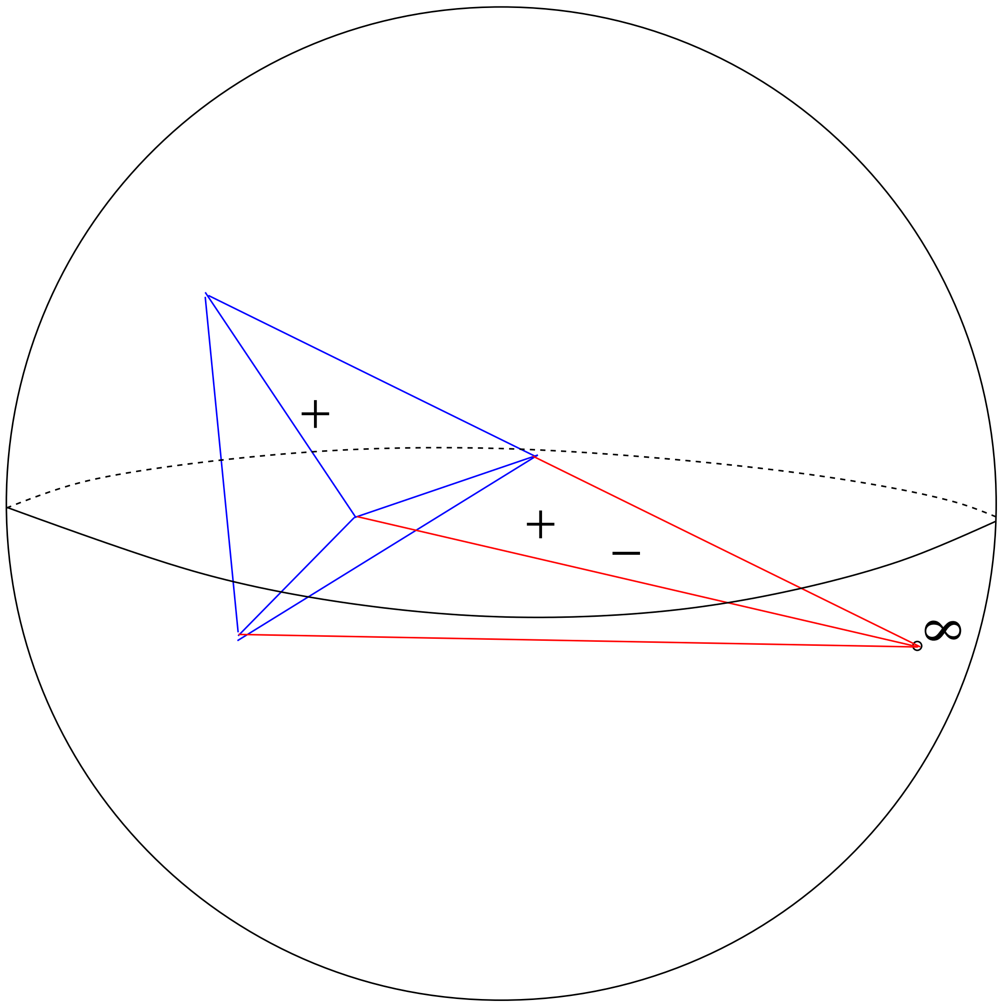
From Finite to Infinite

Let us make a finite vertex infinite. First extend an edge to infinity.



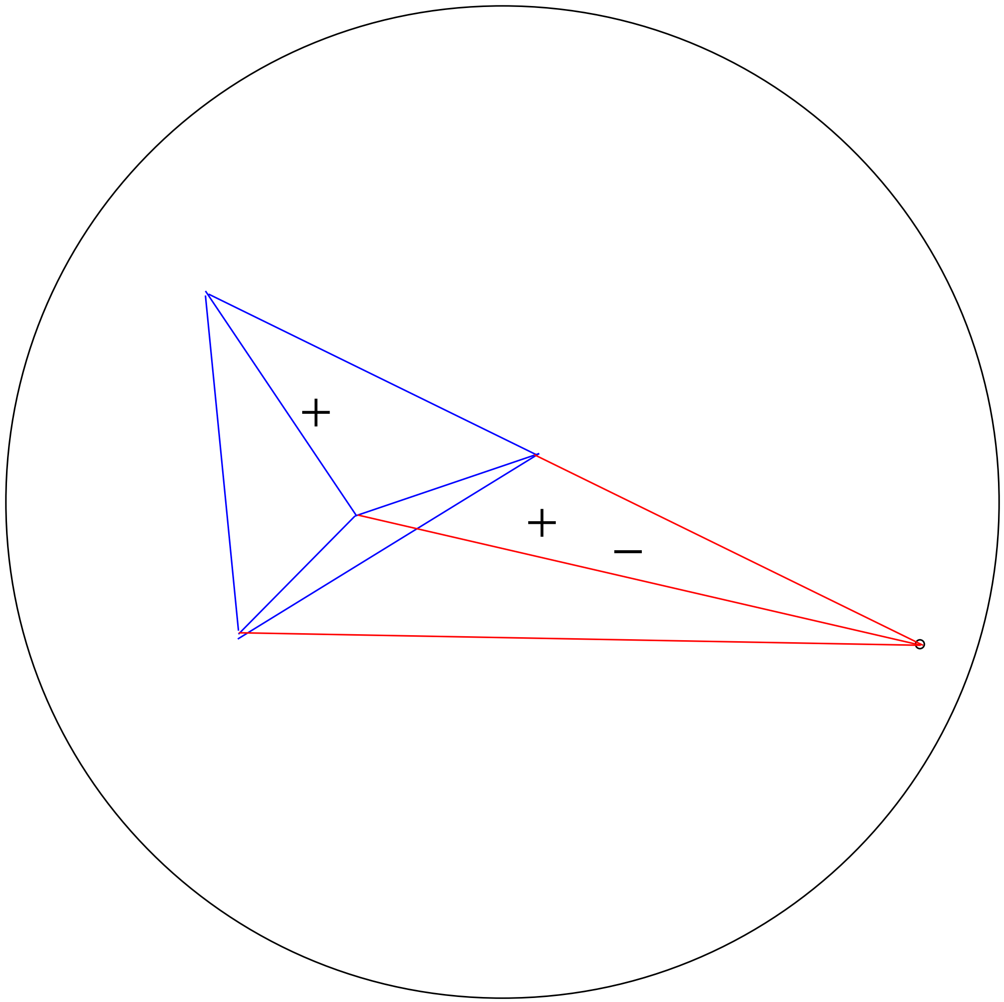
From Finite to Infinite

Then form the red tetrahedra, with an ideal vertex.



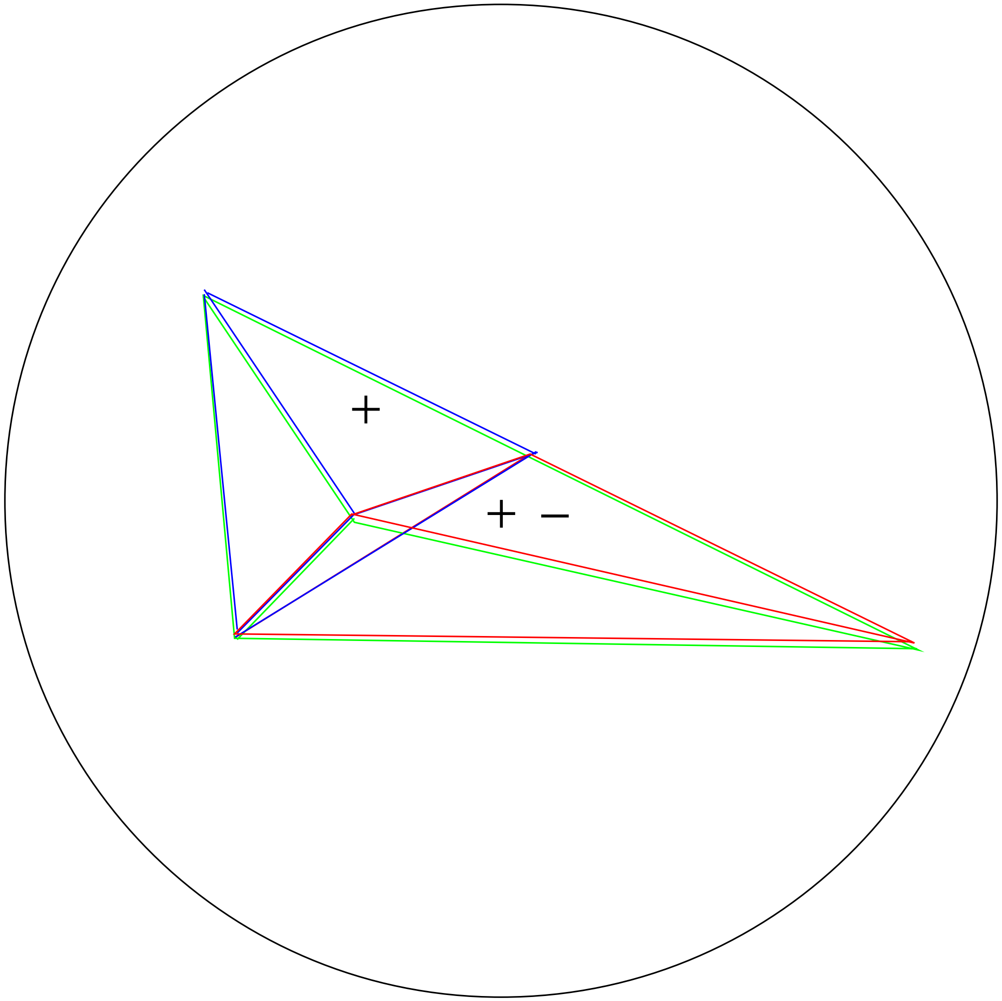
From Finite to Infinite

and note...



From Finite to Infinite

and note...



From Finite to Infinite

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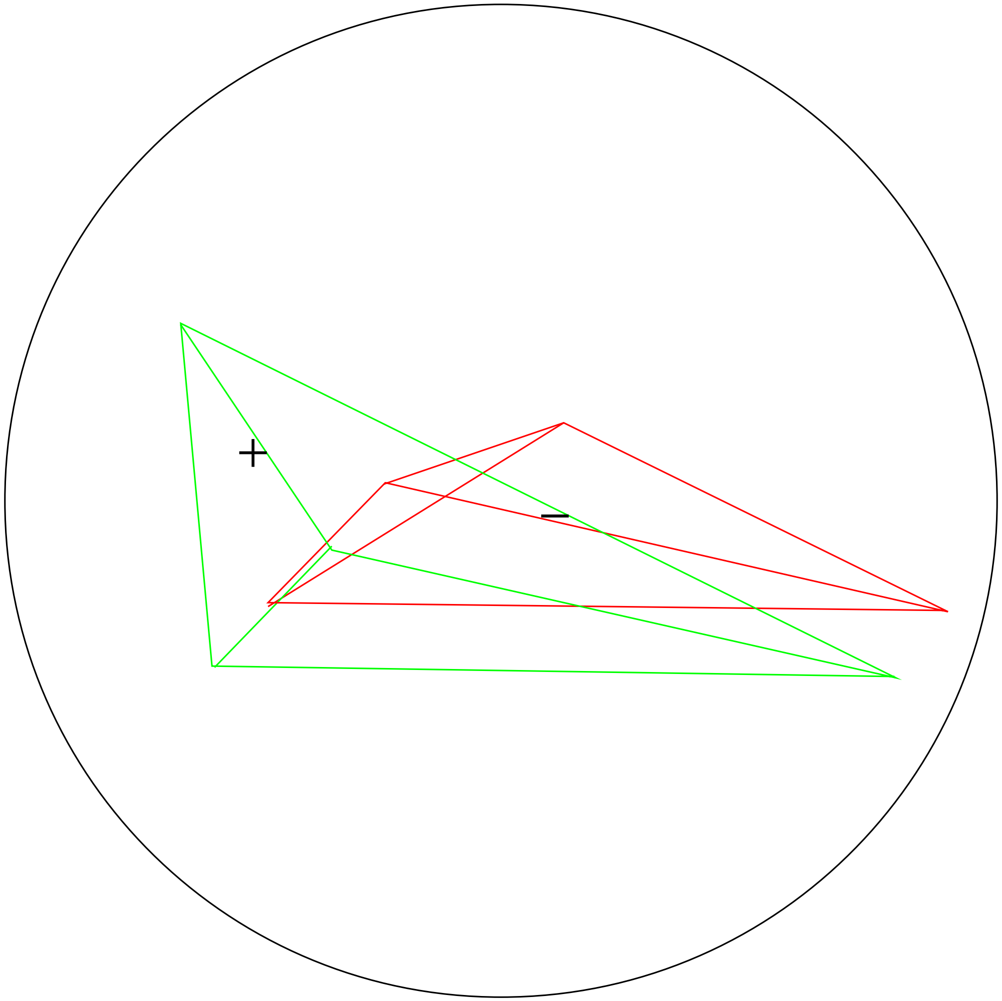

