From: Henk J. M. Bos, Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction (Springer: NY, 2001)

Chapter 23

Descartes' solution of Pappus' problem

23.1 The problem

Locus problems

Descartes first studied Pappus' problem during late 1631 and early 1632, on the instigation of Golius. In Chapter 19 I argued that the confrontation with the problem was decisive for the final stage of the development of his programmatic ideas on geometry. I now turn to his treatment of the problem in the *Geometry*, where he used it as the central example for illustrating his techniques and showing their power.

Pappus' problem was a locus problem, that is, an indeterminate problem whose infinitely many solutions form a one-dimensional locus.[1] Such loci are curves (or sometimes straight lines). Descartes' primary approach to locus problems was (cf. Section 22.3) to find a pointwise construction of the locus. He prescribed assuming arbitrary values for one of the two unknowns in the final indeterminate equation, and to determine the corresponding values for the other unknown by the methods suitable for determinate problems. The pairs of values thus constructed were coordinates of points on the locus. In principle, any number of such points could be determined; hence, the result of this procedure was a pointwise construction of the required locus.[2]

The problem

I recall that Pappus' problem is as follows (cf. Problem 19.1):[3]

[1] Cf. Descartes' own characterization of locus problems, [Descartes 1637] pp. 334–335: "Car ces lieux ne sont autre chose, sinon que lors qu'il est question de trouver quelque point auquel il manque une condition pour estre entierement determiné."

[2] Cf. Table 4.2 item 3.2 and Notes 10 and 12 of Chapter 22.

[3] Cf. also Notes 3 and 5 of Chapter 19.

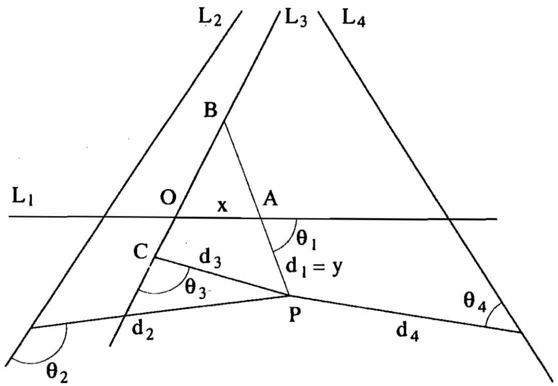

Figure 23.1: Pappus' problem

Problem 23.1 (Pappus' problem)[4]
Given n straight lines L_i in the plane (see Figure 23.1), n angles θ_i, and a line segment a. For any point P in the plane, the oblique distances d_i to the lines L_i are defined as the (positive) lengths of segments that are drawn from P toward L_i making the angle θ_i with L_i. It is required to find the locus of points P for which a certain ratio, involving the d_i and depending on the number of lines, is constant. The relevant ratios are:

For 3 lines:	d_1^2	:	$d_2 d_3$	(23.1)
For 4 lines:	$d_1 d_2$	:	$d_3 d_4$	(23.2)
For 5 lines:	$d_1 d_2 d_3$	:	$a d_4 d_5$	(23.3)
For 6 lines:	$d_1 d_2 d_3$	:	$d_4 d_5 d_6$	(23.4)
In general for an even number $2k$ of lines:	$d_1 \dots d_k$	:	$d_{k+1} \dots d_{2k}$,	(23.5)
and for an uneven number $2k+1$:	$d_1 \dots d_{k+1}$	:	$a d_{k+2} \dots d_{2k+1}$.	(23.6)

[4][Pappus Collection] pp. 507–510; cf. [Pappus 1876–1878] vol. 2, pp. 676–681 and [Pappus 1986] vol. 1 pp. 118–123.

