PROBLEM LISTS ON SPATIAL GRAPHS

INTERNATIONAL WORKSHOP ON SPATIAL GRAPHS 2010

Abstract. International Workshop on Spatial Graphs 2010 was held on August 17-21, 2010 at Waseda University, Tokyo, JAPAN. This problem lists gather several open problems and conjectures which are raised by the authors.

Introduction

Spatial graph theory is relatively new field which places between graph theory and knot theory, and it is growing up and now forming a theory. This problem lists would be a guiding principle to study spatial graphs, and we hope it will contribute to progress of spatial graph theory. There are other problem lists and survey on spatial graphs. For example, [41] deals with regular projections of spatial graphs and [22] deals with intrinsically knotted and linked spatial graphs, [50] surveys what is known until 2005.

Contents

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1. Ramin Naimi

1.1. For which "measures of complexity" is intrinsic knotting or linking arbitrarily complex?

Problem 1.1. *For each given "measure of complexity" in knotting and linking, such as bridge number, tunnel number, width, volume, etc., determine whether the following is true: Given any m, every embedding of a sufficiently large complete graph contains a knot or link with complexity greater than m.*

Example 1.2. It is known that for any m, every embedding of a sufficiently large complete graph contains a 2-component link with linking number $> m$, a knot with crossing number $> m$, an m-component link such that every component K has $a_2(K) > m$ and every pair of components has linking number $> m$.

1.2. Arfless implies Knotless?

Conjecture 1.3. *Suppose a graph G has an embedding such that every knot in it has Arf invariant zero. Does G have a knotless embedding?*

2. Joel Foisy

Definition 2.1 (c.f. [8])**.** A graph has the *disjoint linking property* if it contains, in every spatial embedding, two disjoint pairs of cycles that form non split links.

Problem 2.2. *Does $K_{4,4,4}$ have the disjoint linking property?*

Problem 2.3 (c.f. [6], [13], [15])**.** *Does there exist an intrinsically 3-linked graph that embeds on the torus?*

Conjecture 2.4. *For high enough n and m, $C_n \times C_m$ is such a graph.*

3. Ryo Nikkuni

3.1. Questions on intrinsically knotted graphs.

An embedding f of a graph G into the 3-sphere is called a *spatial embedding* of G. A graph G is said to be *intrinsically linked* if for every spatial embedding f of G, $f(G)$ contains a nonsplittable 2-component link. It is well known that K_6 is intrinsically linked [12, 53]. On the other hand, a graph G is said to be *intrinsically knotted* if for every spatial embedding f of G, $f(G)$ contains a nontrivial knot. It is also well known that K_7 is intrinsically knotted [12]. A $\triangle Y$*-exchange* is an operation to obtain a new graph G_Y from a graph $G_\triangle$ by removing all edges of a cycle $\triangle$ of $G_\triangle$ with exactly three edges $\overline{uv}, \overline{vw}$ and $\overline{wu}$, and adding a new vertex x and connecting it to each of the vertices u, v and w as illustrated in Figure 1. A $Y\triangle$-exchange is the reverse of this operation. It is known that if $G_\triangle$ is intrinsically linked (resp. intrinsically knotted) then G_Y is also intrinsically linked (resp. intrinsically knotted) [40]. In particular, if G_Y is intrinsically linked then $G_\triangle$ is also intrinsically linked, and the set of all minor-minimal intrinsically linked graphs coincides with the *Petersen family*, which is the set of all seven graphs obtained from K_6 by a finite sequence of $\triangle Y$ and $Y\triangle$-exchanges [52]. On the other hand, if G_Y is intrinsically knotted then $G_\triangle$ may not always be intrinsically knotted. Actually Flapan-Naimi showed that there exists a graph which is obtained from K_7 by five times of $\triangle Y$-exchanges and twice $Y\triangle$-exchanges such that it is not intrinsically knotted [14]. We call the set of all twenty graphs obtained from K_7 by a finite sequence of $\triangle Y$ and $Y\triangle$-exchanges the

Heawood family. Hanaki-Nikkuni-Taniyama-Yamazaki determined when a graph G in the Heawood family is intrinsically knotted as follows.

