MODIFIED $6j$-SYMBOLS AND 3-MANIFOLD INVARIANTS

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This work is joint with Bertrand Patureau-Mirand and Vladimir Turaev.

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Quantum Invariants of 3-manifolds

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Turaev-Viro Invariants $\xleftrightarrow{\text{Modular Categories}}$ Reshetikhin-Turaev invariants

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–Non-symmetric 6j-symbols

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  - –Jones Polynomial specialized to a root of unity

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3. Give a modified T-V invariant for nilpotent modules.

Let $q = e^{i\frac{\pi}{N}}$ with N odd. Let $U_q(\mathfrak{sl}_2)$ be the $\mathbb{C}$-algebra generated by E, F, K, K^{-1} satisfying the relation:

$$KK^{-1} = K^{-1}K = 1,\ KEK^{-1} = q^2E, KFK^{-1} = q^{-2}F,\ [E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \quad (1)$$

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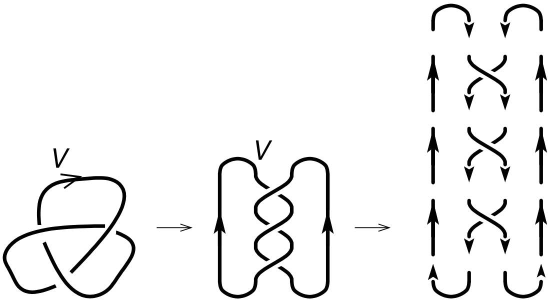
Ex. 2: $I = (\mathbb{C} \setminus \mathbb{Z}) \cup N\mathbb{Z}$ if $\mathcal{C} = U^H$-mod.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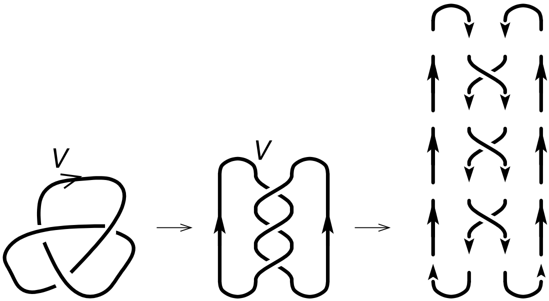

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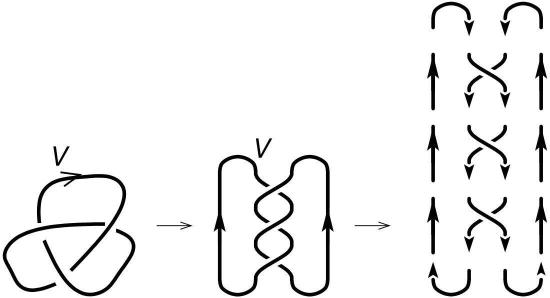

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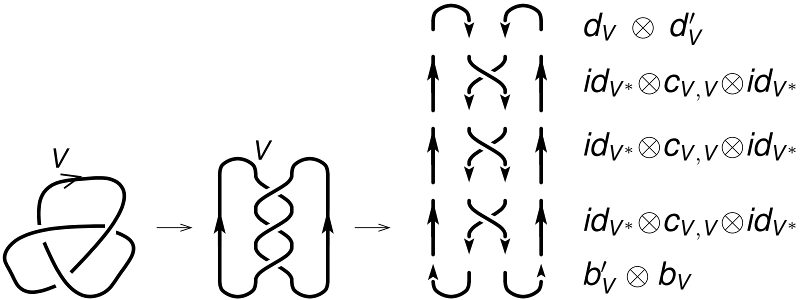
So the trefoil knot can be associated with:

$$F(K_{3_1}^V) := (d_V \otimes d'_V) \circ (id_{V^*} \otimes c_{V,V} \otimes id_{V^*})^3 \circ (b'_V \otimes b_V) \in End(\mathbb{C}) \cong \mathbb{C}.$$



