Perfectly contractile graphs and quadratic toric rings

Akiyoshi Tsuchiya (University of Tokyo)

joint work with Hidefumi Ohsugi and Kazuki Shibata
**Perfect graph**

$G$ : a finite simple graph
(no loops and no multiple edges)
with vertex set $[d] = \{1, 2, \ldots, d\}$ and edge set $E$

A **clique** in $G$ is a set of pairwise adjacent vertices in $G$.

$\omega(G) :=$ the clique number of $G$

\[ \omega(G) = \max\{|C| : C \text{ is a clique of } G\} \]

$\chi(G) :=$ the chromatic number of $G$

In general,

\[ \omega(G) \leq \chi(G) \]

**Definition**

We say that $G$ is perfect if for any induced subgraph $H$ of $G$,

\[ \omega(H) = \chi(H) \]

e.g., bipartite graph, chordal graph
Example 1

\[ \omega(G) = 2 \ < \ \chi(G) = 3 \]
Example II

$G$:

$\omega(G) = \chi(G) = 3$

But $G$ is NOT perfect.
Perfect Graph Theorem

$\overline{G}$ := the complement graph of $G$

**Theorem (Weak Perfect Graph Theorem, Lovász)**

$G$ is perfect if and only if $\overline{G}$ is perfect.

An odd hole is an induced odd cycle of length $\geq 5$.
An odd antihole is the complement graph of an odd hole.

**Theorem (Strong Perfect Graph Theorem, Chudnovsky–Robertson–Seymour–Thomas)**

$G$ is perfect if and only if $G$ contains no odd holes and no odd antiholes as induced subgraphs.
Stable set

$S \subset [d]$ is a stable set or an independent set of $G$
if for $\forall i, j \in S$, $\{i, j\} \notin E$.

$S(G) :=$ the set of stable sets of $G$.

$S(G) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}\}$

$\{1, 3\}$ is stable.  $\{2, 4, 5\}$ is NOT stable.
Algebraic Characterization of Perfect Graph I

\( K : \text{field.} \)

\[ K[t_{\pm}^{\pm \pm 1}, s] := K[t_{1}^{\pm 1}, \ldots, t_{d}^{\pm 1}, s]. \]

\[ K[G] := K[(\prod_{i \in S} t_{i}) s : S \in S(G)] \subset K[t_{\pm}^{\pm 1}, s]. \]

\[ R[G] := K[x_{S} : S \in S(G)] \text{ with } \deg x_{S} = 1. \]

\[ \pi : R(G) \rightarrow K[G] \text{ defined by } x_{S} \mapsto (\prod_{i \in S} t_{i}) s. \]

\( I_{G} = \ker \pi. \)

Theorem (Ohsugi–Hibi)

TFAE:

1. \( G \) is perfect;
2. The initial ideal of \( I_{G} \) with respect to any reverse lexicographic order is squarefree;
3. The initial ideal of \( I_{G} \) with respect to a reverse lexicographic order such that \( x_\emptyset \) is the smallest variable is squarefree.
Algebraic Characterization of Perfect Graph II

\[ K[\Gamma(G)] := K[(\prod_{i \in S} t_i) s, (\prod_{i \in S} t_i^{-1}) s : S \in S(G)]. \]
\[ K[\Omega(G)] := K[(\prod_{i \in S} t_i) u s, (\prod_{i \in S} t_i^{-1}) u^{-1} s, s : S \in S(G)]. \]

Theorem (Ohsugi–Hibi, Hibi–T)

TFAE:

1. \( G \) is perfect;
2. \( K[\Gamma(G)] \) is (normal) Gorenstein;
3. \( K[\Gamma(G)] \) is normal Gorenstein;
4. \( K[\Omega(G)] \) is normal;
5. \( K[\Omega(G)] \) is normal Gorenstein.
Quadratic toric rings

$G$ : a perfect graph.

Question

*When is $I_G$ generated by quadratic binomials? When does $I_G$ possess a quadratic initial ideal?*

e.g.,

- comparability graphs;
- almost bipartite graphs;
- chordal graphs;
- ring graphs;
- the complement graphs of chordal bipartite graphs.
Even antihole

An even hole is an induced even cycle of length $\geq 6$. An even antihole is the complement graph of an even hole.