23.2 The general solution: equations and constructions

Descartes' solution of the problem as developed in the *Geometry* started in the first book[5] with the general derivation of the equation of the locus. I have briefly sketched his procedure in Section 19.2; I now analyze it in more detail. Descartes introduced a coordinate system with its origin O at the intersection of L_1 and one of the other lines (L_3 in the figure), its X-axis along L_1 and its ordinate angle equal to θ_1. With respect to that system d_1 is equal to y. He showed that for any point P with coordinates x and y, each d_i could be written as *Equations*

$$d_i = \alpha_i x + \beta_i y + \gamma_i \,, \tag{23.7}$$

in which the α_i, β_i, and γ_i were constants expressed in terms of ratios of line segments determined by the θ_i and the segments along the lines L_i between their points of mutual intersection;[6] because the θ_i and the positions of the lines L_i in the plane were given, the α_i, β_i, and γ_i were known.[7] Descartes was aware that if all the lines L_i are parallel, the x did not occur in the equation, because $d_1 = y$, $d_i = \beta_i y + \gamma_i$ for all $i > 1$.

The requirement that the given ratio (cf. Equations 23.5 and 23.6) be constant leads to the equation:

$$y(\alpha_2 x + \beta_2 y + \gamma_2)\cdots = \delta\overline{a}(\alpha_l x + \beta_l y + \gamma_l)(\alpha_{l+1} x + \beta_{l+1} y + \gamma_{l+1})\cdots \,; \tag{23.8}$$

with $l = k+1$ or $k+2$ depending on the number of given lines, $\overline{a} = a$ for an uneven number of lines and $= 1$ otherwise, and δ is the given constant value of the ratio. These are polynomial equations[8] in x and y (or in y alone if the

[5][Descartes 1637] pp. 310–314.

[6]Thus to express d_3 in x and y Descartes considered the triangles OAB and CPB (A and B are the intersections of the extension of d_1, with L_1 and L_3 respectively, C is the intersection of d_3 with L_3); he noted that although P was unknown, the angles of these triangles, and thereby the ratios of their sides, were known. Thus if $AB : OA = \lambda$ and $CP : BP = \mu$, λ and μ are known and $d_3 = \mu BP = \mu(y + AB) = \mu(y + \lambda x) = \mu y + \mu\lambda x$. Descartes wrote the ratios not as single letters but as ratios between constant line segments; thus for the μ and λ above he wrote $c : z$ and $b : z$, respectively. Note that, contrary to Descartes' usual practice, z denoted an indeterminate, not an unknown.

[7][Descartes 1637] p. 312; see below, Equations 23.13 and 23.14 for the case of the three- and four-line locus.

[8]If the problem is taken in its strict classical sense, the d_i, as well as the line segment a and the ratio δ, should be interpreted as positive, whence the equation should be

$$|y(\alpha_2 x + \beta_2 y + \gamma_2)\dots| = \delta|\overline{a}(\alpha_l x + \beta_l y + \gamma_l)(\alpha_{l+1} x + \beta_{l+1} y + \gamma_{l+1})\dots| \,, \tag{23.9}$$

which is equivalent to

$$y(\alpha_2 x + \beta_2 y + \gamma_2)\cdots = \pm\delta\overline{a}(\alpha_l x + \beta_l y + \gamma_l)(\alpha_{l+1} x + \beta_{l+1} y + \gamma_{l+1})\cdots \,. \tag{23.10}$$

The solution of one Pappus problem, therefore, consists of two curves. For a given set of n straight lines the collection of Pappus loci with respect to these lines and arbitrary constant values for the ratio thus constitutes a one-parameter family of curves represented by the equation

$$y(\alpha_2 x + \beta_2 y + \gamma_2)\cdots = \delta\overline{a}(\alpha_l x + \beta_l y + \gamma_l)(\alpha_{l+1} x + \beta_{l+1} y + \gamma_{l+1})\cdots \,, \tag{23.11}$$

Number of given lines	Degree of the equation	Highest power of x in the equation	Case of parallel given lines; degree of the equation in y
3	2	2	2
4	2	2	2
5	3	2	3
6	3	3	3
⋮	⋮	⋮	⋮

Table 23.1: Pappus' problem — the degrees of the equations

given lines are parallel). As all d_i are linear in x and/or y, the degrees of these equations depend on the number of given lines. Descartes did not explicitly note these dependencies, but it appears from his further statements about the constructibility of points on the loci that he was aware of them. They are listed in Table 23.1.

