An algorithm for detecting intrinsically knotted graphs, yielding many new minor minimal IK graphs.

Ramin Naimi
Occidental College

We describe an algorithm that detects seemingly most (and possibly all) intrinsically knotted (IK) graphs, and can help find knotless embeddings for graphs that are not IK. We have used this algorithm, implemented as a Mathematica program, to find many new minor minimal IK graphs, 200 of which we have verified "manually" to be minor minimal IK. We have also used this algorithm to find knotless embeddings for several graphs, some of which had previously resisted our attempts to do so. (Joint work with Noam Goldberg, Thomas Mattman, and Jonathan Miller.)

Graphs of 20 edges are 2–apex, hence unknotted

Thomas W. Mattman
California State University, Chico

A graph is 2–apex if it is planar after the deletion of at most two vertices. Such graphs are not intrinsically knotted, IK. We investigate the converse, does not IK imply 2–apex? We determine the simplest possible counterexample, a graph on nine vertices and 21 edges that is neither IK nor 2–apex. In the process, we show that every graph of 20 or fewer edges is 2–apex. This provides a new proof that an IK graph must have at least 21 edges. We also classify IK graphs on nine vertices and 21 edges and find no new examples of minor minimal IK graphs in this set.

Unraveling tangles

Catherine Farkas
University of Illinois at Chicago

In 1972, Kinoshita gave an example of a non-planar $\theta_3$ curve which contained no knots. In 1984, Suzuki generalized this example to show that for every $n \geq 3$ there is a non-planar $\theta_n$-curve every subgraph of which is planar. New interest in this area has been sparked by its relationship to entanglements of polymers as well as by the challenge of synthesizing organic molecules with an entangled structure. Motivated by descriptions of entangled molecular structures known as ravels, we introduce a method of constructing a $\theta_4$ curve from a tangle $T$, and show that if the tangle is algebraic then every such $\theta_4$ curve is either planar or contains a knot.

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On strongly almost trivial embeddings of graphs

Ryo Hanaki
Nara University of Education

A continuous map \( \varphi \) from \( G \) to \( \mathbb{R}^2 \) is called a projection of \( G \) if its multiple points are only finitely many transversal double points away from the vertices. For a spatial embedding \( f \) of \( G \), we say that \( \varphi \) is a projection of \( f \) if there exists a spatial embedding \( f' \) of \( G \) which is equivalent to \( f \) such that \( \varphi = \pi \circ f' \) where \( \pi \) is the natural projection. Then we also say that \( f \) is obtained from \( \varphi \). A projection \( \varphi \) of \( G \) is said to be trivial if only trivial spatial embeddings of \( G \) are obtained from \( \varphi \). A spatial embedding \( f \) of \( G \) is said to be strongly almost trivial if \( f \) is nontrivial and there exists a projection \( \hat{f} \) of \( f \) such that \( f|_{H} \) is trivial for any proper subgraph \( H \) of \( G \).

In this talk, we present new classes of graphs which have a strongly almost trivial embedding and that of graphs which have no strongly almost trivial embeddings. Then we show that both a property that a graph has a strongly almost trivial embedding and a property that a graph has no strongly almost trivial embeddings are not inherited by minors.

– Lunch time –

Topological symmetry groups and local knotting

Erica Flapan
Pomona College

If a graph \( \Gamma \) is embedded in \( S^3 \), then the symmetries of \( \Gamma \) can be represented by automorphisms of the abstract graph that are induced by a homeomorphism of \( S^3 \). The topological symmetry group, \( TSG_+(\Gamma) \), is the subgroup of the automorphism group of \( \Gamma \) consisting of those automorphisms induced by orientation preserving homeomorphisms of \( S^3 \).

In general, we are interested in which groups can occur as \( TSG_+(\Gamma) \) for some graph \( \Gamma \) embedded in \( S^3 \). Frucht showed that every finite group is the automorphism group of some abstract graph. By contrast, we showed that while every finite abelian group and every symmetric group can occur as \( TSG_+(\Gamma) \) for some embedded graph \( \Gamma \), the alternating group \( A_5 \) and the cyclic groups of prime order are the only simple groups which can occur. However, it is not known what groups can occur as \( TSG_+(\Gamma) \) for any embedding of a particular graph \( \Gamma \) in \( S^3 \).