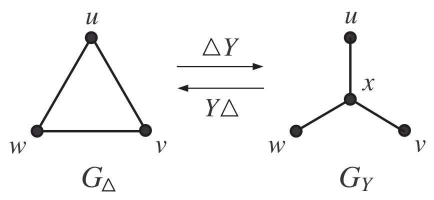

FIGURE 1

Theorem 3.1 ([27]). *Let G be a graph in the Heawood family. Then the following are equivalent:*

(1) *G is intrinsically knotted,*
(2) *G is obtained from K_7 by a finite sequence of $\triangle Y$-exchanges,*
(3) *G does not contain a triple of mutually disjoint cycles.*

Note that there exist fourteen graphs in the Heawood family which are intrinsically knotted, and all of them are minor-minimal [33]. Moreover, each of the other six graphs is minor-minimal with respect to another intrinsic nontriviality. We say that a graph G is *intrinsically knotted or completely 3-linked* if for every spatial embedding f of G, $f(G)$ contains a nontrivial knot or a 3-component link any of whose 2-component sublink is nonsplittable. It is clear that an intinsically knotted graph is intrinsically knotted or completely 3-linked. Then the following is known.

Theorem 3.2 ([27]). *All of the graphs in the Heawood family are minor-minimal intrinsically knotted or completely 3-linked graphs.*

It is known that the set of all minor-minimal intrinsically knotted graphs does not coincide with the set of all graphs obtained from K_7 by a finite sequence of $\triangle Y$-exchanges. For example, although $K_{3,3,1,1}$ cannot be obtained from K_7 by a finite sequence of $\triangle Y$-exchanges, it is minor-minimal intrinsically knotted graphs [18]. There exist fifty eight graphs obtained from $K_{3,3,1,1}$ by a finite sequence of $\triangle Y$ and $Y\triangle$-exchanges. Moreover, there exist twenty six graphs obtained from $K_{3,3,1,1}$ by a finite sequence of $\triangle Y$-exchanges, and it is known that all of them are also minor-minimal intrinsically knotted graphs [33]. Then we ask the following question.[1]

Question 3.3. *Let G be a graph obtained from $K_{3,3,1,1}$ by a finite sequence of $\triangle Y$ and $Y\triangle$-exchanges.*

(1) *What is the necessary and sufficient condition for G to be intrinsically knotted?*
(2) *Assume that G is not intrinsically knotted. Is G intrinsically knotted or completely 3-linked?*

[1]R. Naimi announced in his talk at this workshop that recently Goldberg-Mattman-Miller-Naimi showed that every graph obtained from $K_{3,3,1,1}$ by a finite sequence of $\triangle Y$ and $Y\triangle$-exchanges is a minor-minimal intrinsically knotted graph.

As we saw in Theorem 3.1, a graph G in the Heawood family is intrinsically knotted if and only if it is obtained from K_7 by a finite sequence of $\triangle Y$-exchanges. Then, the following is known. Here, $\Gamma(G)$ denotes the set of all cycles of a graph G, and $\Gamma^{(2)}(G)$ denotes the set of all unions of disjoint two cycles of G.

Theorem 3.4 ([45]). *Let G be a graph which is obtained from K_7 by a finite sequence of $\triangle Y$-exchanges. Then there exists a map ω from $\Gamma(G)$ to $\mathbb{Z}$ and a map ξ from $\Gamma^{(2)}(G)$ to $\mathbb{Z}$ such that for any spatial embedding f of G, it follows that*

$$\sum_{\gamma\in\Gamma(G)} \omega(\gamma)a_2(f(\gamma)) = 2 \sum_{\lambda\in\Gamma^{(2)}(G)} \xi(\lambda)\mathrm{lk}(f(\lambda))^2 - 21,$$

where lk *denotes the linking number, and a_2 denotes the second coefficient of the Conway polynomial.*

Note that Theorem 3.4 was first proved by Nikkuni in the case of $G = K_7$ [43]. As a corollary of Theorem 3.4, we have the following.