Constructions

On the basis of the numbers in Table 23.1 Descartes was able to make general statements about the means of construction (plane, solid, or higher-order) necessary in the pointwise construction of the loci. Points on the loci could be constructed (cf. Section 22.3) by giving arbitrary values to y and constructing the corresponding xs as root(s) of the resulting equations. Choosing fixed values for y had the advantage that in the case of five, seven, nine, etc., lines, the resulting equations had degrees two, three, four, etc., whereas for fixed values of x the degrees were higher, namely, three, four, five, etc. The special cases in which all given lines were parallel led, as is easily seen, to equations in one unknown only, namely y. The resulting loci consisted of straight lines parallel to the given ones; the positions of these lines were determined by the roots of this equation in y.

The degrees of the final equations in x, or, for parallel lines, in y, are listed in the last two columns of Table 23.1. These degrees determined by what means the roots could be constructed. Descartes explained the relation between degree and constructibility in the third book of the *Geometry* (cf. Chapter 26), but anticipating his results there, he classified in the first book the cases of Pappus'

where δ now ranges over all (positive and negative) values. Usually Descartes started by considering one point on the locus and adjusted the coordinate system such that its x and y coordinates were positive. He then read off the values of the coefficients α_i, β_i, γ_i from the figure and tacitly assumed that the expressions thus gained applied generally. Moreover, he usually took the constant δ to be 1. The effect of these choices was that in dealing with a Pappus problem he considered one solution curve only. Yet the figures he provided suggest that he was well aware of the other solutions and realized that an obvious adjustment of the equation would produce them.

Number of given lines:	Degree of the equation in one unknown:	Points on the locus constructible by:
3, 4, 5 but not 5 parallel	2	plane means (circles and straight lines)
5 parallel, 6, 7, 8, 9 but not 9 parallel	3 or 4	solid means (conic sections)
9 parallel, 10, 11, 12, 13 but not 13 parallel	5 or 6	circles and a curve "only one degree more composite than the conics"
	etc.	

Table 23.2: Pappus' loci — pointwise constructibility

problem according to the means necessary for their pointwise construction. I summarize his classification[9] in Table 23.2. Descartes postponed the explanation of the expression "a curve only one degree more composite than the conics:"[10] he had the "Cartesian parabola" in mind, cf. Section 26.3.

These results concluded the first book of the *Geometry*. They convincingly illustrated the power of Descartes' method by surveying the various cases of a difficult problem, classifying these, and determining the status as to constructibility of each class.

23.3 The three- and four-line Pappus problem

The problem

However impressive, the result reached at the end of Book I was a classification only, it did not provide the actual constructions. In Book II Descartes dealt in much more detail with one Pappus problem, namely, the problem in three or four lines:

Problem 23.2 (Pappus' problem in three and four lines)[11]
Given four straight lines L_i in the plane (see Figures 23.1 and 23.2, the problem "in three lines" arises if two of the given lines coincide) and four angles θ_i. For any point P in the plane, the oblique distances d_i to the lines L_i are defined as in Problem 23.1. It is required to find the locus of points P for which

$$d_1 d_2 = d_3 d_4 \,. \tag{23.12}$$

[9] [Descartes 1637] pp. 313–314.
[10] [Descartes 1637] p. 314: ". . . une ligne, qui n'est que d'un degré plus composée que les sections coniques, en la façon que i'expliqueray cy aprés."
[11] [Descartes 1637] pp. 324–334