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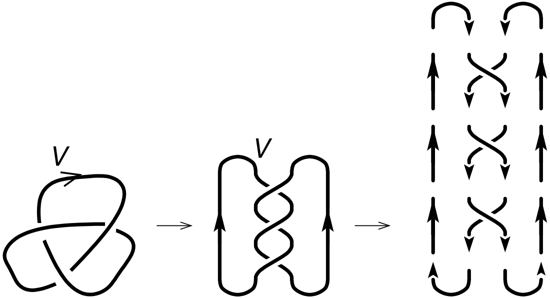

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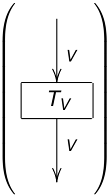
R3 holds because $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ implies $(c_{V,\widetilde{V}} \otimes Id) \circ (Id \circ c_{V,V}) \circ (c_{V,V} \otimes Id) = (Id \circ c_{V,V}) \circ (c_{V,V} \otimes Id) \circ (Id \circ c_{V,V})$.

If V simple object of $\mathcal{C}$ and T_V be a 1-1 tangle whose open string is colored by V then

F 

$\in \mathrm{End}(V) = \mathbb{C}\,\mathrm{Id}_V$

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$$F\left(\boxed{T_V}\right) \in \mathrm{End}(V) = \mathbb{C}\,\mathrm{Id}_V \text{ define } \left\langle \boxed{T_V} \right\rangle \text{ by } F\left(\boxed{T_V}\right) = \left\langle \boxed{T_V} \right\rangle \mathrm{Id}_V\,.$$

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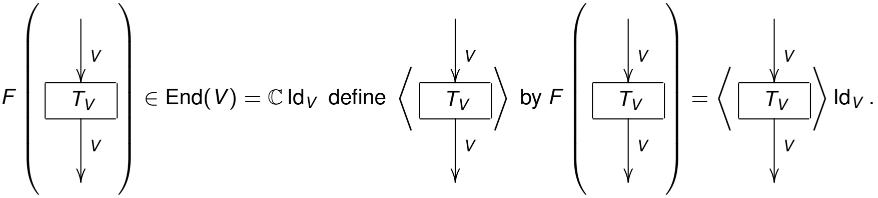
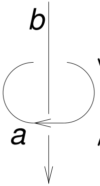


THEOREM

Let L be a framed $\mathcal{C}$-colored link with at least one edge colored by an element V in I. Cutting such an edge, we obtain a colored (1,1)-ribbon graph T_V whose closure is L. There exists $\mathrm{d} : I \to \mathbb{C}$ *such that*

$$F'(L) := \mathrm{d}(V) < T_V > \in \mathbb{C}$$

is independent of the choice of the edge to be cut and yields a well defined invariant of L.

Let $a, b \in I$. Define $S'(a, b) = \Big\langle$ 

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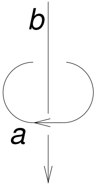
Let $a, b \in I$. Define $S'(a, b) = \Big\langle$ 

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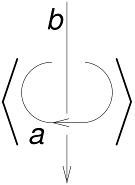
For all $a, b \in I$, $S'(a, b) \neq 0$.

Let $a, b \in I$. Define $S'(a,b) =$  .

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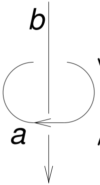
Let $a, b \in I$. Define $S'(a,b) = \Big\langle$  $\Big\rangle$.

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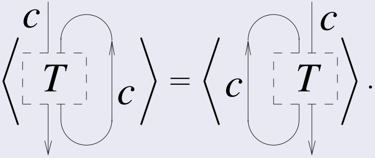
DEFINITION

Let $a \in I$. Define $\mathrm{d}(a) = \frac{S'(a,c)}{S'(c,a)}$.