Corollary 3.5 ([45]). *Let G be a graph which is obtained from K_7 by a finite sequence of $\triangle Y$-exchanges. Then there exists a subset Γ of $\Gamma(G)$ such that for any spatial embedding f of G, it follows that*

$$\sum_{\gamma\in\Gamma} a_2(f(\gamma)) \equiv 1 \pmod 2.$$

Namely, Corollary 3.5 is a Conway-Gordon type theorem for any graph G which is obtained from K_7 by a finite sequence of $\triangle Y$-exchanges, and Theorem 3.4 is an integral lift of it. Now we ask the following question.

Question 3.6. *Let G be an intrinsically knotted graph which is obtained from $K_{3,3,1,1}$ by a finite sequence of $\triangle Y$ and $Y\triangle$-exchanges. Does there exist a subset Γ of $\Gamma(G)$ such that for any spatial embedding f of G, it follows that*

$$\sum_{\gamma\in\Gamma} a_2(f(\gamma)) \equiv 1 \pmod 2? \tag{3.1}$$

Assume that there exists such a subset Γ. Then, does there exist any integral lift of (3.1) in terms of the square of the linking number and the second coefficient of the Conway polynomial?

4. Thomas Mattman

4.1. Intrinsic Knotting Problems.

Problem 4.1. *Classify knotting of graphs on nine vertices.*

Graphs on eight vertices were classified independently by [7] and [3]. Morris [39] gives a computer proof that shows there are 11 MMIK (minor minimal intrinsically knotted) graphs on nine or fewer vertices: four (K_7, H_8, H_9, F_9) obtained from K_7 by Δ–Y moves, three obtained from $K_{3,3,1,1}$ by Δ–Y moves and four others (included in Morris's list of indeterminate graphs – see [23]). The idea of this problem is to turn the computer proof into something understandable by humans.

Problem 4.2. *Classify knotting of bipartite graphs on 11 vertices.*

Bipartite graphs on 10 or fewer vertices are classified in [29].

Prove (or give a counterexample to) the following:

Conjecture 4.3. *A bipartite graph on n vertices with at least five vertices in each part and with $4n-17$ or more edges is IK.*

The conjecture is known to hold for graphs with five or six vertices in one of the parts by [29]. For graphs in general, $5n-14$ edges suffices (assuming $n \geq 7$) by [7].

Problem 4.4. *Classify knotting of multi–partite graphs that are three edges shy of being complete.*

Graphs that are two (or fewer) shy are classified in [7].

Definition 4.5. Graph G has *Property A* if G is IK and removing any vertex gives an intrinsically linked (IL) graph.

Problem 4.6. *Find graphs that are minor minimal with respect to Property A.*

See Unsolved Question 3 of Section 8.2 of Adams's *Knot Book* [2].

Prove or give a counterexample to the following two conjectures.

Conjecture 4.7. *The graphs derived from K_7 and $K_{3,3,1,1}$ by Δ–Y moves are minor minimal graphs with respect to Property A.*

This would follow from:

Conjecture 4.8. Δ - *Y moves preserve Property A.*

Definition 4.9. G is *n–apex*, if it is planar after the deletion of at most n vertices.

A theorem of Sachs [53] shows:

Theorem 4.10. *If graph G is 1–apex then G is not IL.*

Problem 4.11. *Find the simplest example of a graph that is neither IL nor 1–apex.*

Such a graph has at least 8 vertices and at least 15 edges (see [36]). The disjoint union of two $K_{3,3}$ graphs is a candidate (simplest in terms of number of edges?) while Böhme's [5] J_1 is likely the simplest in terms of number of vertices.