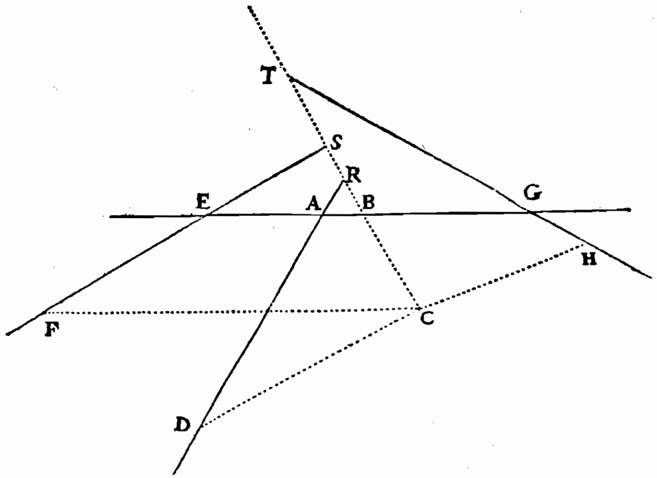

Figure 23.2: Pappus' problem in four lines (*Geometry* p. 309)

Equation 23.12 implies that Descartes took the given ratio δ to be equal to 1; he did not comment on this point, in fact his analysis can be easily adjusted to other values of δ.

In dealing with the three- and four-line locus Descartes did not pursue the approach by pointwise constructions; rather he provided constructions of the loci based on Apollonius' theory of conic sections. Descartes dealt with almost[12] all variants of the three- and four-line locus, and he presented his results in the classical form of constructions with proofs. His presentation was rather involved because of the many case distinctions he made. Descartes did not explain how he had arrived at the construction, but from the construction and the proof it is clear that he had used some indeterminate coefficient procedure.

I illustrate his analysis and construction by following one case, the one in which the locus is an ellipse. Afterward I briefly reconstruct the indeterminate coefficients method by which he probably found it.

The equation Descartes had explained the method for deriving the equation of the locus in the first book (cf. Analysis 19.2). Now he introduced letters for the unknowns and the various given parameters in the four-line Pappus problem as follows[13]

[12]Cf. Note 18.

[13]Contrary to the procedure explained in Section 1.7, I have not changed Descartes' lettering, partly because there are so many letters that some complexity cannot be avoided, partly to facilitate comparison with the original, which, because the *Geometry* is readily available, more readers may want to do than in the case of my other sources. I retain the notation L_i

(see Figure 23.2):

$$\begin{aligned} x &= AB \\ y &= BC \\ z:b &= AB:BR \\ z:c &= CR:CD \\ k &= AE \\ z:d &= BE:BS \\ z:e &= CS:CF \\ l &= AG \\ z:f &= BG:BT\,. \end{aligned} \tag{23.13}$$

Note that z was not an unknown but an indeterminate serving as common term in all the given ratios. He then expressed[14] the segments CB, CF, CD, and CH in terms of the known and unknown line segments introduced in Equation 23.13:

$$\begin{aligned} CB &= y \\ CF &= \frac{ezy + dek + dex}{z^2} \\ CD &= \frac{cyz + bcx}{z^2} \\ CH &= \frac{gzy + fgl - fgx}{z^2}\,. \end{aligned} \tag{23.14}$$

From these results, which were already given in Book I, the equation of the curve was readily calculated by inserting the values above in the defining property $CB \times CF = CD \times CH$ of the locus:[15]

$$y^2 = \frac{(cfglz - dekz^2)y - (dez^2 + cfgz - bcgz)xy + bcfglx - bcfgx^2}{ez^3 - cgz^2}\,. \tag{23.15}$$

Descartes then introduced further letters for abbreviation:[16]

$$\begin{aligned} 2m &= \frac{cfglz - dekz^2}{ez^3 - cgz^2} \\ \frac{2n}{z} &= \frac{dez^2 + cfgz - bcgz}{ez^3 - cgz^2} \end{aligned} \tag{23.16}$$

for the given lines and d_i for the distances; later on I introduce new coordinates u and v and letters r, t, and s for certain terms and line segments; they do not overlap with Descartes' lettering.

[14]Cf. Note 6.