We are interested in whether all of the subgroups of a given \( TSG_+(\Gamma) \) can occur as \( TSG_+(\Gamma') \), for some re-embedding \( \Gamma' \) of \( \Gamma \) in \( S^3 \). We prove the uniqueness up to isotopy of a ball containing all of the local knots on an edge of an embedded graph, and use this result to give a positive answer to this question in many cases. This is joint work with Blake Mellor and Ramin Naimi.

Topological symmetry groups of complete graphs

Blake Mellor
Loyola Marymount University

Given an embedding \( \Gamma \) of a graph in \( S^3 \), the topological symmetry group \( TSG(\Gamma) \) is the subgroup of the automorphism group of the abstract graph which is induced by orientation-preserving homeomorphisms of the pair \( (S^3, \Gamma) \).
Complete graphs are an interesting class of graphs to consider, since they have the largest possible automorphism groups. Flapan, Naimi and Tanvakis have shown that the only possible topological symmetry groups for complete graphs are finite subgroups of $SO(3)$ or $D_m \times D_m$ (with $m$ odd). However, their result does not specify which complete graphs admit embeddings realizing particular topological symmetry groups.

In this talk, we answer this question for the symmetry groups of the regular polyhedra, $A_4$, $S_4$ and $A_5$. For each of these groups, we will determine exactly which complete graphs have an embedding which realizes that group as its topological symmetry group. As part of our investigation, we develop an Edge Embedding Lemma which can be applied to many other groups and graphs as well. This is joint work with Erica Flapan and Ramin Naimi.

**Topological symmetry groups of $K_1$ to $K_6$ and $K_{4r+3}$**

Dwayne Chambers
Claremont Graduate University

We focus on the complete graphs on $n$ vertices, $K_n$, embedded in $S^3$ for $n$ less than 7. In particular we characterize all the groups which can occur as the topological symmetry group of $K_n$. We also summarize results showing that only certain cyclic, dihedral, and products of two cyclic groups of odd order can occur as the topological symmetry group of an embedding of the complete graph $K_{4r+3}$ in $S^3$.

**Achirality and linking numbers of links**

Saori Matsuoka
Tokyo Woman’s Christian University

An ordered and oriented $n$-component link $L$ in the 3-sphere is said to be achiral if it is ambient isotopic to its mirror image ignoring the orientation and ordering of the components. For an ordered and oriented $n$-component link $L$, let $\lambda_L$ be the product of linking numbers of all 2-component sublinks in $L$. For $n = 4m + 3$, where $m$ is a non-negative integer, we show that if $L$ is achiral then $\lambda_L = 0$. And for $n \neq 4m + 3$, we show that there exists an $n$-component achiral link $L$ with $\lambda_L \neq 0$ by using achiral embeddings of complete graphs with $n$ vertices $K_n$.

**$C_k$-classifications on string links and spatial graphs with canonical disk/band surfaces**

Akira Yasuhara
Tokyo Gakugei University

This talk is based on joint works with Jean-Baptiste Meilhan and with Ryo Nikkuni.

A family of finite type invariants of string link is given by the HOMFLYPT polynomial of knots that are obtained by various closure operations on (cabled) string links. We call such invariants closure invariants. These closure invariants are useful for $C_k$-classification of spatial graphs that have ‘canonical’ disk/band surfaces. Roughly speaking, if the $C_k$-classification of string links is given by closure invariants, then the $C_k$-classification of such spatial graphs is also given by the
closure invariants. In particular, for $k \leq 5$, the $C_k$-classification of spatial complete graphs on 4 vertices is given by closure invariants.

– August 19 –

**A sufficient condition for intrinsic linking**

Ryan Ottman  
University of California, Santa Barbara

I will provide a proof that any graph with $n$ vertices and at least $4n-9$ edges is intrinsically linked. I will also mention a similar result for intrinsic knotting. Then I will discuss the key results which were necessary to classify the intrinsic knotting and linking of 1 and 2 deficient complete partite graphs. All of these results are from the paper “Intrinsic Knotting and Linking of Almost Complete Graphs” which grew out of work done in an REU.