From Finite to Infinite

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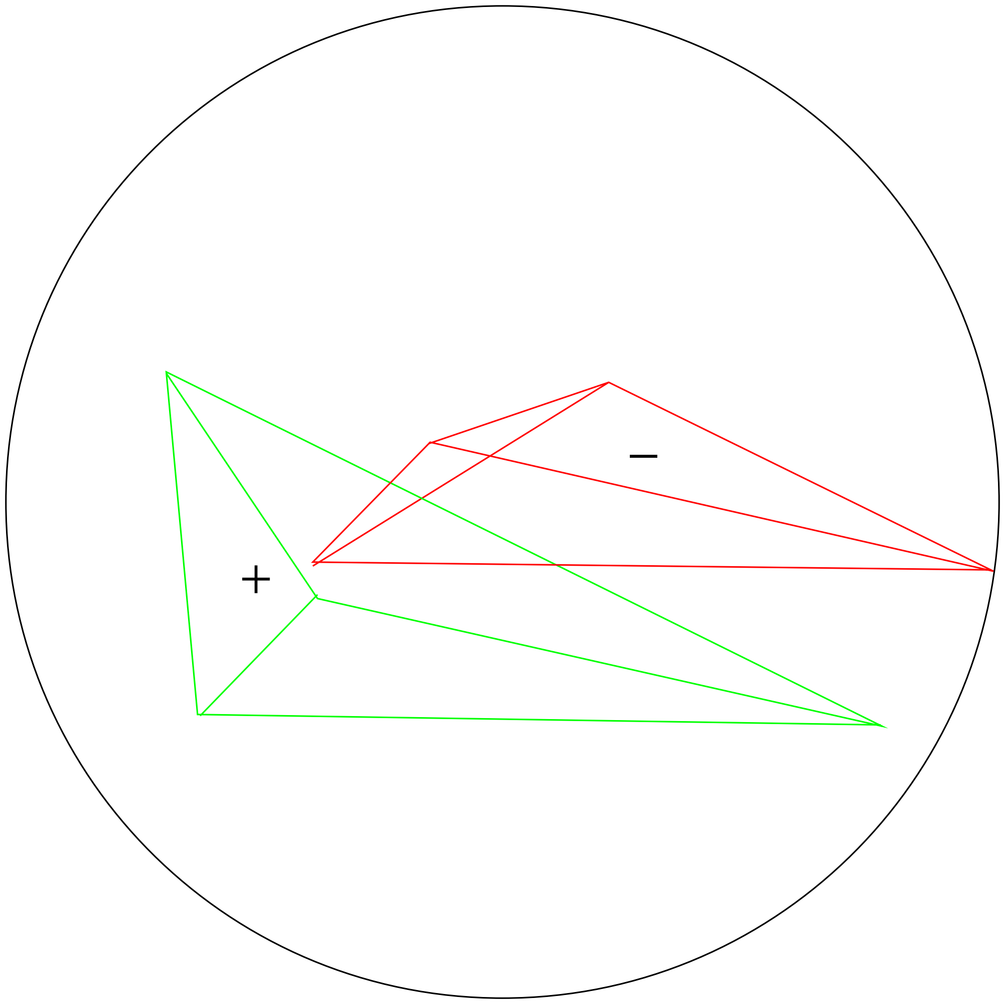

From Finite to Infinite

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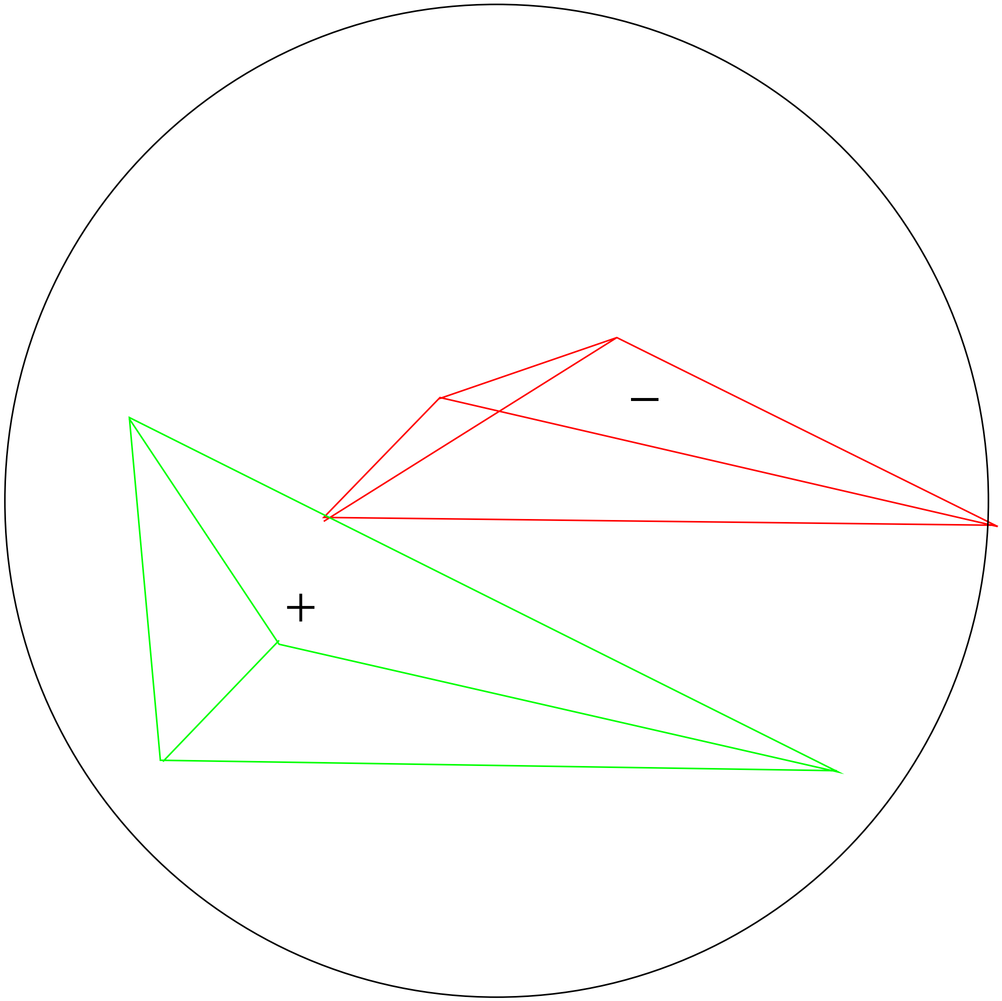
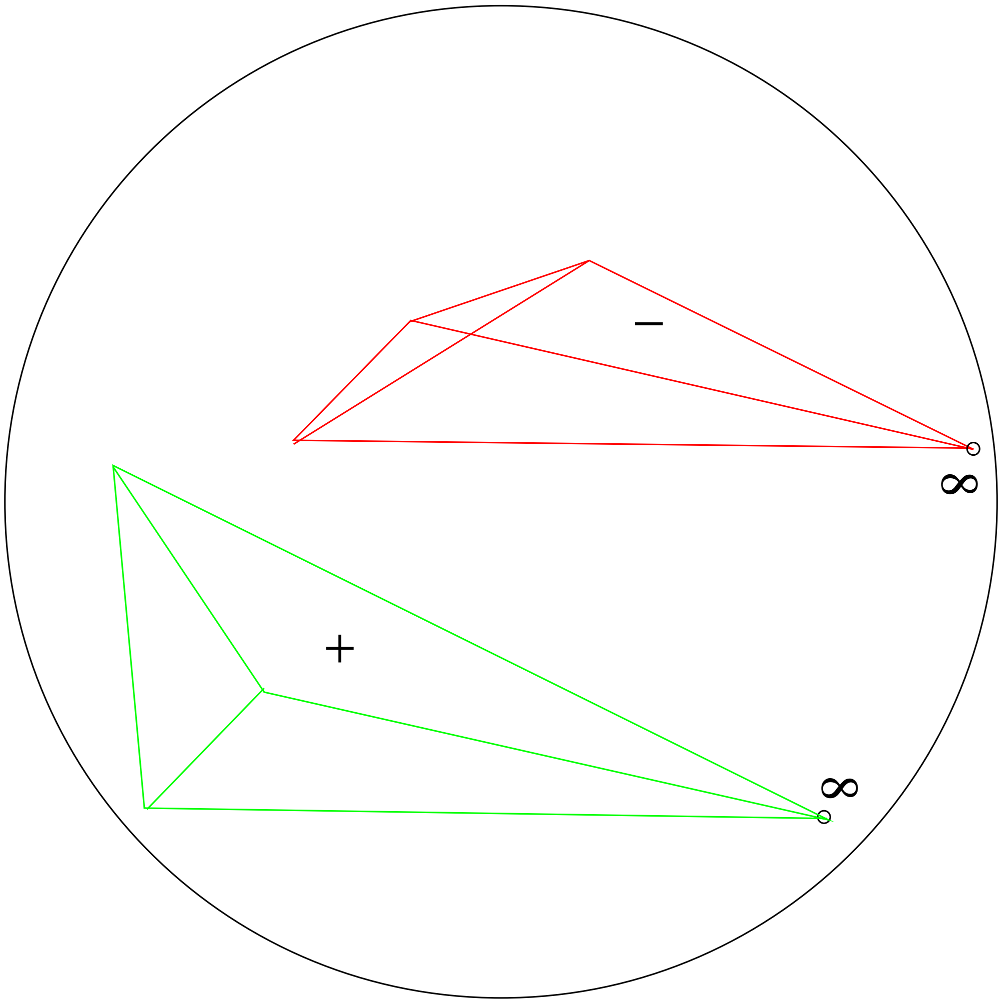


From Finite to Infinite

Hence, we have expressed the finite tetrahedron using two ideal tetrahedra each with only 3 finite vertices.



From Finite to Infinite

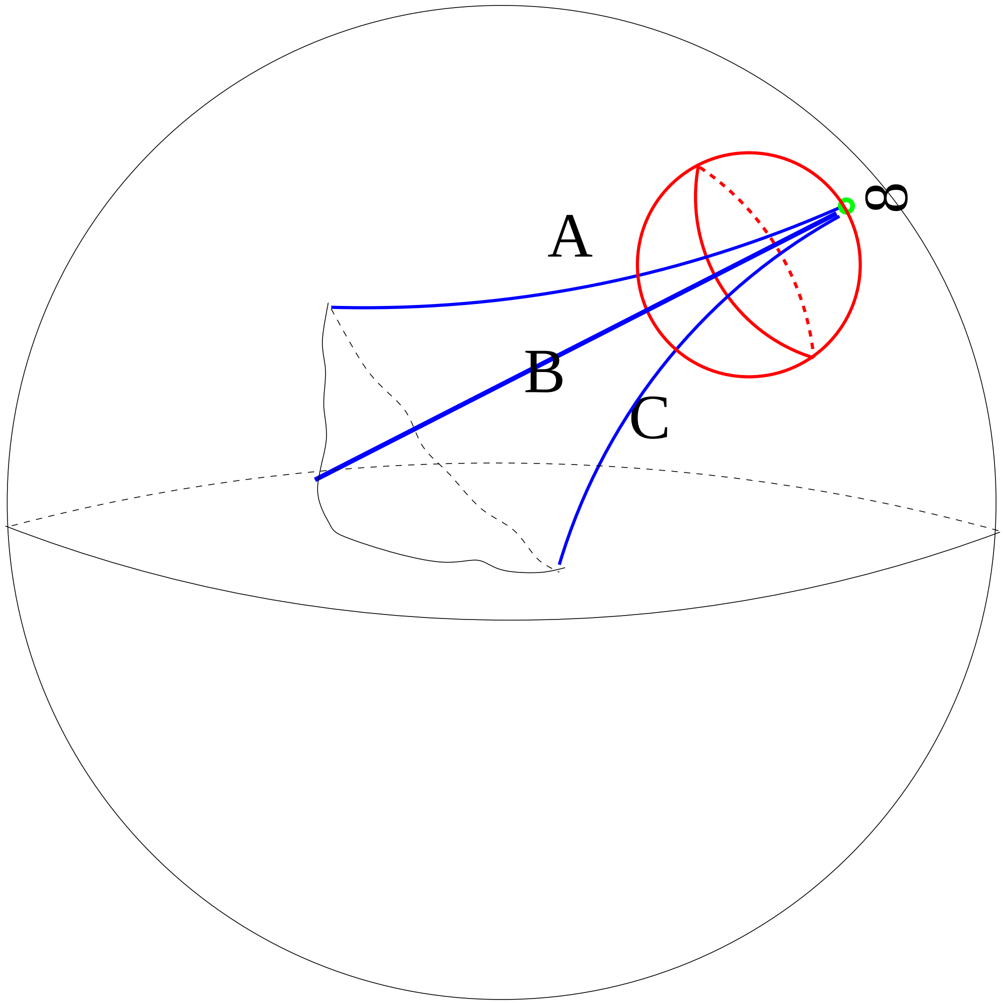
One can continue this till one is using only ideal tetrahedra. The hard step is removing the final vertex. The best known method to do this is due to Yana Mohanty (2003).