[15][Descartes 1637] p. 325. Descartes did not separately discuss the case in which $ez^3 - cgz^2$ is zero, which leads to an equation without y^2-term; cf. Note 18 below.

[16]I have added a minus sign in the left-hand term of the last equation, where the text has $\frac{p}{m}$ ([Descartes 1637] p. 326). However, as Tannery has remarked (cf. [Descartes 1964–1974] vol. 6 p. 399), Descartes calculated further as if the left-hand term was $\frac{-p}{m}$. In their translation Smith and Latham stick to the + sign, whereby their formulae don't agree with those of the original.

$$
\begin{aligned}
o &= \frac{-2mn}{z} + \frac{bcfgl}{ez^3 - cgz^2} \\
\frac{-p}{m} &= \frac{n^2}{z^2} - \frac{bcfg}{ez^3 - cgz^2} \,.
\end{aligned}
\tag{23.17}
$$

Note that Descartes did not introduce a unit length (cf. Section 21.3); as a result all his equations were homogeneous. Inserting the abbreviations and solving with respect to y, Equation 23.15 became[17]

$$
y = m - \frac{n}{z}x + \sqrt{m^2 + ox - \frac{p}{m}x^2}\,, \tag{23.18}
$$

which constituted the end result of the analysis of the three- and four-line problem.

The construction Descartes then turned to the construction of the locus on the basis of this equation. He did so by constructing, within the given configuration of lines, a conic section whose position and parameters depended on the coefficients of Equation 23.18. For the actual construction of this conic section he refered to the classical constructions of conic sections with given center, diameter direction, ordinate angle, and parameters as explained in Apollonius' *Conics*. He then proved that the constructed conic section was the required locus by showing that its equation coincided with the equation of the locus (i.e., equation 23.18). The total argument (construction and proof) implied an almost complete[18] proof that all quadratic equations in two unknowns represent conic sections and that therefore all three- and four-line loci are conic sections. It also implied a classification of the different cases (straight line, parabola, hyperbola, ellipse, circle).

Reading Descartes' argument is complicated by the fact that his terminology was based on the assumption that all letters in formulas denoted positive magnitudes and that therefore it was only the sign of a term in an equation that determined whether it should be added or subtracted. For instance in step 1 of Construction 23.3 below Descartes took $BK = m$ along BC downward from B "because here there is $+m$," and he stated that K should be taken upward from B "if there had been $-m$." This formulation ignored the possibility that m itself could be negative, whereas the definition of m in Equation 23.16 not at all excluded that possibility. Descartes was surely aware that terms such as m could turn out to be negative, but his terminology was not developed far enough to distinguish between the sign preceding a term and the positivity (or negativity) of that term.

[17] [Descartes 1637] p. 326.

[18] Actually, as he noted himself in a letter to Debeaune of 20 II 1639 ([Descartes 1964–1974] vol. 2 p. 511), he had overlooked the case in which the coefficient of y^2 in the equation of the curve is zero; cf. Note 15 above. In the letter he stated (correctly) that in that case the locus is a hyperbola; he also claimed that one of its asymptotes was parallel to the line AB, which is incorrect; it should be BC.

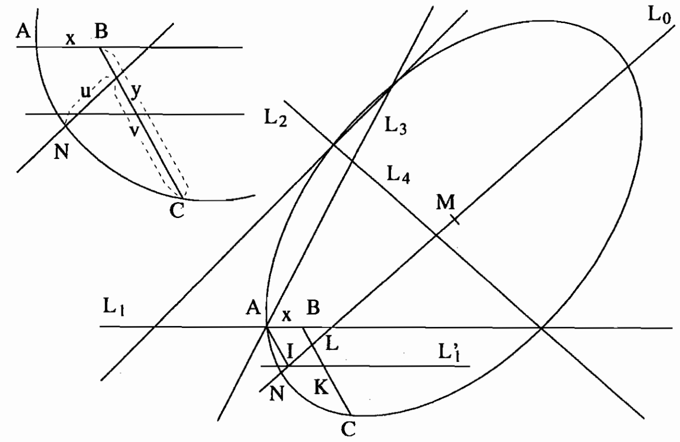

Figure 23.3: Pappus' problem in four lines — construction in the case of an ellipse

In rendering his construction and proof I paraphrase only one of the cases he distinguished, namely, the one in which the locus was an ellipse. It was as follows:

Construction 23.3 (Pappus' problem in three or four lines — case ellipse)[19]

Given and required: see Problem 23.2; the analysis of the problem has led to Equation 23.18.