Lemma A

Let c be as in lemma above. Then we have



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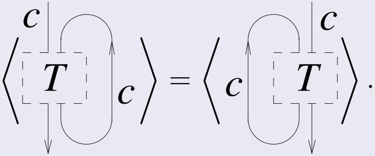

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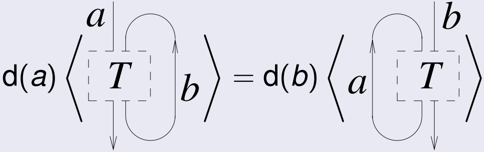
Let $a, b \in I$. Then we have



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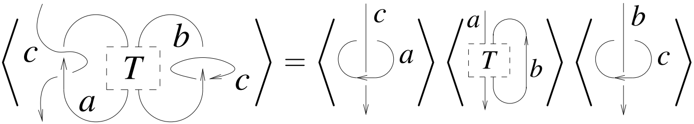
Proof of Lemma B. By definition we have



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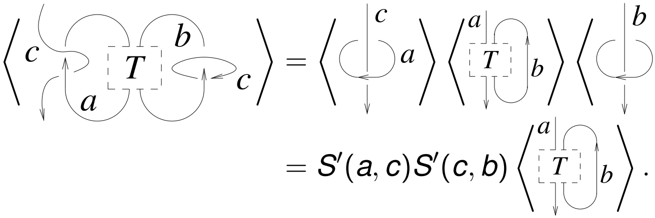

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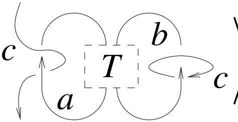
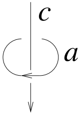
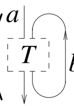
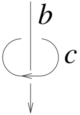
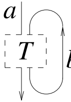
$\Big\langle$ 

$\Big\rangle = \Big\langle$ 

$\Big\rangle \Big\langle$ 

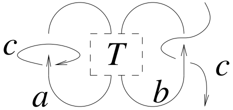
$\Big\rangle \Big\langle$ 

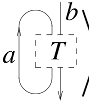
$\Big\rangle$

$= S'(a,c)S'(c,b) \Big\langle$ 

$\Big\rangle .$

Similarly,

$\Big\langle$ 

$\Big\rangle = S'(b,c)S'(c,a) \Big\langle$ 

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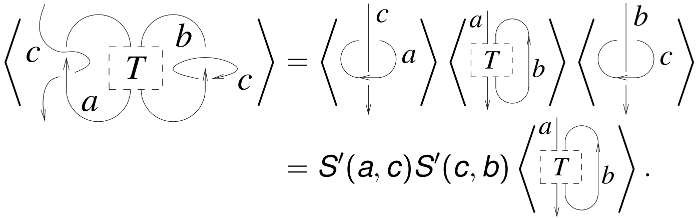

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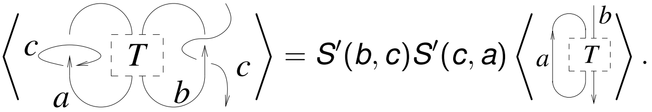

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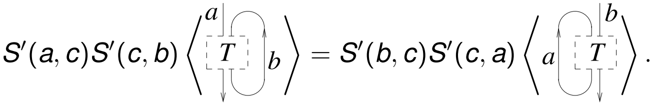

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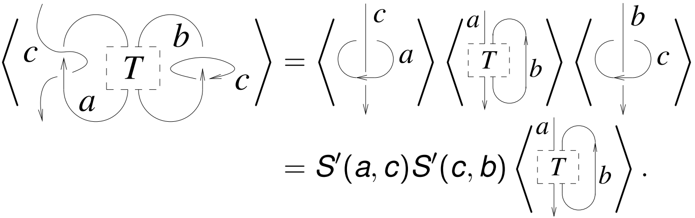

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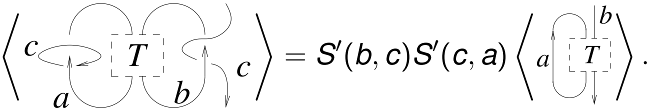

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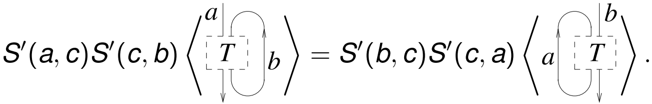

Then the proof follows from $\mathrm{d}(a) := S'(a,c)/S'(c,a)$.