Getting a Grip on Volume

At this point, we see that understanding hyperbolic volume can be reduced to understanding the volume of an ideal tetrahedron. To this it useful to take a close look at the ideal tetrahedron's angles.

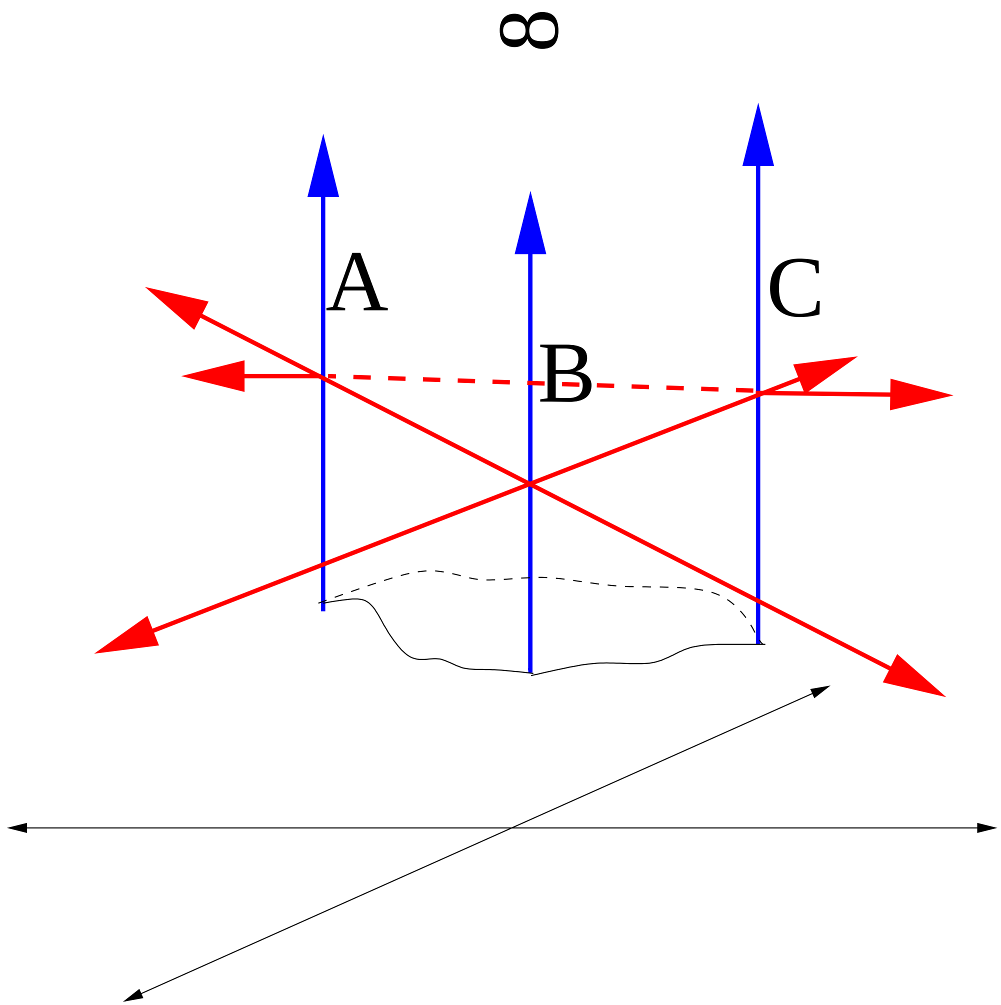
Ideal Tetrahedron's Angles

Given any ideal polyhedron, at each ideal vertex we see this. The red sphere is a horosphere.



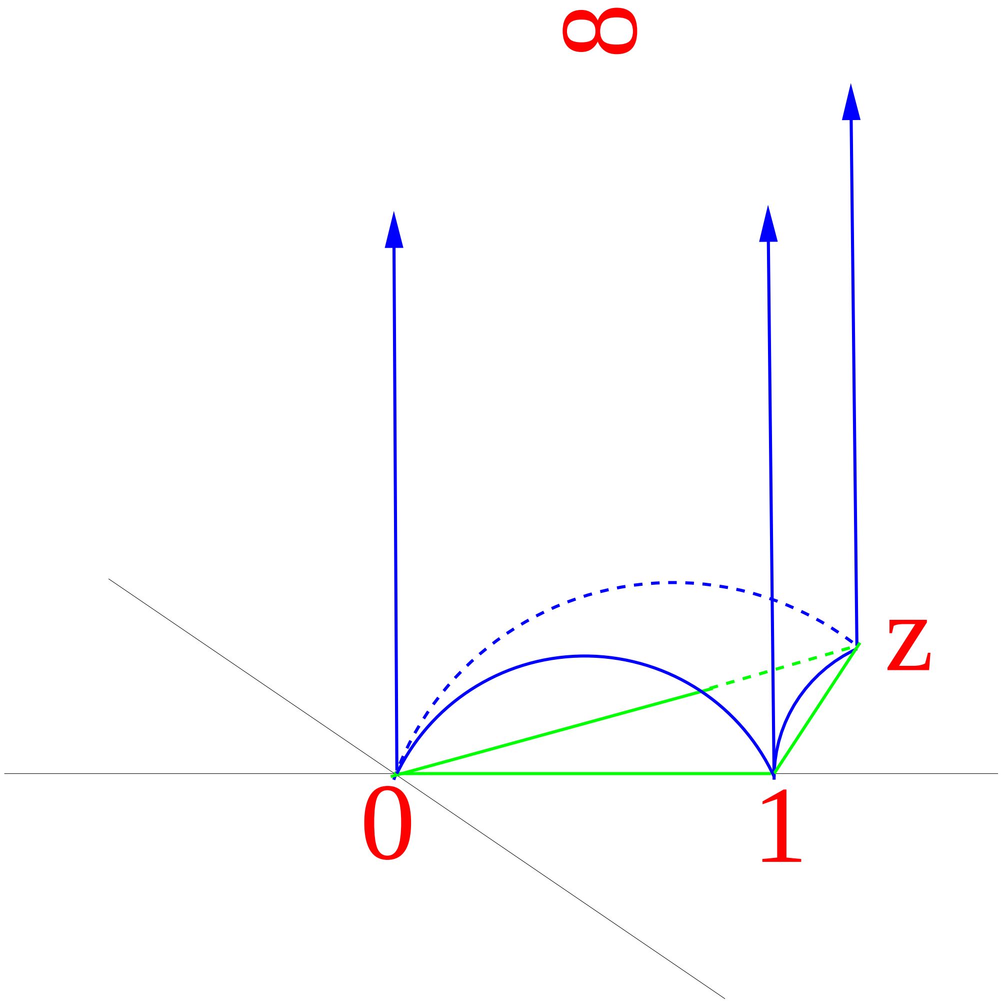
Euclidean Angles

Sending the ideal vertex to the point at infinity in the upper-half space model, we find that the angles at an ideal vertex are Euclidean.



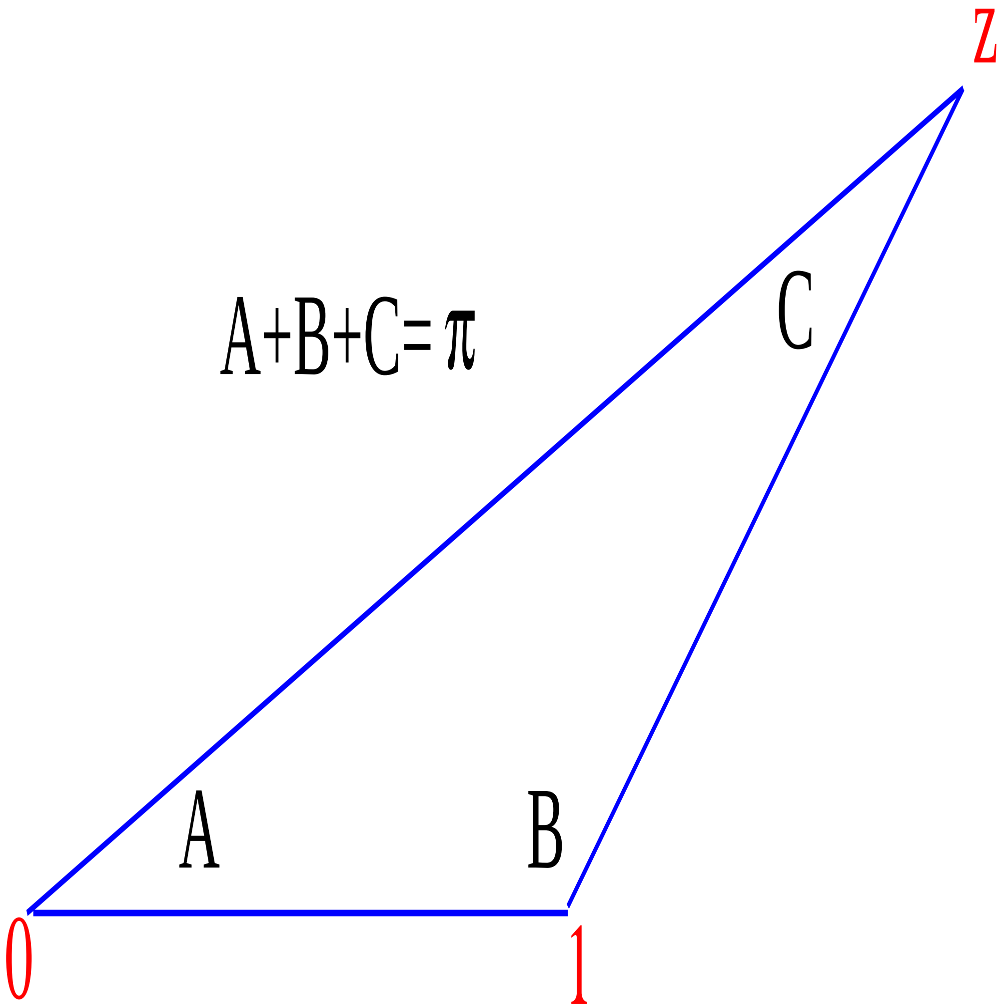
Ideal Tetrahedron's Angles

We view our tetrahedron in the upper-half space model.



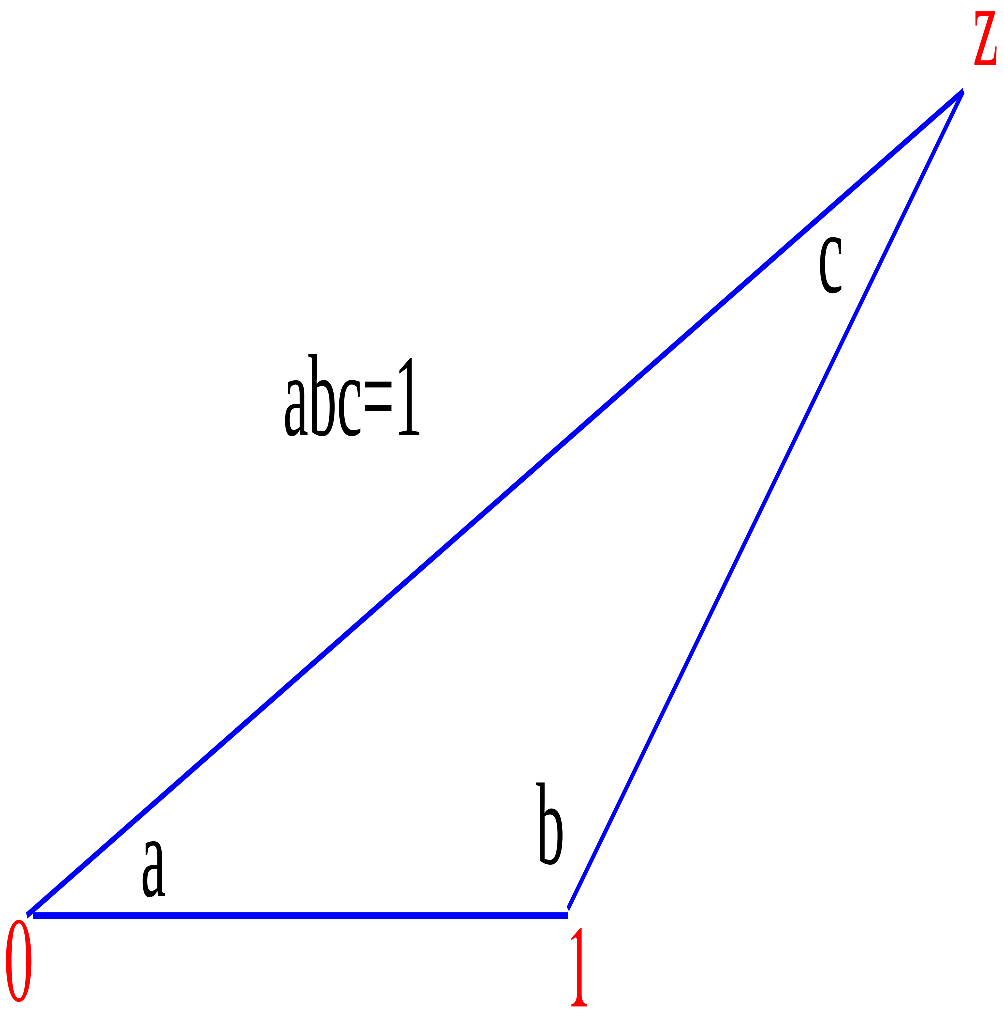
Ideal Tetrahedron's Angles

Looking down from infinity we see.