Similarly, a theorem of [3] and [48] shows

Theorem 4.12. *If graph G is 2–apex then G is not IK.*

Problem 4.13. *Classify the graphs that are neither IK nor 2–apex.*

Note that E_9 is the simplest example of such a graph [36].

Problem 4.14. *In each of the problems above, replace IK with "not 2–apex."*

Prove or give a counterexample to the following:

Conjecture 4.15. *Y – Δ moves do not preserve the 2–apex property.*

Problem 4.16. *What can be said about n–apex graphs for $n \geq 3$? That is, 1–apex is related to intrinsic linking and 2–apex to intrinsic knotting. Are there topological ways to characterize n–apex for larger n?*

Definition 4.17. We say that graphs related by a sequence of Δ–Y and Y–Δ moves are *cousins* and call a graph G along with its set of cousins the graph's *family*.

These families can become quite large [23].

It is possible to have a family consisting of a single graph: decorate a Δ-free graph by adding degree 1 vertices adjacent to any degree 3 vertices. However, in adding such vertices G loses its MMIK property (assuming it was MMIK to start with).

Problem 4.18. *Is there an MMIK graph that is its own family? In other words is there a MMIK graph that is both Δ-free and of minimal degree four or more?*

If not, we can ask,

Problem 4.19. *What is the smallest family of an MMIK graph?*

The smallest family we know of are the seven graphs in the family of Foisy's [19] (13,30) graph.

5. Blake Mellor

5.1. Minimal numbers of links and knots.

Definition 5.1. Given a graph G, the *minimal number of links (knots) in* G, denoted $mnl(G)$ ($mnk(G)$), is the minimum, over all spatial embeddings Γ of G, of the number of distinct non-trivial links (knots) in Γ.

Problem 5.2. *Given a graph G, what are $mnl(G)$ and $mnk(G)$? In particular, what if $G = K_n$ or $G = K_{n,n}$?*

Conjecture 5.3. *$mnl(K_{n,m})$ is realized by the fan embedding. The fan embedding of $K_{4,4}$ is shown in Figure 2.*

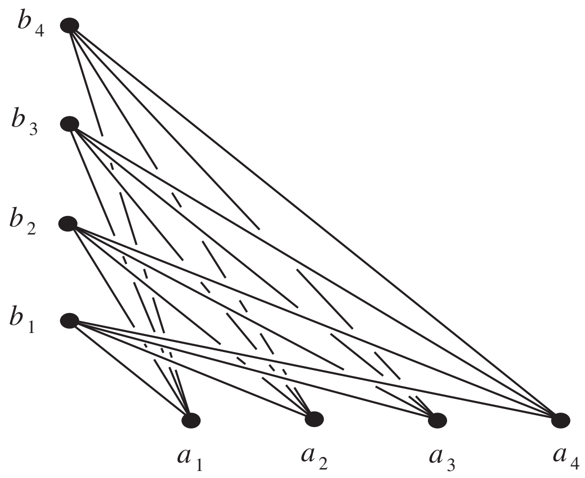

Figure 2. The fan embedding of $K_{4,4}$

This problem has been investigated for complete partite graphs with 9 or fewer vertices [1, 17]. A few of the results are listed below:

Theorem 5.4 ([1, 17]). *Some results for the minimal number of links:*

(1) $mnl(K_7) = 21$
(2) $mnl(K_{4,4}) = 2$
(3) $mnl(K_{n,4}) = 2\binom{n}{4}$
(4) $mnl(K_{3,3,1,1}) = 25$
(5) $217 \leq mnl(K_8) \leq 305$
(6) $mnl(K_{4,4,1}) = 74$

Some results for the minimal number of knots:

1. $mnk(K_{2,2,1,1,1,1}) = 2$
2. $18 \leq mnk(K_8) \leq 29$

Remark 5.5. The number of knotted Hamiltonian cycles in K_n, and particularly in K_8, has also been studied in [4] and [28].