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Ex. 1: $I \subset \{0, ..., N-1\}$ representing simples of $\mathcal{C} = \mathcal{M}(U\text{-mod})$,
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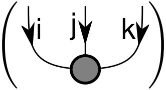
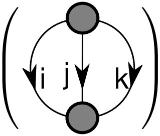
We choose basis F 

$\in H(i, j, k)$ globally dual with respect to the natural duality $H(i, j, k) \otimes H(k^*, j^*, i^*) \to \mathbb{C}$:

F' 

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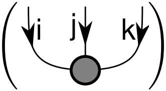
We choose basis F 

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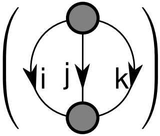
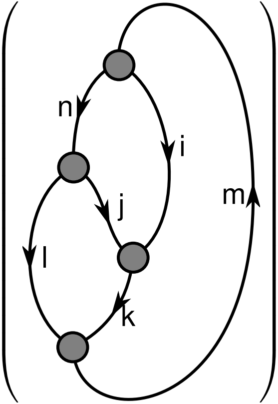
F' 

$= 1$.

Then the 6j-symbols are the scalars:

$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = F'$ 

$\in \mathbb{C}$

Symmetries of the oriented tetrahedron:

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} = \begin{vmatrix} j & k^* & i^* \\ m & n & l \end{vmatrix} = \begin{vmatrix} k & l & m \\ n^* & i & j^* \end{vmatrix}.$$

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Under certain admissibility conditions on $j_1, ..., j_9 \in I$ we have Biendenharn-Elliott type identity:

$$\begin{vmatrix} j_5 & j_3 & j_6 \\ j_4 & j_0 & j_8 \end{vmatrix} \begin{vmatrix} j_1 & j_2 & j_5 \\ j_8 & j_0 & j_7 \end{vmatrix} = \sum_{j \in I} \mathrm{d}(j) \begin{vmatrix} j_1 & j_2 & j_5 \\ j_3 & j_6 & j \end{vmatrix} \begin{vmatrix} j_1 & j & j_6 \\ j_4 & j_0 & j_7 \end{vmatrix} \begin{vmatrix} j_2 & j_3 & j \\ j_4 & j_7 & j_8 \end{vmatrix}.$$

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$$\mathrm{d}(k) \sum_{n \in I} \mathrm{d}(n) \begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix} \begin{vmatrix} k & j^* & i \\ n & m & l \end{vmatrix} = \dim(H(k, l, m^*)).$$

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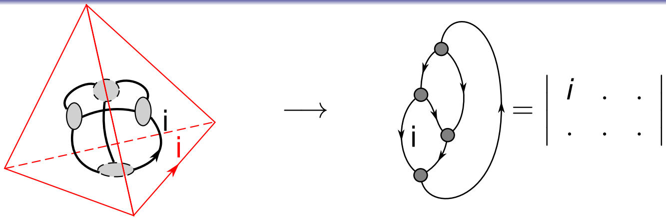
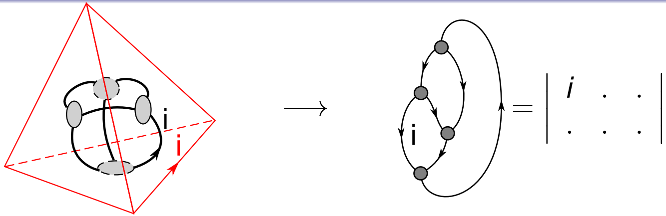
Admissibility conditions are required because the category U^H-mod is not semi-simple. Also, note that the admissibility conditions imply that both sums are finite.



Here the red edges are the edges of the tetrahedron.



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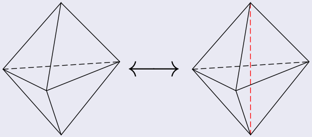
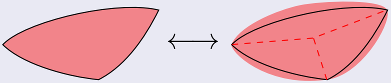
DEFINITION (PACHNER MOVES)



(P23)



(Bubble)

Allow triangulations where two tetrahedrons can have more than one face in common.