Construction:

1. (Construction of the point (I) for which $x = 0$ and $y = m$.) Draw a line $L'_1 \parallel L_1$ (see Figure 23.3[20]) intersecting BC in K and such that $BK = m$ (K taken below B because of $+$ sign of m, see above); take point I on it such that $IK = x$.

(Descartes took the line BC as generic ordinate. The point I is independent of the value of x and therefore well defined; $\angle BAI$ is equal to the given angle θ_1 between d_1 and L_1, and $AI = m$. It

[19] [Descartes 1637] pp. 327–332.

[20] The figure is adapted from Descartes' figure in [Descartes 1637] p. 327. The locus is there drawn as a circle; I have stressed the generality of the case by drawing an explicit ellipse. I have removed the letters that don't occur in the argument and I have added the L_i to indicate the given and constructed straight lines, the x and y that Descartes did not incorporate in the figure, and the u and v that I use in rendering the argument.

is not clear why Descartes didn't introduce the point I directly by taking $AI = m$ on a line through A parallel to the ordinate direction BC.)

2. (Construction of the straight line along which the diameter of the ellipse is situated.) Draw a straight line L_0 through I, intersecting BK at L such that $IK : KL = z : n$.

(L_0 passes through the center of the ellipse hence its part within the ellipse is a diameter; the ordinates remain parallel to BC, the ordinate angle, therefore, is equal to $\angle AIL$. With $BC = y$ we have $LC = y - BK + KL = y - m + \frac{nx}{z}$, and therefore (cf. Equation 23.18)

$$LC = \sqrt{m^2 + ox - \frac{p}{m}x^2} \,. \qquad (23.19)$$

LC is the ordinate of the ellipse with respect to the diameter along L_0 and the ordinate angle $\angle AIL$.)

2a. (Classification) Descartes noted at this point that if the square root in Equation 23.19 was zero, the locus coincided with the straight line L_0, and that if the root could be extracted (meaning $m^2 + ox - \frac{p}{m}x^2 = (\alpha x + \beta)^2$ for some α and β), the locus was "another straight line not harder to find than IL."[21] He then claimed that in all other cases the locus would be a conic section, namely:[22] a parabola if the term $\frac{p}{m}x^2$ was zero, a hyperbola if that term was preceded by the sign $+$, an ellipse if preceded by a $-$, and, in particular, a circle if $\angle ILC$ was a right angle and $a^2m = pz^2$. Descartes' construction and his proof for the ellipse was as follows:

3. (Construction of the center.) Introduce a new parameter a defined by $KL : IL = n : a$; then, because $KL = \frac{nx}{z}$, the abscissa IL along L_0 measured from I is $\frac{a}{z}x$. Take M on L_0 with $IM = \frac{aom}{2pz}$; M is the center of the ellipse.

4. (Construction of the parameters.) Take $t = \frac{ma}{pz}\sqrt{o^2 + 4pm}$ as *latus transversum* and $r = \frac{z}{a}\sqrt{o^2 + 4pm}$ as *latus rectum* of the ellipse.[23]

5. (Construction of the ellipse.) Use the constructions from Apollonius' *Conics*[24] to construct an ellipse with center M, diameter along L_0, ordinate angle equal to $\angle AIL$, *latus rectum* r, and *latus transversum* t. This ellipse is the required locus.