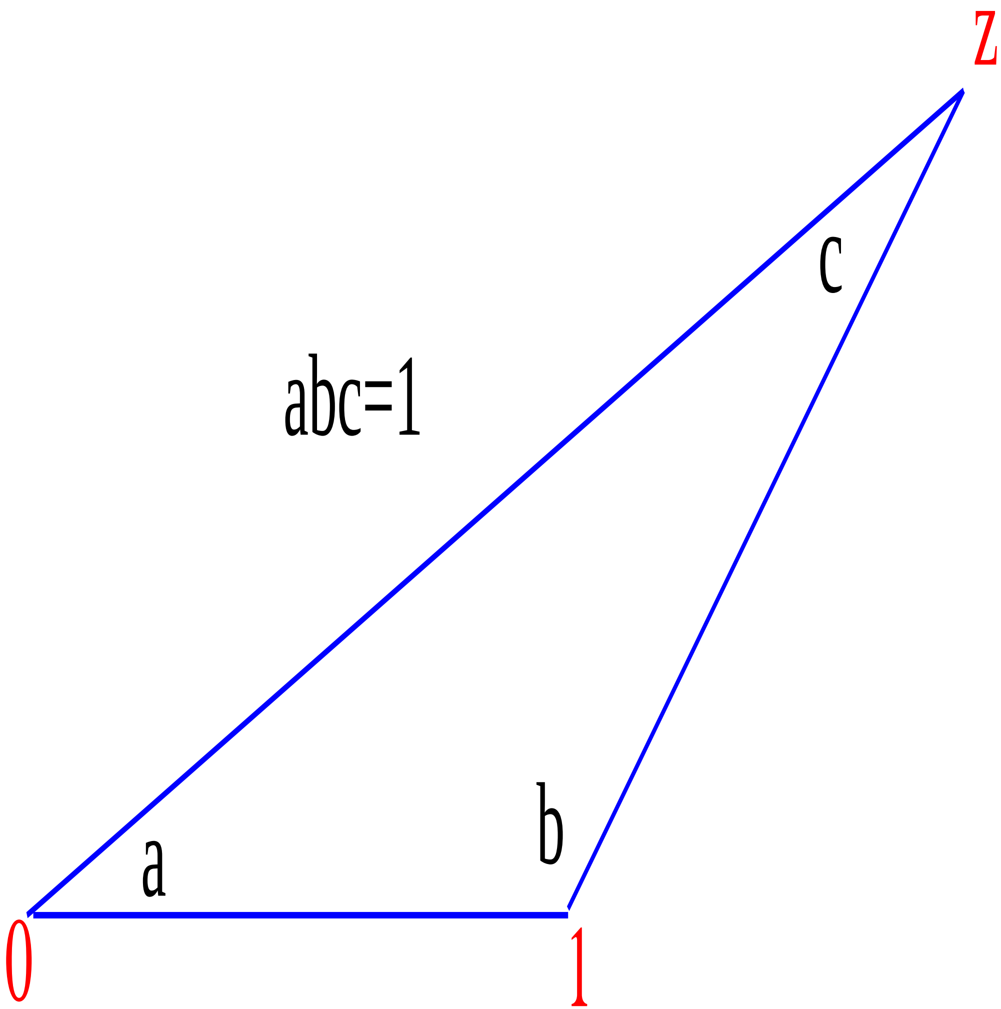
Ideal Tetrahedron's Clinants

It is best not to think in terms of the dihedral angles but rather the dihedral *clinants*. Namely $e^{2I\theta}$ is the clinant associated to the angle θ.



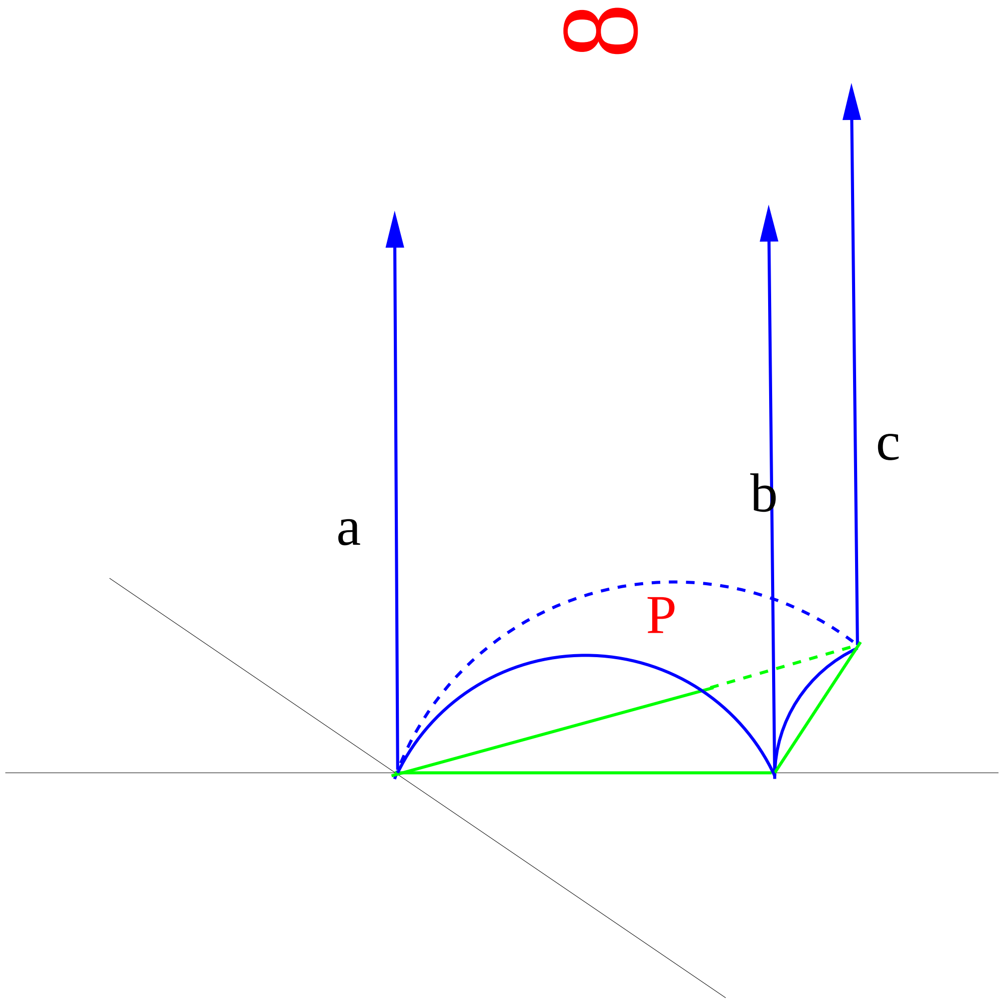
Ideal Tetrahedron's Clinants

The compactification of the space of ideal tetrahedra is all clinants triples (a, b, c) such that $abc = 1$, "blown up" at $(1, 1, 1)$. To see this, note that the z coordinate equals $\frac{1-a}{1-b}$.



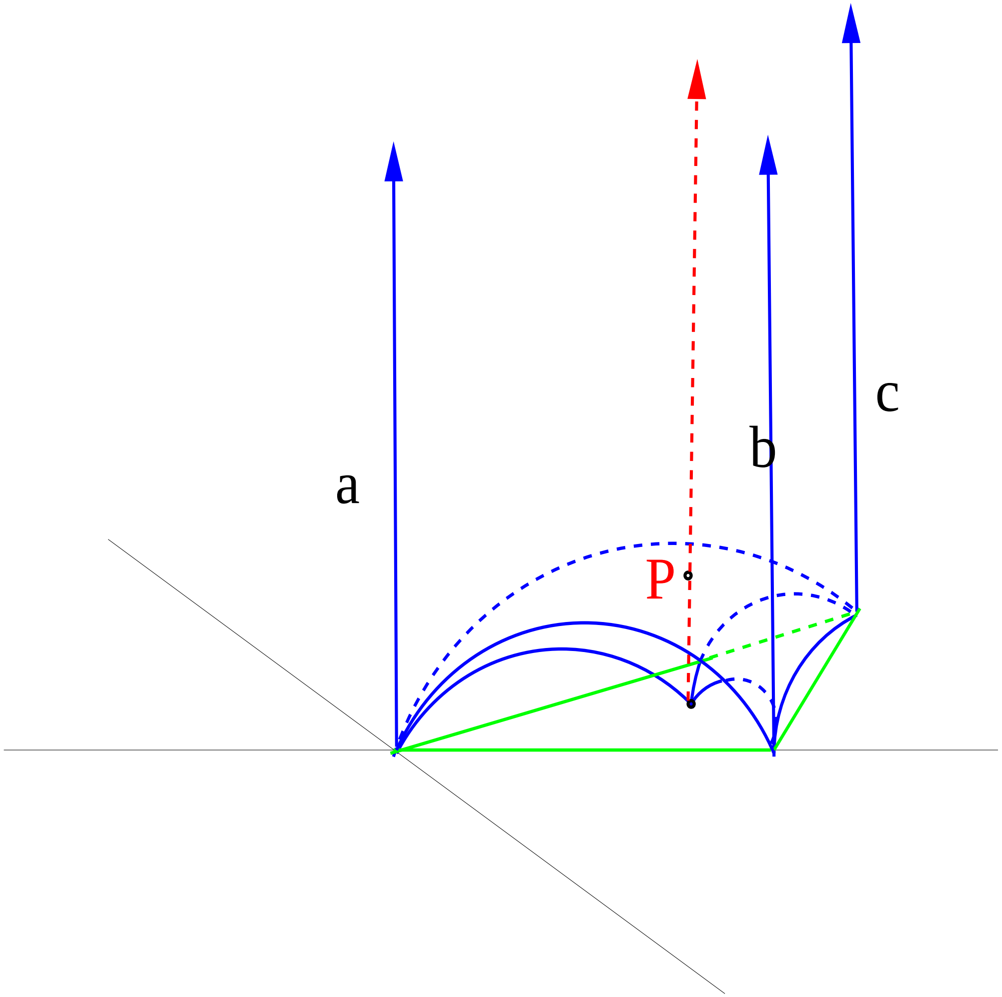
Decomposing Ideal Tetrahedron

We need one more decomposition. Start with an ideal tetrahedron...