Theorem

Any two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of M are related by a finite sequence of Pachner moves

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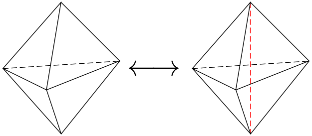

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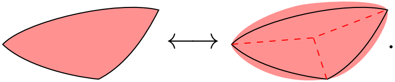

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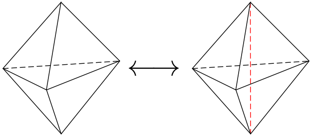

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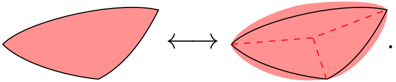
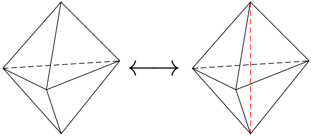


(Bubble)

THEOREM

If $\mathcal{T}$ and $\mathcal{T}'$ two triangulations of M which differ by a Pachner move then $TV(\mathcal{T}) = TV(\mathcal{T}')$.
Note here we have $\mathcal{C} = \mathcal{M}(U\text{-mod})$.

The first move



(P23)

correspond to the Biendenharn-Elliott identity of 6j-symbols:

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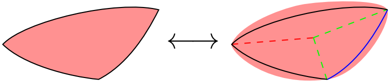
$$\sum_{j \in I} \mathrm{qdim}(j) \begin{vmatrix} j_1 & j_2 & j_5 \\ j_3 & j_6 & j \end{vmatrix} \begin{vmatrix} j_1 & j & j_6 \\ j_4 & j_0 & j_7 \end{vmatrix} \begin{vmatrix} j_2 & j_3 & j \\ j_4 & j_7 & j_8 \end{vmatrix}$$

where the red indicates the added edge and corresponding color.

The bubble move



(Bubble)

is a consequence of the orthonormality relation of 6*j*-symbols:
if $\dim(H(i, j, k^*)) = 1$ then

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Here we fix the color of the edge which is blue.

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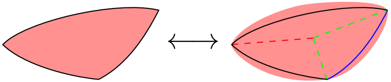

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Here we fix the color of the edge which is blue. Thus, the state sum of the right hand side of the bubble move is $\mathcal{D}^2$ times the state sum of the left hand side.

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Obstructions to applying T-V construction to U^H-mod:

1. qdim and $|T|_\sigma$ are zero for elements of I.

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3. $\mathcal{D}^2 = \sum_{i \in I} \text{qdim}(i)^2$ is zero. Solution: Add a link in M and modify the state sum and bubble move.

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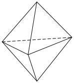
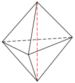
$\mathcal{T} =$  $\longleftrightarrow \mathcal{T}' =$  (P23)

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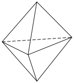
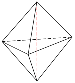
$\mathcal{T} =$  $\longleftrightarrow \mathcal{T}' =$  (P23)

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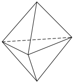
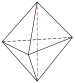
$\mathcal{T} =$  $\longleftrightarrow \mathcal{T}' =$  (P23)

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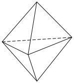
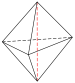
$\mathcal{T} =$  $\longleftrightarrow \mathcal{T}' =$  (P23)

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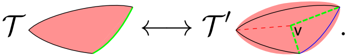

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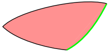
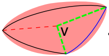
$\mathcal{T}$  $\longleftrightarrow \mathcal{T}'$ 

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And we modify the state sum as follow:

$$TV(\mathcal{T}, \mathcal{L}, \sigma) = N^{-2v} \sum_{\sigma \in S_{\overline{\sigma}}} \left(\prod_{e \in \mathcal{T}_1 \setminus \mathcal{L}} \mathrm{d}(\sigma(e)) \right) \left(\prod_{T \in \mathcal{T}_3} |T|_\sigma \right).$$

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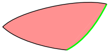
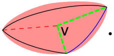
$\mathcal{T}$  $\longleftrightarrow$ $\mathcal{T}'$ 

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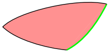
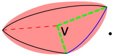
$\mathcal{T}$  $\longleftrightarrow$ $\mathcal{T}'$ 

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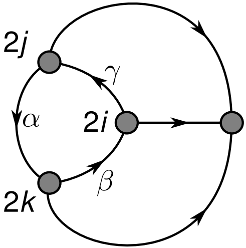
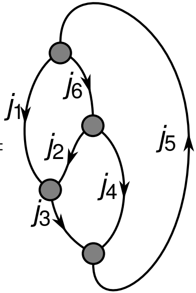
THEOREM