(Apollonius' construction, to which Descartes referred explicitly, proceeds by locating a cone in space that intersects the plane according

[21] [Descartes 1637] p. 328: "une autre ligne droite qui ne seroit pas plus malaysée a trouver qu'IL." He did not work out this (degenerate) case in more detail and did not note that the solution would actually consist of two straight lines.

[22] [Descartes 1637], cf. the remark on Descartes' terminology above.

[23] The letters r and t are mine and I have simplified the expressions; Descartes gave them as $\sqrt{\frac{a^2o^2m^2}{ppz^2} + \frac{4a^2m^3}{pz^2}}$ and $\sqrt{\frac{o^2z^2}{a^2} + \frac{4mpz^2}{a^2}}$, respectively.

[24] Propositons I-52–60 of [Apollonius Conics] contain the constructions of the conic sections; Props. 56–58 concern the ellipse.

the required ellipse.)

Proof:[25]

6. Take from M towards I a distance along L_0 equal to half the *latus transversum*, call its endpoint N; so $NM = \frac{1}{2}t$. N is the vertex of the ellipse corresponding to the diameter along L_0. Consider point C as on the ellipse. The Apollonian theory of conics yields the relation

$$v^2 = ru - \frac{r}{t}u^2 \,, \qquad (23.20)$$

in which u is the abscissa NL measured from the vertex N, v the ordinate LC, r the *latus rectum*, and t the *latus transversum* (cf. Figure 23.3, detail).

7. Now insert for u the values used in the construction, taking $x = IK$, $y = BC$:

$$\begin{aligned} u &= NL = (NM - IM + IL) = \qquad (23.21) \\ &= \frac{1}{2}t - \frac{aom}{2pz} + \frac{a}{z}x \,. \end{aligned}$$

Inserting this result in Equation 23.20 leads, after some calculation, to

$$v^2 = m^2 + ox - \frac{p}{m}x^2 \,. \qquad (23.22)$$

Hence,

$$LC = v = \sqrt{m^2 + ox - \frac{p}{m}x^2} \,. \qquad (23.23)$$

8. As $LC = y - m + \frac{nx}{z}$ (see **2**) Equation 23.23 yields

$$y = m - \frac{n}{z}x + \sqrt{m^2 + ox - \frac{p}{m}x^2} \,. \qquad (23.24)$$

Hence the points on the ellipse satisfy the equation of the locus derived in the analysis (i.e., Equation 23.18), so the locus is an ellipse.

Derivation of the solution

The formulas and the argument of Descartes' proof strongly suggest that he found the construction as follows by means of a indeterminate coefficients procedure. From the equation of the curve

$$y = m - \frac{n}{z}x + \sqrt{m^2 + ox - \frac{p}{m}x^2} \,, \qquad (23.25)$$

Descartes could recognize the line defined by

$$y = m - \frac{n}{z}x \qquad (23.26)$$

[25] [Descartes 1637] pp. 332–333.

as the diameter of the curve; items **1** and **2** of his construction locate this diameter (L_0) with respect to the given straight lines and fix the point I corresponding to $x = 0$. The abscissae as measured from I along the diameter then are $\frac{a}{z}x$ with a as introduced in **3**. Calling u the abscissa as measured from a vertex of a conic along the diameter, and v the corresponding ordinate, r the *latus rectum*, and t the *latus transversum*, the general equation of the conic is

$$v^2 = ru \pm \frac{r}{t}u^2 \,. \qquad (23.27)$$

Now $u = (a/z)x - s$ for some line segment s, so the right-hand side of Equation 23.27 can be considered as a second-degree polynomial in x. But we also have $v = y - m + (n/z)x$, hence it follows from Equation 23.25 that

$$v^2 = m^2 + ox - \frac{p}{m}x^2 \,. \qquad (23.28)$$

Equating the coefficients of the powers of x on the right-hand sides of Equations 23.27 and 23.28 provides three equations from which r, t, and s can be determined; the location of M also follows immediately. The values found are precisely the ones Descartes used in his construction and proof.