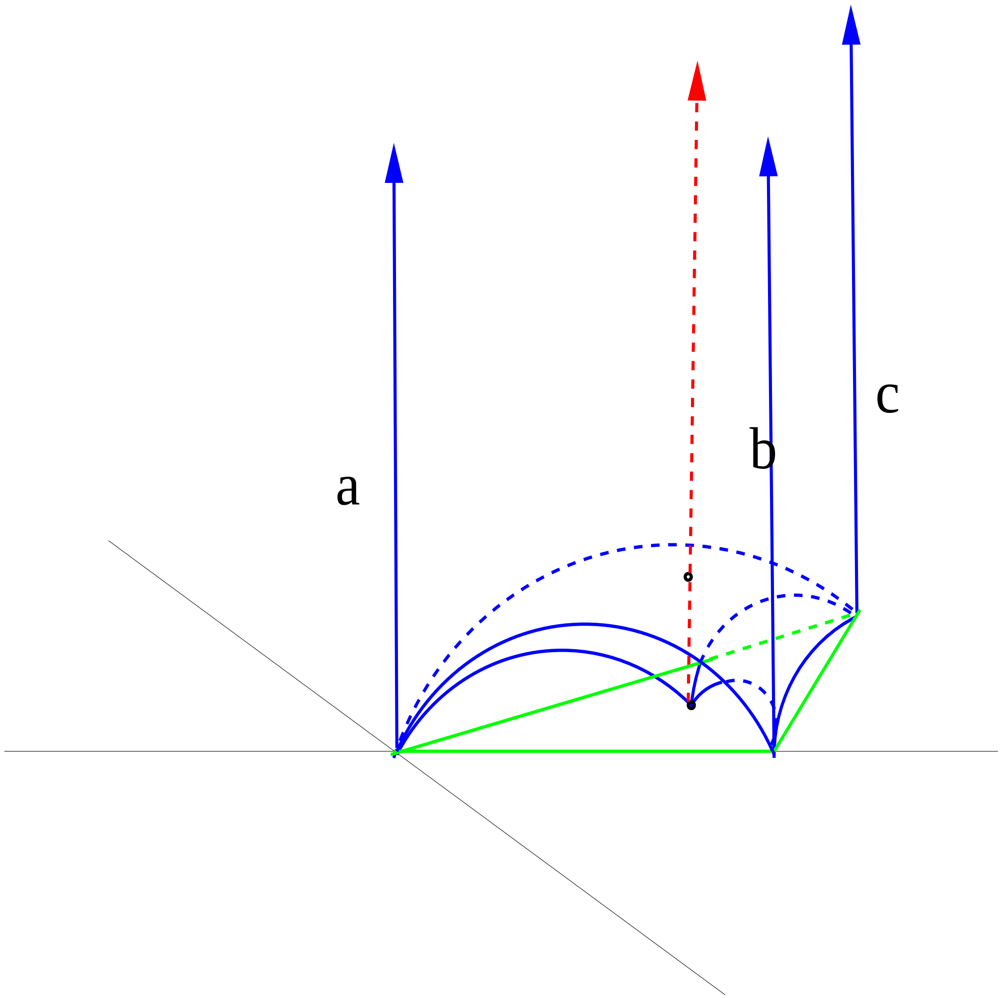
Decomposing Ideal Tetrahedron

and double it.



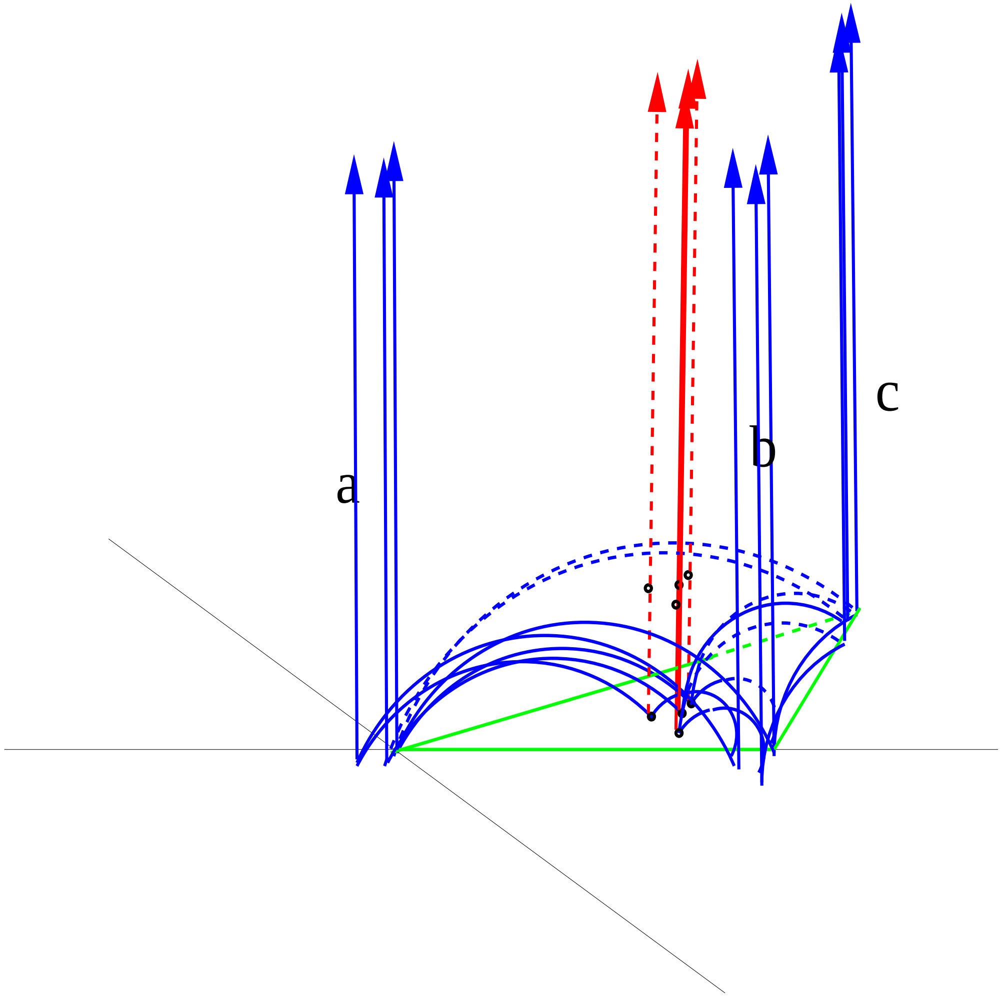
Decomposing Ideal Tetrahedron

Firepole this doubled ideal tetrahedron.



Decomposing Ideal Tetrahedron

Then we have our 2-3 move which....



Decomposing Ideal Tetrahedron

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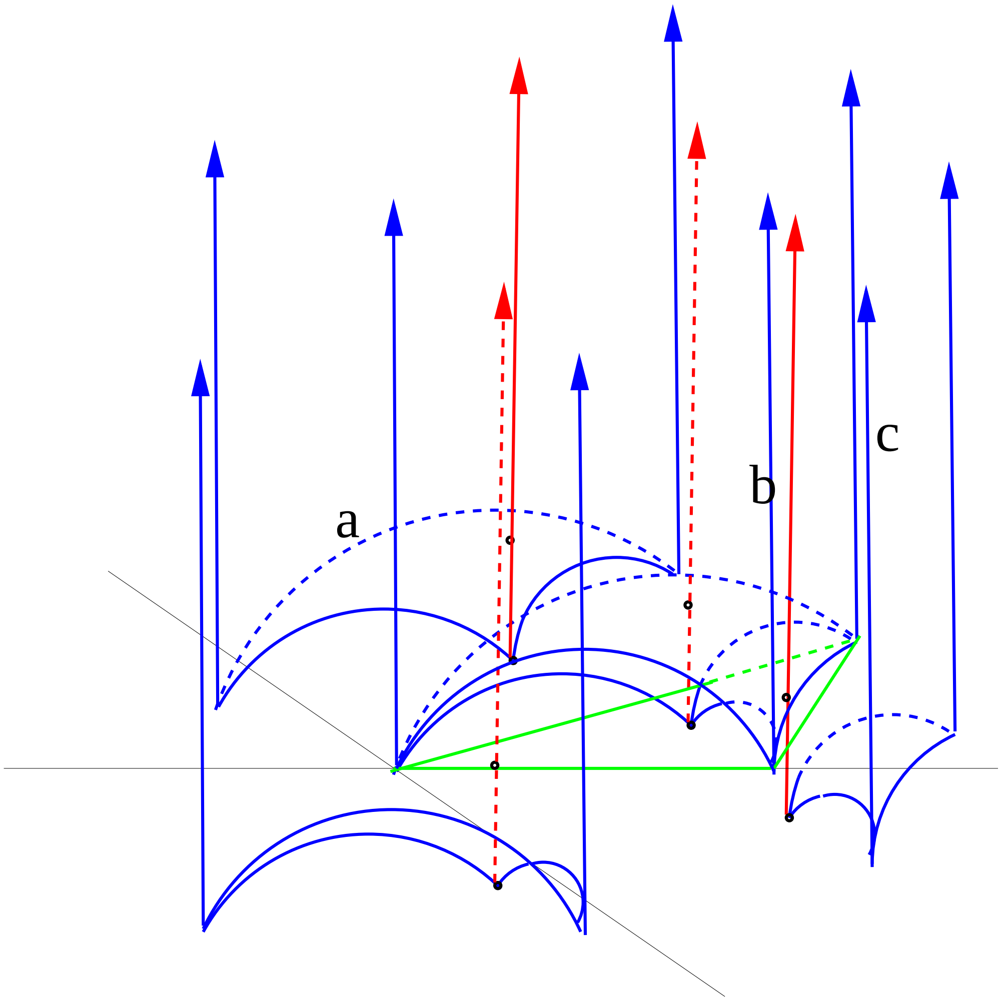

Decomposing Ideal Tetrahedron

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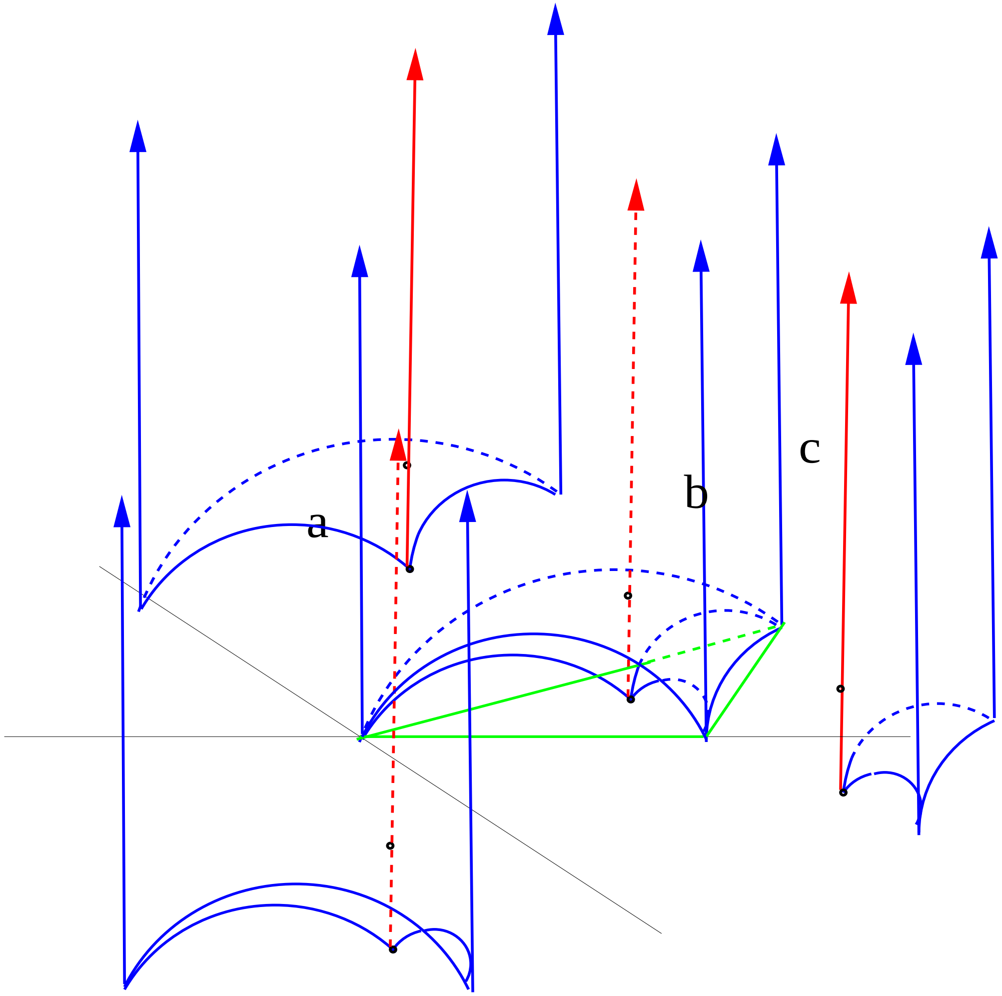