**Similarities between flat and planar graphs**

Joel Foisy  
State University of New York at Potsdam

Recall that a graph $G$ is flat if $G$ has a spatial embedding $\phi(G)$ for which every cycle of $G$ bounds a disk whose interior is disjoint from $\phi(G)$. In this talk we discuss similarities between flat graphs and planar graphs. In particular, a 3–connected planar graph has the property that, in a planar embedding, at every vertex the graph looks like a wheel with spokes connected at the center. A similar property can be seen to hold for flat graphs. Studying this property will hopefully lead to a better understanding of flat graphs and maybe even to a new proof of Sachs’ linkless embedding conjecture.

**A knotted handlebody and a spatial graph**

Atsushi Ishii  
University of Tsukuba

A knotted handlebody is a handlebody embedded in a 3-dimensional manifold $M$. We assume that $M = S^3$ in this talk. A regular neighborhood of a spatial graph is a knotted handlebody. Conversely, any knotted handlebody can be obtained from a spatial graph. I will explain a fundamental relationship between them, which shows when two spatial graphs represent the same knotted handlebody. Then I give an invariant for knotted handlebodies. We note that it is also an invariant for spatial graphs.

**Bridge position and the representativity of spatial graphs**

Makoto Ozawa  
Komazawa University

First, we extend Otal’s result for the trivial knot to trivial spatial graphs, namely, we show that for any bridge tangle decomposing sphere $S^2$ for a trivial spatial graph $\Gamma$, there exists a 2-sphere $F$ such that $F$ contains $\Gamma$ and $F$ intersects $S^2$ in a single loop. This characterizes the bridge position of a trivial spatial graph.

Next, we introduce two invariants for spatial graphs. As a generalization of the bridge number for knots, we define the *bridge string number* $bs(\Gamma)$ of a spatial graph
Γ as the minimal number of \(|Γ \cap S^2|\) for all bridge tangle decomposing sphere \(S^2\).

As a spatial version of the representativity for a graph embedded in a surface, we define the representativity of a non-trivial spatial graph Γ as

\[
r(Γ) = \max_{F \in \mathcal{F}} \min_{D \in \mathcal{D}_F} |\partial D \cap \Gamma|,
\]

where \(\mathcal{F}\) is the set of all closed surfaces containing Γ and \(\mathcal{D}_F\) is the set of all compressing disks for \(F\) in \(S^3\). Then we show that the representativity \(r(Γ)\) does not exceed the half of the bridge string number \(bs(Γ)\). In particular, if Γ is a knot, then \(r(Γ) \leq b(Γ)\), where \(b(Γ)\) denotes the bridge number. This generalizes the Schubert’s result on torus knots.

Next, we show that for any 2-connected graph \(G\), any positive genus closed surface \(F\) embedded in \(S^3\) \((g(F) \geq g(G))\) and any integer \(n\), there exists a spatial graph \(Γ\) of \(G\) contained in \(F\) such that \(r(F, Γ) \geq n\), where \(r(F, Γ) = \min_{D \in \mathcal{D}_F} |\partial D \cap Γ|\).

Finally, we show that if \(r(Γ) > β_1(G)\), then Γ contains a non-free connected subgraph, where \(β_1(G)\) denotes the first Betti number of \(G\).

Knots and the cyclic polytope

Jorge Ramírez Alfonsín
Université Montpellier 2

The main purpose of this talk is to present a geometric method to explore some knot representation problems. We shall discuss in particular how the cyclic polytope can be used to investigate linear spatial representations and ropes of knots.

Intrinsic linking and knotting in straight-edge embeddings of complete graphs

Lew Ludwig
Denison University

It is well known that the complete graph on six vertices, \(K_6\), is intrinsically knotted and \(K_7\) is intrinsically linked. During a knot theory workshop conducted by Colin Adams, it was asked how many such links or knots would occur in complete graphs if the embeddings consisted of straight edges? In this presentation, we will discuss a three-year project with undergraduate students where we enumerated the number of links in straightedge embeddings of \(K_6\) and \(K_7\) as well as the number of knots in straight-edge embeddings of \(K_7\). We will also discuss several open questions including triple-linking and linear ambient isotopy in straightedge embeddings of complete graphs.