Significance

Descartes realized that his solution of the general three- and four-line locus problem had a significance beyond the special sphere of the Pappus problems. He wrote at the end of his solution:

> Finally, because all equations of degree not higher than the second are included in the discussion just given, not only is the problem of the ancients relating to three or four lines completely solved, but also the whole problem of what they called the composition of solid loci, and consequently that of plane loci, since they are included among the solid loci. . . . The ancients attempted nothing beyond the composition of solid loci, and it would appear that the sole aim of Apollonius in his treatise on the conic sections was the solution of problems of solid loci.[26]

The reference is to the introduction of the third book of Apollonius' *Conics* where "many surprising theorems" are announced, "that are useful for the syntheses of the solid loci and for the diorisms." Solid loci[27] were curves obtained by the intersection of spheres, cylinders, or cones with planes, so they were conic sections. The synthesis of these loci was necessary in the solution of solid problems (cf. Section 5.5). Once the analysis of such a problem had revealed that the required point was on two solid loci, each defined by a certain property, the synthesis (construction) of the problem required that these loci were in fact constructed. This meant that the nature of the locus (ellipse, hyperbola, parabola) had to be determined and that a vertex had to be given in position, together with the direction of the corresponding diameter, while the ordinate angle, the *latus rectum*, and the *latus transversum* had to be given in magnitude. Given these elements the loci could indeed be constructed by Propositions I-52–60 of Apollonius' *Conics*. Thus the synthesis of solid loci consisted in the determination, given the locus-property, of the nature of the conic, its diameter, ordinate angle, vertex (or center), *latus rectum*, and *latus transversum*. This was indeed precisely what Descartes did.

[26] [Descartes 1637] p. 334–335: "Au reste a cause que les equations, qui ne montent que iusques au quarré, sont toutes comprises en ce que ie viens d'expliquer; non seulement le problesme des anciens en 3 et 4 lignes est icy entierement achevé; mais aussy tout ce qui appartient à ce qu'ils nommoient la composition des lieux solides; et par consequent aussy a celle des lieux plans, a cause qu'ils sont compris dans les solides. . . . Mais le plus haut but qu'ayent eu les anciens en cete matiere a esté de parvenir a la composition des lieux solides: Et il semble que tout ce qu'Apollonius a escrit des sections coniques n'a esté qu'à dessein de la chercher."

[27] Cf. A.Jones' essay "The loci of Aristaeus, Euclid, and Eratosthenes" in his Pappus edition, [Pappus 1986] vol. 2 pp. 573–599; the two short quotations above are from his translation *ibid.* p. 585.

Constructibility

When presenting his general results on Pappus' problem in Book I, Descartes had concentrated on the constructibility of points on the locus by plane, or solid, or higher-order means (cf. Table 23.2). However, such pointwise constructions beg the question in what sense they provide the whole locus. In his treatment of Pappus' problem in three and four lines, discussed above, Descartes achieved the required locus by Apollonian constructions, which did provide the whole conic section, namely, as the intersection figure of a cone and a plane. Thus in a sense his solution in this case was stronger than in the general case. On the other hand, Apollonius' constructions presupposed the possibility of locating a cone in a prescribed position with respect to a plane. This is not a method of construction that immediately presents itself to the mind as clear and distinct. The more evident alternative was to generate a curve by motions in the plane. As we will see in the next section, Descartes solved some instances of the five-line locus by specifying such a generation of the locus. It is remarkable that in the case of the three- and four-line locus he did not do so. He may have considered it superfluous to work out a complete method of tracing conic sections by motion, knowing several instances of such procedures; elsewhere in the *Geometry* he gave a tracing procedure for the hyperbola (cf. Section 19.4, Problem 19.5) and referred to the familiar one (by means of strings) for the ellipse (cf. Section 24.4). However this may have been, the solution of the three- and four-line locus raised the question of the relative acceptability of the various ways of generating curves in geometry. We will see in Chapter 24 that Descartes devoted a considerable part of his book to this issue.