Decomposing Ideal Tetrahedron

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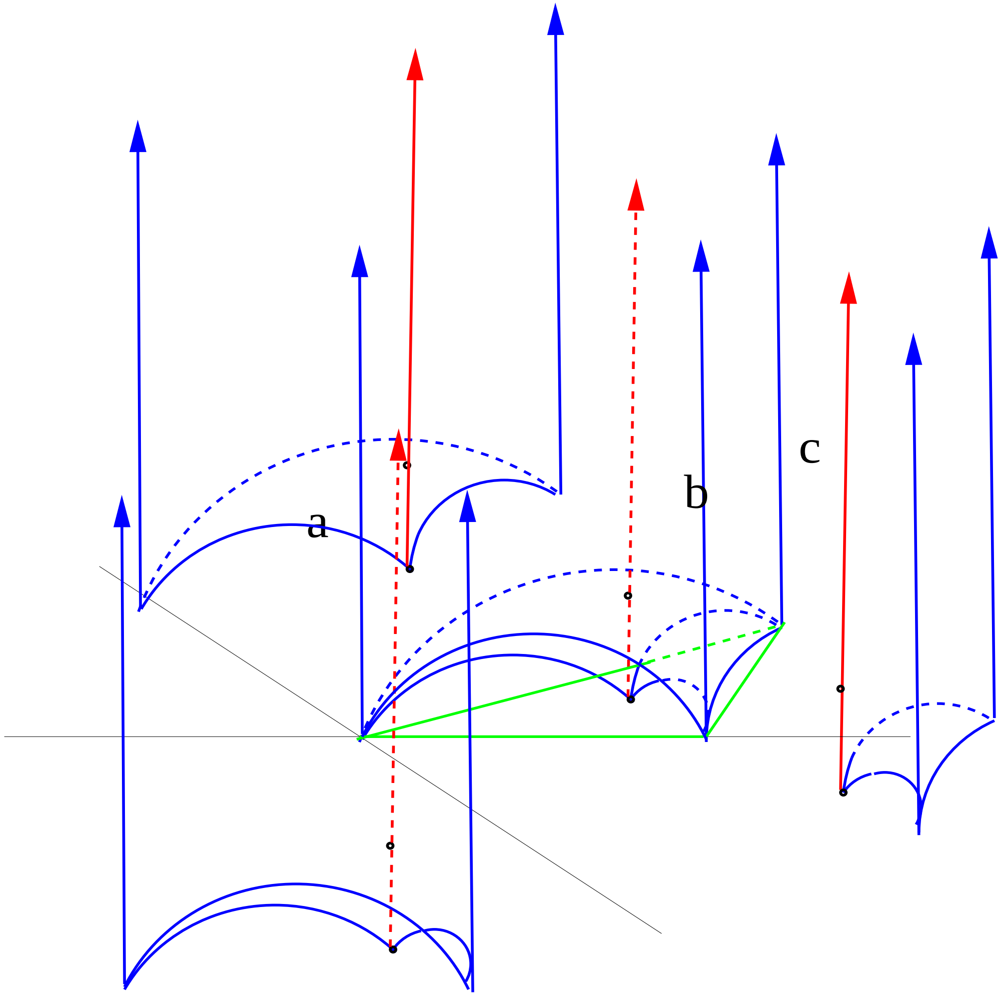

Decomposing Ideal Tetrahedron

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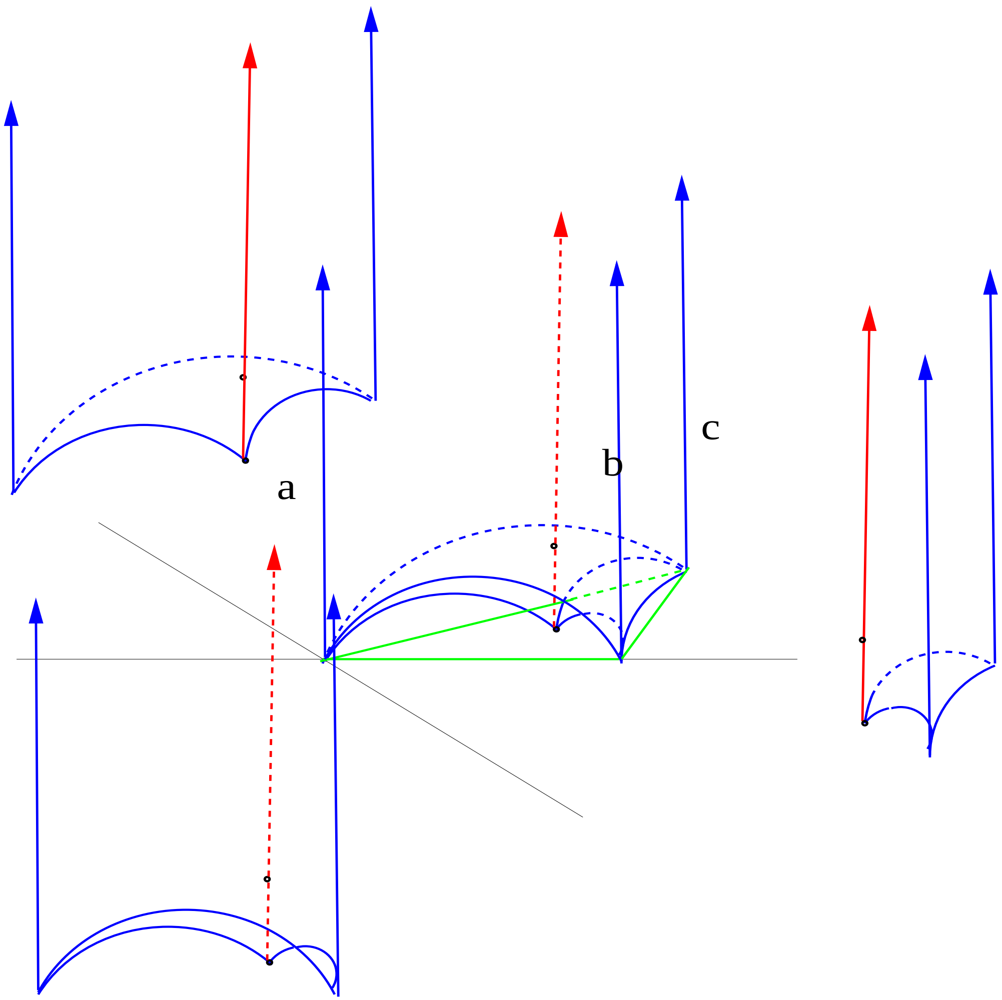
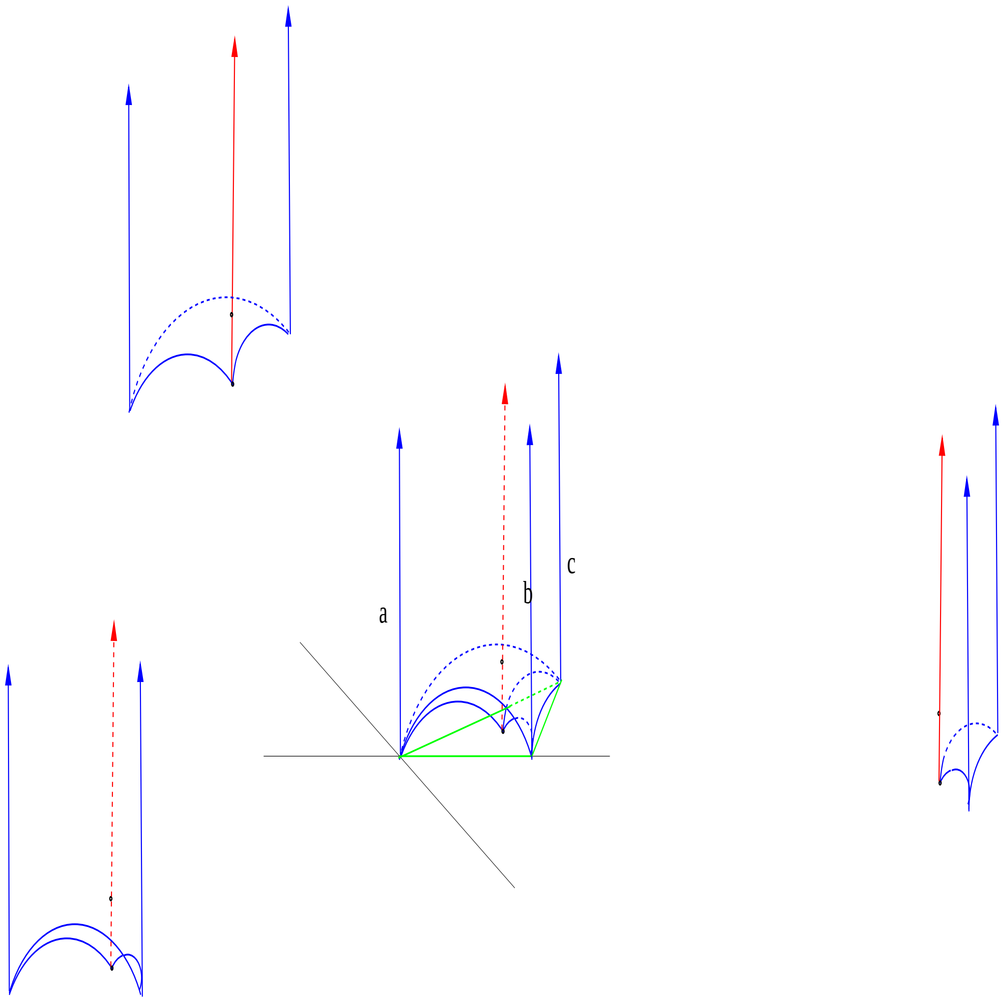


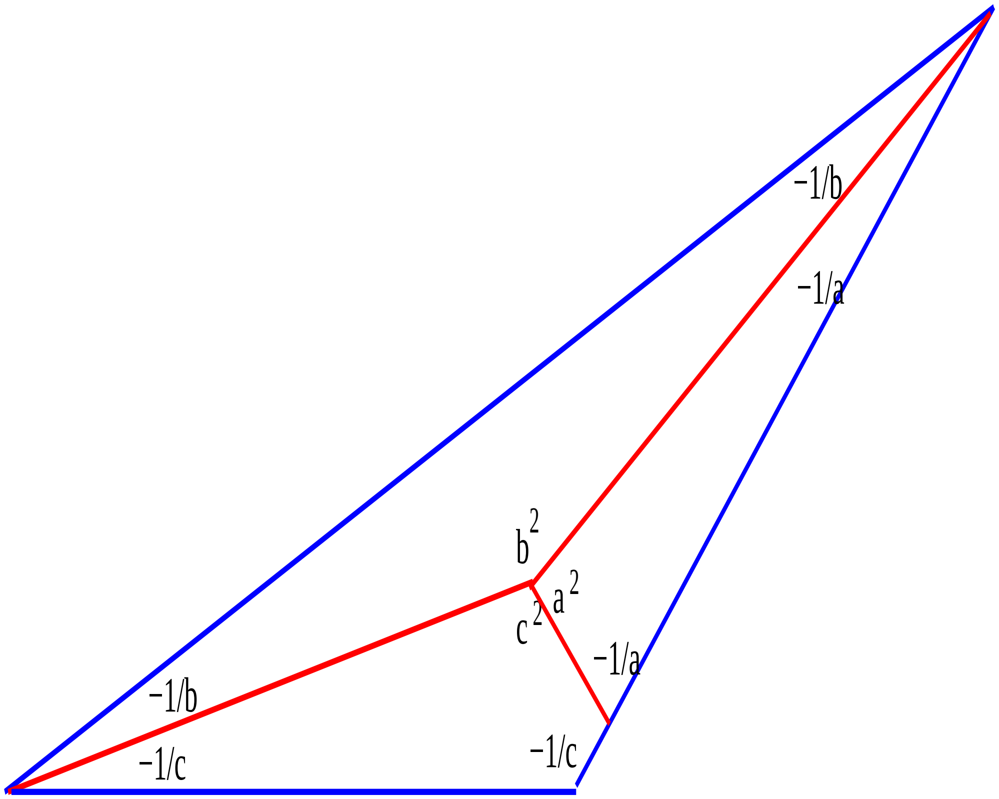
Decomposing Ideal Tetrahedron

allows to view our double tetrahedron as three ideal tetrahedra.



Decomposing Ideal Tetrahedron

From infinity we see these three ideal tetrahedra are very special ideal tetrahedra, the *isosceles ideal tetrahedron*.