Let $\mathcal{T},\mathcal{T}'$ be triangulations of M, $\mathcal{L}\subset\mathcal{T}$, $\mathcal{L}'\subset\mathcal{T}'$ isotopic links in M and σ, σ' be 1-cocycle on $\mathcal{T}$ and $\mathcal{T}'$ such that $[\sigma]=[\sigma']$ in $H^1(M,\mathbb{C}/2\mathbb{Z})$ then $(\mathcal{T},\mathcal{L},\sigma)$ and $(\mathcal{T}',\mathcal{L}',\sigma')$ are related by a finite sequence of L-bubble moves, "Pachner 2-3 moves outside the link" and isotopies of M. Thus, $TV(\mathcal{T},\mathcal{L},\sigma)=TV(\mathcal{T}',\mathcal{L}',\sigma')$.

$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix} =$$



$$= \begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix} =$$



with
$$j_1 = \alpha \qquad j_2 = -\beta \qquad j_6 = -\gamma$$
$$j_4 = \beta - \gamma - 2i \qquad j_5 = \alpha - \gamma + 2j \qquad j_3 = \alpha - \beta - 2k$$

Lemma

Let $N' \in \mathbb{N}$ such that $N = 2N' - 1$. If $i \leq N'$ and $j \geq -N'$ then

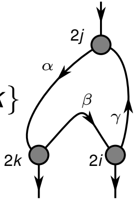
$$\{i+N'\}\{\beta-\gamma-i+N'+2\}\begin{bmatrix} 2i-2 & \alpha \\ 2j+2 & \beta \\ 2k & \gamma \end{bmatrix} = \{\gamma+i+N'-1\}\{\alpha+j+N'+1\}\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma-2 \end{bmatrix} + \{\gamma-1\}\{\alpha-k-N'\}\begin{bmatrix} 2i & \alpha+1 \\ 2j & \beta+1 \\ 2k & \gamma-1 \end{bmatrix}$$

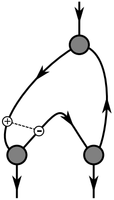
Let $(i,j,k) \in \mathbb{Z}$ such that $-N' \leq i,j,k,i+j+k \leq N'$, and $i,k \leq i+j+k$. Set $M = N' - i - j - k$.
Then the following MAIN FORMULA holds:

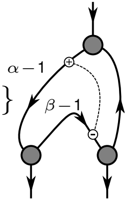
$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix} = \{i,j,k\}\{\alpha-N'-k;j+k\}!\{\gamma-N'-j;i+j\}!\left(\sum_{n=0}^{M}\begin{bmatrix} M \\ n \end{bmatrix}\times \{\beta+k-\alpha+N'+1;M-n\}!\{\beta+k-\gamma-i+j-N';M-n\}! \{\alpha-\beta-2k-M;n\}!\{\gamma-\beta+i+N'+1;n\}!\{\beta-N'-i-n;N'-j\}!\right).$$

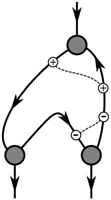
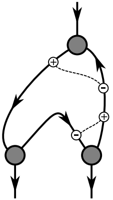
where $\delta = \beta-\gamma-2i, \epsilon = \gamma-\alpha-2j, \phi = \alpha-\beta-2k$. A similar relation holds for $j,k \geq i+j+k$.

Graphical proof of Lemma:

$\{\gamma\}\{\alpha - N' - k\}$ 

$= -\{\gamma\}$ 
$=$

$\{-\gamma\}$ 

$=$ 
$-$ 
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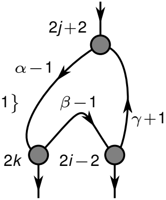
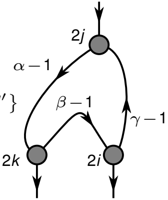
$\{i + N'\}\{\gamma - \beta + i + N' - 1\}$ 

$-\{\gamma + i + N'\}\{\alpha + j + N'\}$ 

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