$\overline{C_6}$:

Proposition

Let $G$ be a perfect graph. If $I_G$ is generated by quadratic binomials, then $G$ contains no even antiholes.
Odd stretcher

An odd stretcher $G_{s,t,u}$ is a graph on the vertex set

$$\{i_1, i_2, \ldots, i_{2s}, j_1, j_2, \ldots, j_{2t}, k_1, k_2, \ldots, k_{2u}\}$$

with edges

$$\{i_1, j_1\}, \{i_1, k_1\}, \{j_1, k_1\}, \{i_{2s}, j_{2t}\}, \{i_{2s}, k_{2t}\}, \{j_{2t}, k_{2s}\},$$

$$\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{2s-1}, i_{2s}\},$$

$$\{j_1, j_2\}, \{j_2, j_3\}, \ldots, \{j_{2t-1}, j_{2t}\},$$

$$\{k_1, k_2\}, \{k_2, k_3\}, \ldots, \{k_{2u-1}, k_{2u}\}.$$
Perfectly contractile graph

An even pair in a graph $G$ is a pair of non-adjacent vertices of $G$ such that the length of all chordless paths between them is even.

Contracting a pair of vertices $\{x, y\}$ an in a graph $G$ means removing $x$ and $y$ and adding a new vertex $z$ with edges to every neighbor of $x$ or $y$.

A graph $G$ is called even contractile if there is a sequence $G_0, \ldots, G_k$ of graphs such that $G = G_0$, each $G_i$ is obtained from $G_{i-1}$ by contracting an even pair of $G_{i-1}$, and $G_k$ is a complete graph.

Definition (Bertschi)

We say that $G$ is perfectly contractile if any induced subgraphs of $G$ are even contractile.

Theorem (Bertschi)

Every perfectly contractile graph is perfect.
Example 1

\[ G: \]

\[ G \] is even contractile.

In fact, \( G \) is perfectly contractile.
Example II

$C_5$: 

$C_6$: 

$C_5$ and $\overline{C_6}$ have no even pairs, hence, they are NOT even contractile.
Example III

\[ G: \]

\[ G \] is even contractile.

But \[ G \] is NOT perfectly contractile.
Combinatorial characterization of perfectly contractile graph (conjecture)

Conjecture (Everett–Reed)

\[ G \text{ is perfectly contractile if and only if } G \text{ contains no odd holes, no antiholes and no odd stretchers as induced subgraphs.} \]

Proposition

\[ \text{If } G \text{ is perfectly contractile, then } G \text{ contains no odd holes, no antiholes and no odd stretchers as induced subgraphs.} \]
Algebraic characterization of perfectly contractile graph (conjecture)

Proposition

Let $G$ be a perfect graph. If $I_G$ is generated by quadratic binomials, then $G$ contains no even antiholes and no odd stretchers as induced subgraphs.

Conjecture

Let $G$ be a perfect graph. TFAE:

1. $G$ is perfectly contractile;
2. $I_G$ is generated by quadratic binomials;
3. $G$ contains no even antiholes and no odd stretchers as induced subgraphs.
Meyniel graph

Definition
A graph is called Meyniel or very strongly perfect if any odd cycle of length $\geq 5$ has at least two chords.

Theorem (Bertschi)
Every Meyniel graph is perfectly contractile.

Theorem (Ohsugi–Shibata–T)
For each Meyniel graph $G$, $I_G$ is generated by quadratic binomials.
Perfectly orderable graph

\( G \) : a graph on the vertex set \( \{v_1, \ldots, v_n\} \).

An ordering \( v_1 < \cdots < v_n \) of the vertex set of \( G \) is called perfect if \( G \) contains no \( P_4 \ abcd \) such that \( a < b \) and \( d < c \).

\[
P_4 \ abcd: \hspace{1cm} a \quad b \quad c \quad d
\]

**Definition**

We say that \( G \) is **perfect orderable** if it has a perfect ordering \( v_1 < \cdots < v_n \) of the vertex set.

**Theorem (Bertschi)**

*Every perfectly orderable graph is perfectly contractile.*
Perfectly orderable graph

Theorem (Ohsugi–Shibata–T)
For any perfectly orderable graph $G$, the initial ideal of $I_G$ with respect to a reverse lexicographic order is squarefree and quadratic.

Remark
The following graphs are perfectly orderable:
- comparability graphs;
- chordal graphs;
- the complement graphs of chordal graphs.

Hence this theorem is a generalization of results on several toric ideals.
Clique separable graph

Definition
We say that a graph is **clique separable** if it is obtained by successive gluing along cliques starting with graphs of Type 1 or 2:

1. The join of a bipartite graph with more than 3 vertices with a complete graph;
2. A complete multipartite graph.

Theorem (Bertschi)

*Every clique separable graph is perfectly contractile.*

Theorem (Ohsugi–Shibata–T)

*For any clique separable graph $G$, the initial ideal of $I_G$ with respect to a reverse lexicographic order is squarefree and quadratic.*