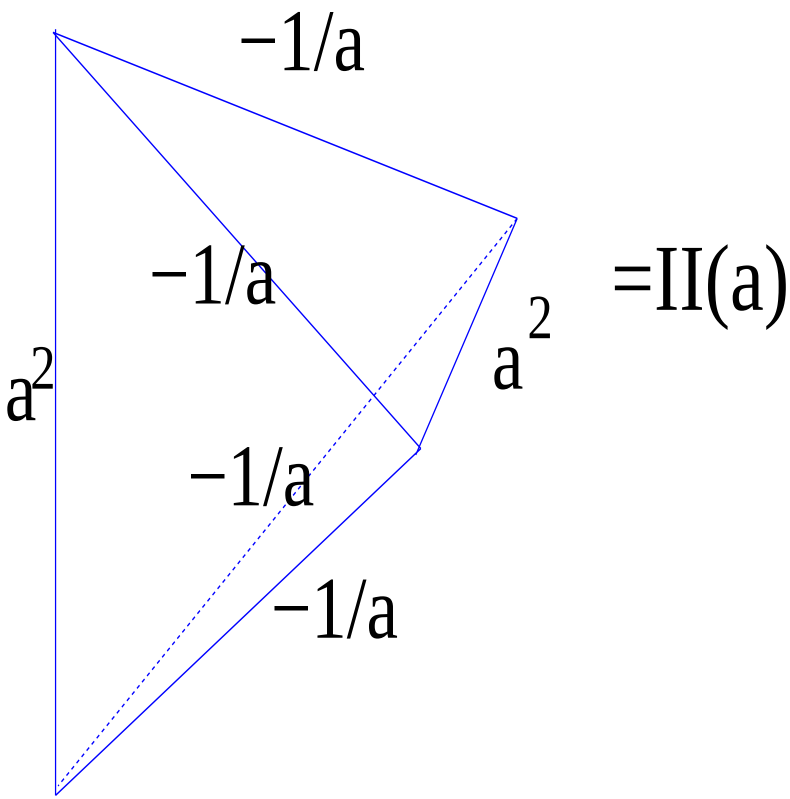


The Isosceles Ideal Tetrahedron

Let us denote this isosceles ideal tetrahedron as $II(a)$. We have just proved

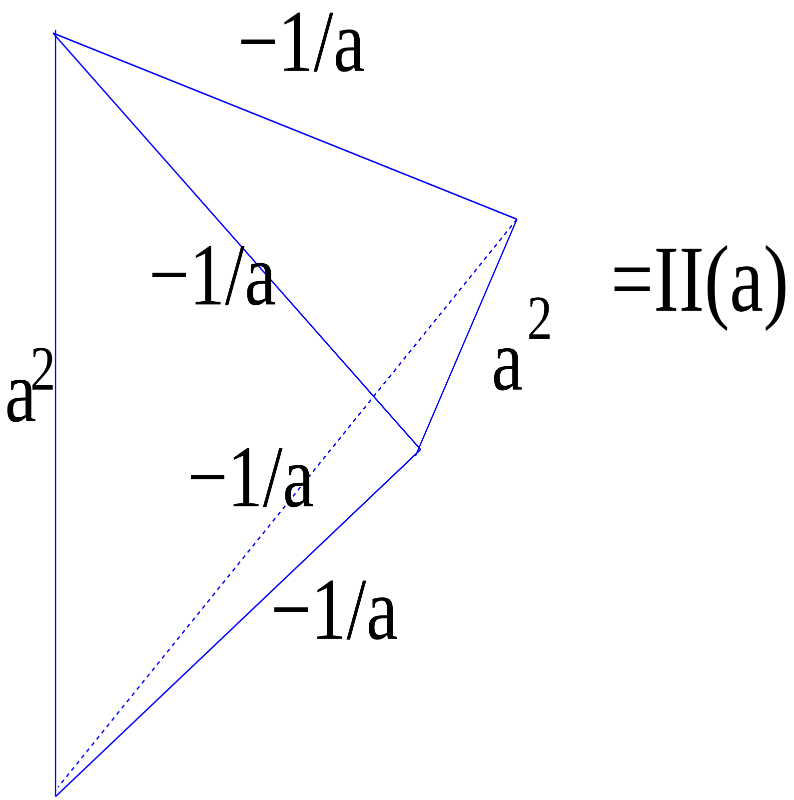
$$IT(a, b, c) = II(a) + II(b) + II(c).$$

So we have reduced finding the volume of an ideal tetrahedron to finding the volume of an isosceles ideal tetrahedron.



The Isosceles Ideal Tetrahedron

Equally important is that the z coordinate of an Isosceles ideal tetrahedron $II(a)$ i s a itself, and a z coordinate corresponds to an isosceles ideal tetrahedron if and only if it is unit sized.



A Tetrahedron's Root

Theorem:(Dupont, Sah)

$$[z^n] = n \sum_{k=1}^{n} [e^{\frac{ik2\pi}{n}} z]$$

In particular

Corollary:(Kubert)

$$Vol(z^n) = n \sum_{k=1}^{n} Vol(e^{\frac{ik2\pi}{n}} z)$$

Milnor's Theorem

Theorem:(Milnor) A continuous function

$$f : S^1 \to \mathbf{R}$$

that satisfies

$$f(z) = f(\bar{z})$$

and

$$f(z^n) = n \sum_{k=1}^{n} f(e^{\frac{ik2\pi}{n}} z)$$

must be equal

$$c\Im(Li_2(z)).$$

$Li_2(\zeta)$ is the Euler dilogarithm

$$Li_2(\zeta) = \int_0^\zeta \frac{\log(1-s)}{s} ds.$$

After normalizing, we have a formula due to Lobachevski,

$$2Vol(IT(a,b,c)) = \Im(a) + \Im(b) + \Im(c).$$

The Milnor Conjecture

Let

$$M = span_{\mathbf{Q}}\{[e^{\frac{i2\pi p}{q}}]\}$$

and view the volume as a map, Vol, from M to $\mathrm{R}_{\mathbf{Q}}$. **Conjecture:** $ker(Vol)$ is the $\mathbf{Q}$ span of elements in the from

$$[e^{\frac{i2\pi p}{q}}] - n\sum_{k=1}^{n}[e^{\frac{ik2\pi}{n}}e^{\frac{i2\pi p}{nq}}]$$

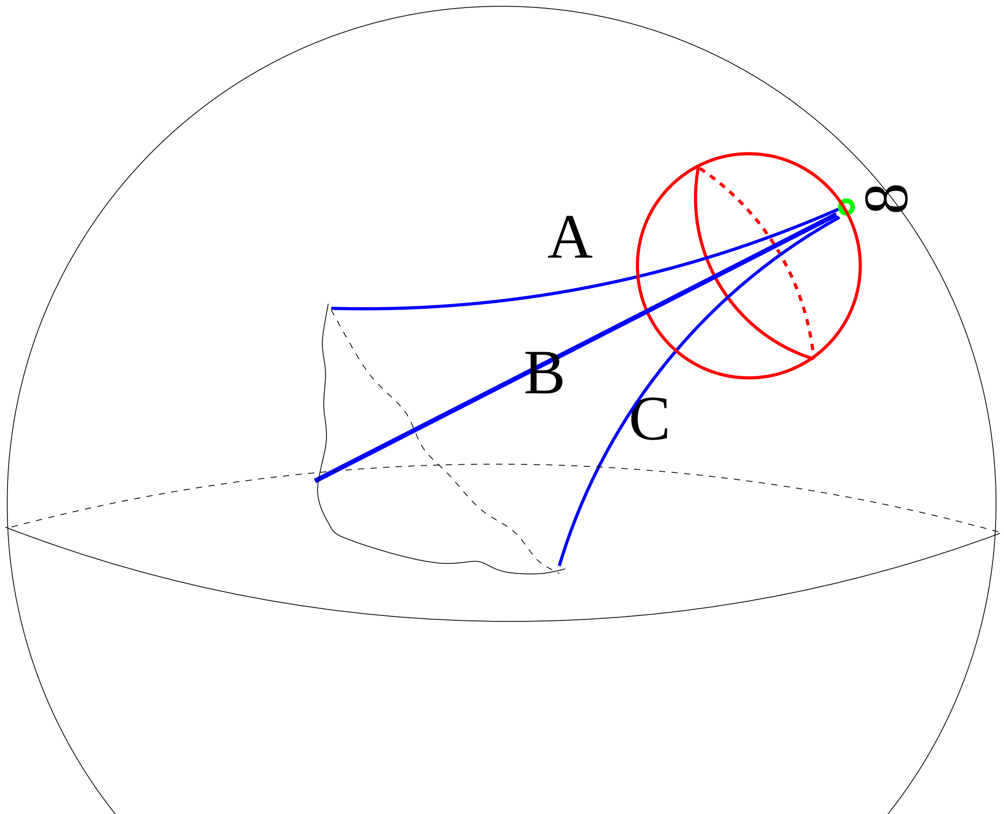
The Milnor conjecture

In words: all rational relations are consequences of the Kubert identities.

Dehn Invariant

Recall, $Sis(H^n) \equiv \frac{<C>}{T}$. Let us extend the Dehn invariant to $\frac{<C>}{T}$. If we have an ideal points cut off with a horoball, we may use the cut off lengths to define

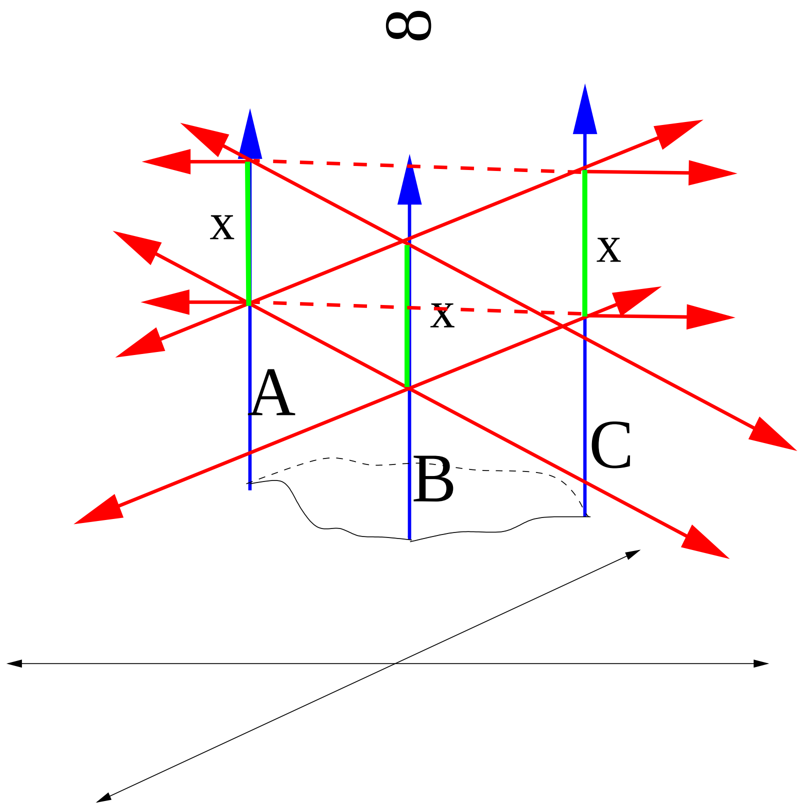
$$Dehn(P) = \sum_{e \in P} l(e) \otimes \theta(e).$$



Dehn Invariant

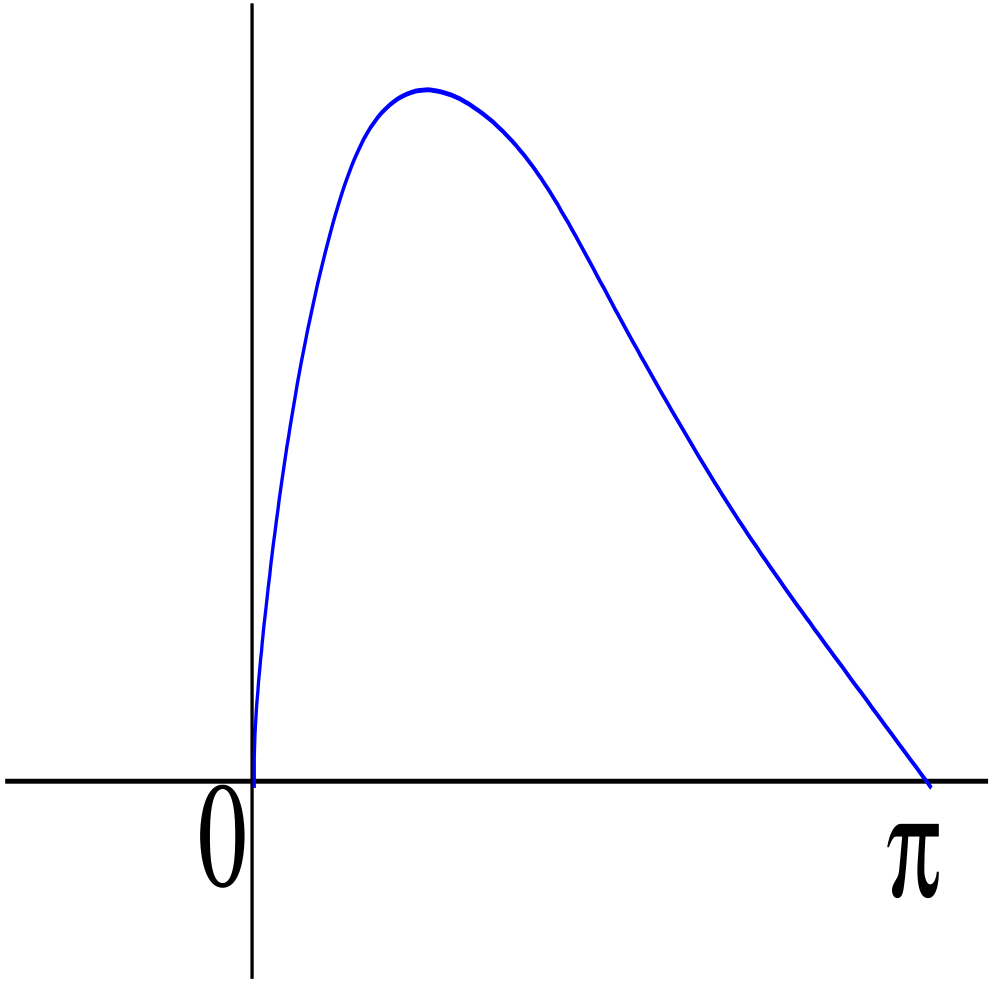
s Notice this is well defined since if you use a different horosphere, then the difference of our two candidate Dehn Invariants is