Number of knots and links in linear \(K_7\)

Choonbae Jeon
Daeduk University

We present a computational result about the number of knots and links in linear embeddings of the complete graph \(K_7\) into \(\mathbb{R}^3\). From this result, we can determine the maximal number of non-trivial knots and links in linear embeddings of \(K_7\).

Counting links in complete graphs
Intrinsic linking has been extensively studied and generalized in many ways, for example to intrinsic n-linking. These generalizations have focused on the minimum complexity of the links contained in graph embeddings. We will consider a different approach: counting the minimum number of nonsplit links (of any type) that must occur in an embedding. We will present results on complete partite graphs on 7 and 8 vertices.

**Planar graphs producing knotted projections with three double points**

Youngsik Huh
Hanyang University

A generic immersion of a planar graph into $\mathbb{R}^2$ is said to be knotted, if it is not a projection of any planar embedding of the graph in $\mathbb{R}^3$. It is known that the number of double points of any knotted projection is at least three, which is a previous work of the speaker. In this talk we determine the planar graphs which have knotted projections with three double points. This is a joint-work with Ryo Nikkuni.

**Multiplicity distance of spatial graphs**

Kouki Taniyama
Waseda University

We define multiplicity and multiplicity distance in a category. We define neighbourhood category of spatial graphs and discuss its multiplicity distance. Here the distance is defined on the set of all spatial embeddings of all finite graphs.

**Prime knots whose arc index is smaller than the crossing number**

Gyo Taek Jin
Korea Advanced Institute of Science and Technology

It is known that the arc index of prime alternating knots is the minimal crossing number plus two and the arc index of prime nonalternating knots is less than or equal to the minimal crossing number. We study some cases when the arc index is strictly less than the minimal crossing number.

**Knotting and linking in the Petersen family**

Danielle O’Donnol
Rice University

An embedding $f$ of a graph $G$ is linked if there is a nontrivial link in $f(G)$. An embedding $f$ of a graph $G$ is algebraically linked if there is a link with nonzero linking number in $f(G)$. We will say an embedding $f$ of a graph $G$ is complexly algebraically linked (CA linked) if there is a link $L$ in $f(G)$ where $|lk(L)| \geq 2$ or if there is more than one link in $f(G)$ with nonzero linking number. An embedding $f$ of a graph $G$ is knotted if there is a nontrivial knot in $f(G)$. This talk will examine the relationship between knotting and linking in the Petersen family, the set of
minor minimal intrinsically linked graphs. We present a linking condition on the spatial graph that results in a knotted embedding.

**On the Conway-Gordon theorems**

Ryo Nikkuni

Tokyo Woman’s Christian University

Let $K_n$ be the complete graph on $n$ vertices. In 1983, Conway-Gordon showed that for every spatial embedding of $K_6$, the sum of the linking numbers over all of the constituent 2-component links is congruent to 1 modulo 2, and for every spatial embedding of $K_7$, the sum of the Arf invariants over all of the Hamiltonian knots is also congruent to 1 modulo 2. In this talk, we introduce the following two topics:

1. We give an integral version of the Conway-Gordon type theorem for every graph obtained from $K_6$ or $K_7$ by a finite sequence of $\triangle Y$-exchanges in terms of the square of the linking number and the second coefficient of the Conway polynomial. This is partially joint work with K. Taniyama.

2. We say that a graph is intrinsically knotted or completely 3-linked if every spatial embedding of the graph contains a nontrivial knot or a 3-component link any of whose 2-component sublink is nonsplittable. We show that a graph obtained from $K_7$ by a finite sequence of $\triangle Y$-exchanges and $Y \triangle$-exchanges is a minor-minimal intrinsically knotted or completely 3-linked graph. This is joint work with R. Hanaki, K. Taniyama and A. Yamazaki.