$$x \otimes \sum_{\theta \in \infty} \theta = x \otimes n\pi = 0.$$



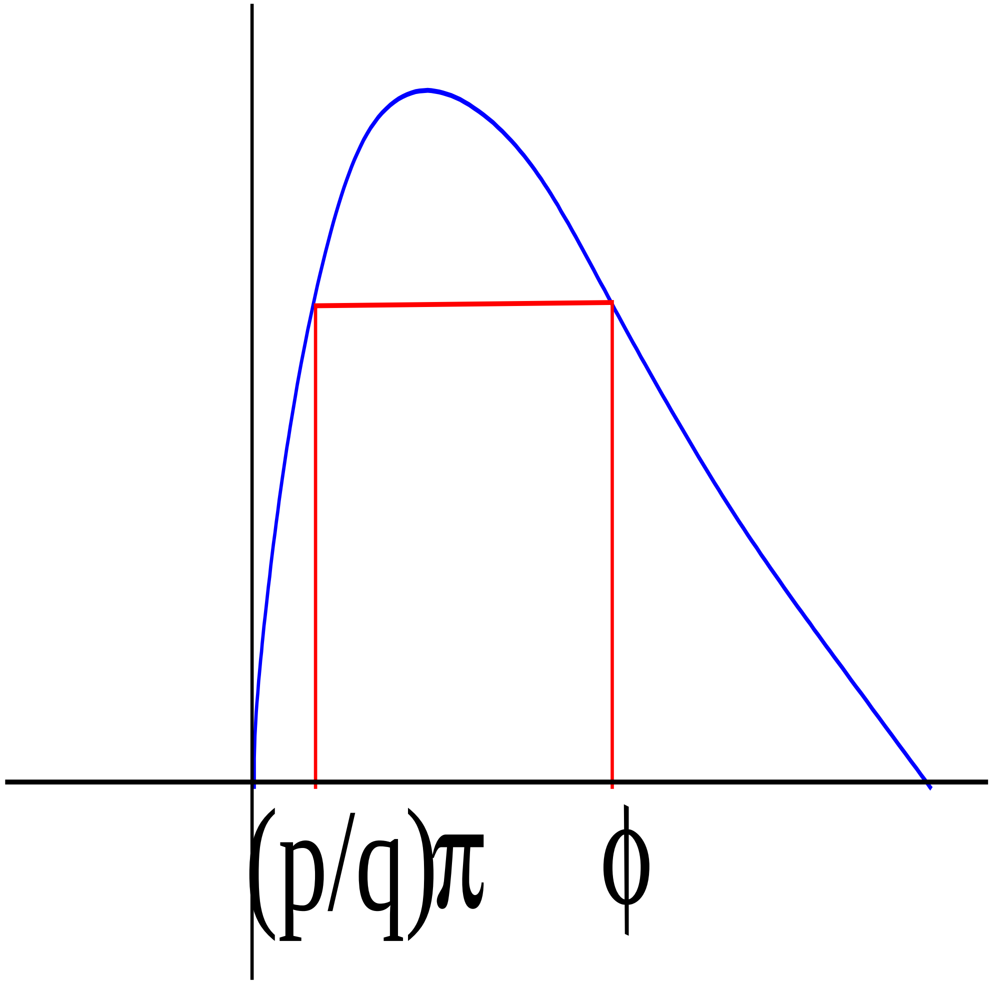
Dehn Example?

There is no know explicit "Dehn counter example" in H^3! Below we have graphed $Vol(II(e^{2I\theta}))$, with respect to θ.



Dehn Example?

We'd like (and expect) that every such $\varphi(p/q)$ is irrational, and hence provides a "Dehn counter example". But not one is known to be! We even have...

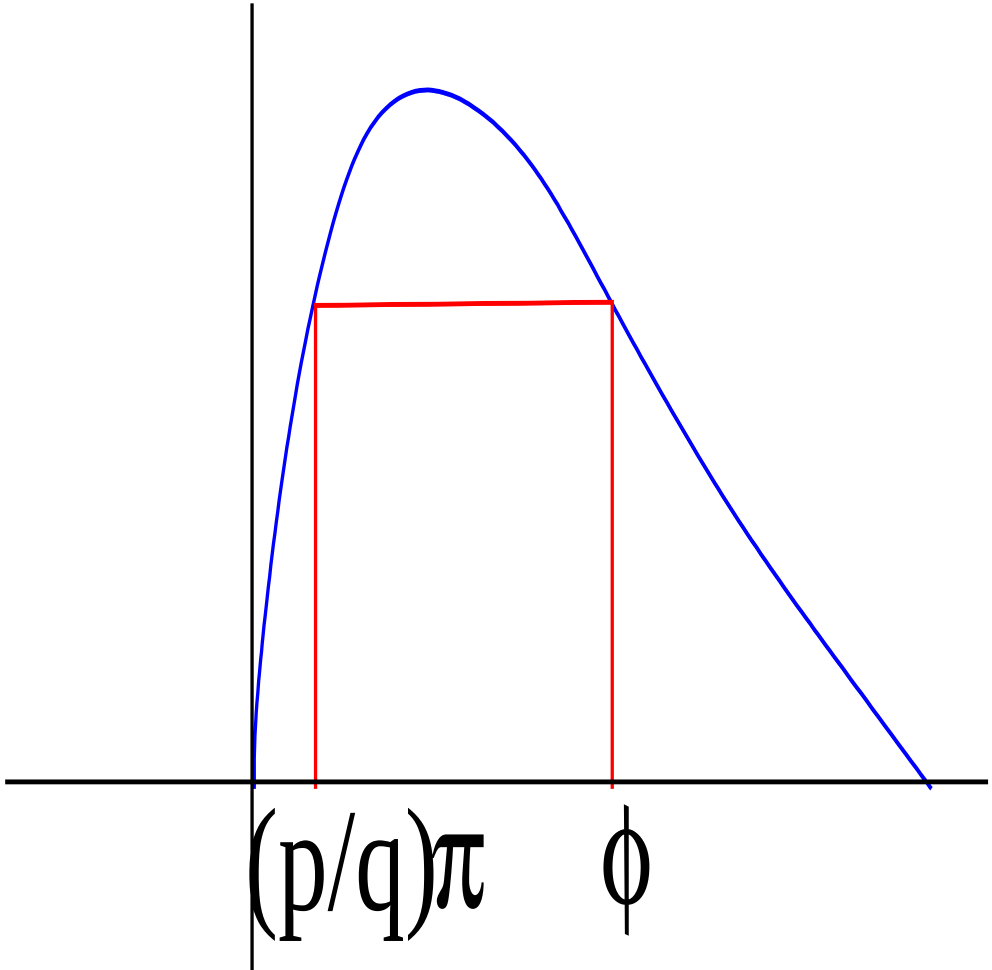


Dehn Example?

Theorem:(Dupont,Sah) If

$$Dehn(\varphi(1/N)) = 0$$

for any $1/N \in (0, 1/6)$, then the Milnor conjecture is false.



Dehn Kernel

Denote the kernel of $Dehn$ restricted to $\frac{<C>}{T}$ as $D(C)$.

Notice: Dehn Sufficiency is equivalent to $(Vol, Dehn)$ being injective.

In other words that $(Vol, Dehn)$ has trivial kernel, or even more simply that Vol is 1-1 when restricted $D(C)$.

Countability Conjectures

Conjecture: Vol is 1-1 when restricted $D(C)$.

Conjecture: $D(C)$ is countable.

Conjecture: $dim_{\mathbf{Q}}(D(C)) > 1$.

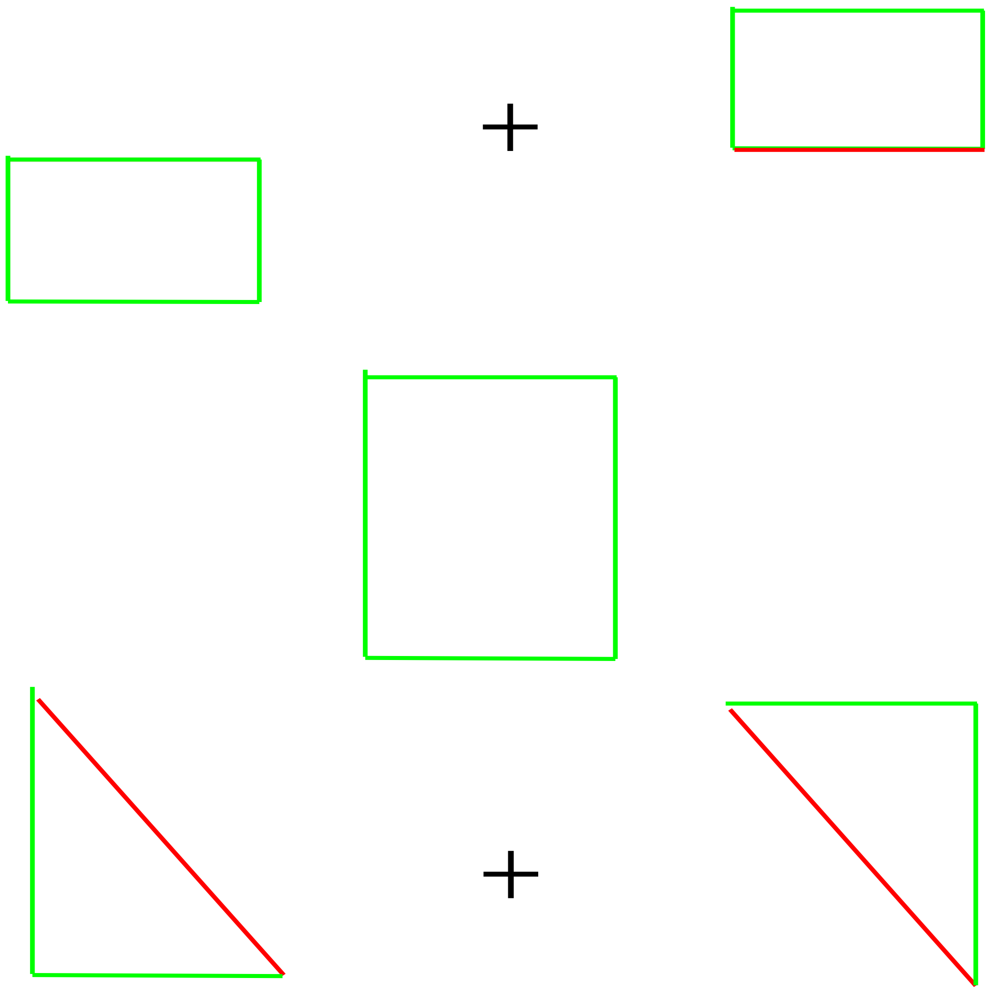
Evidence

Theorem (Suslin) $\frac{<C>}{T}$ has the unique division property.

Theorem: (Dupont, Sah) $Vol(D(C))$ is countable.

Suslin's Theorem

The rectangle on top and the triangle below are both the middle polygon divided by 2.
Hence they are scissors congruent.



Evidence

The unique division property say that for every $[P]$ there exist a class $(1/n)[P] \in Sis(H^n)$ and that if $n[Q] = n[R]$ then $[Q] = [R]$. Notice that

$$[z^n] = n \sum_{k=1}^{n} [e^{\frac{ik2\pi}{n}} z]$$

is a candidate for division. Suslin showed this candidate obeys the $2 - 3$ relation.