6. Erica Flapan

Definition 6.1. Let Γ be a graph embedded in S^3. The topological symmetry group, $\mathrm{TSG}(\Gamma)$, is the subgroup of the automorphism group of Γ consisting of those automorphisms induced by homeomorphisms of (S^3, Γ). Restricting to orientation preserving homeomorphisms of (S^3, Γ) gives us $\mathrm{TSG}_+(\Gamma)$.

Theorem 6.2 ([16])**.** *A finite group G is isomorphic to $\mathrm{TSG}_+(\Gamma)$ for some 3-connected graph embedded in S^3 if and only if G is isomorphic to a finite subgroup of* $\mathrm{SO}(4)$*.*

Problem 6.3. *What groups can occur as $\mathrm{TSG}_+(\Gamma)$ if Γ is not 3-connected?*

Example 6.4. Every finite abelian group and every symmetric group can occur as $\mathrm{TSG}_+(\Gamma)$ for some embedded Γ which is not 3-connected (see Figure 3). However not all of these groups are isomorphic to subgroups of $\mathrm{SO}(4)$.

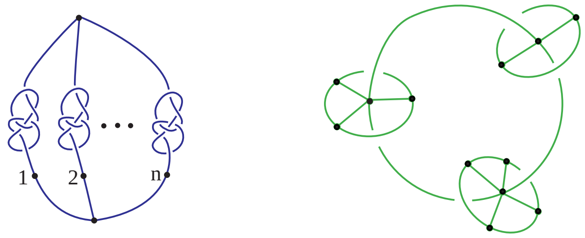

Figure 3. The graph on the left has $\mathrm{TSG}_+(\Gamma) = S_n$. The graph on the right has $\mathrm{TSG}_+(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$

Problem 6.5. *What groups can occur as $\mathrm{TSG}(\Gamma)$?*

Remark 6.6. $\mathrm{TSG}_+(\Gamma)$ is a normal subgroup of $\mathrm{TSG}(\Gamma)$ of index 2.

7. Makoto Ozawa

7.1. Primitive spatial graphs.

Definition 7.1. An embedding ϕ of a graph G in S^3 is said to be *primitive* if for each component G_i of G and any spanning tree T_i of G_i, the bouquet $\phi(G_i)/\phi(T_i)$ obtained from $\phi(G_i)$ by contracting all edges of $\phi(T_i)$ in S^3 is trivial. A graph is primitive if it has a primitive embedding.

Theorem 7.2 ([48])**.** *The property of having a primitive embedding is preserved under taking minors.*

Definition 7.3. Let $\mathcal{C}$ be a property closed under minor-reduction. The *obstruction set* for $\mathcal{C}$, denoted by $\Omega(\mathcal{C})$, is the set of all minor-minimal graphs which do not have $\mathcal{C}$. Let $\mathcal{P}$ be the property that a graph has a primitive embedding.

Theorem 7.4 ([48])**.** *The K_7-family and $K_{3,3,1,1}$-family are contained in the obstruction set $\Omega(\mathcal{P})$ for primitive graphs.*

Problem 7.5. *Determine the obstruction set $\Omega(\mathcal{P})$ for primitive graphs.*

Remark 7.6. Since the Foisy graph F ([19]) is not primitive, there exists a graph $G \in \Omega(P)$ which is a minor of F. This graph G will be a new element of $\Omega(P)$ other than K_7 and $K_{3,3,1,1}$-family, since F has no minor in K_7 nor $K_{3,3,1,1}$-family.

According to [52, (1.7)], flat embeddings of a 4-connected graph are unique up to reflections. We also expect some rigidity of primitive embeddings.

Conjecture 7.7. *Primitive embeddings of a 5-connected graph are unique up to reflections.*

Remark 7.8. It has been shown by the author that the Petersen graph has a unique primitive embedding up to reflections.

The following Figure 4 shows a 'rough summary' for the hierarchy of graphs by spatial embeddings. We refer to [9], [10], [11], [12], [14], [18], [19], [21], [26], [33], [34], [35], [48], [51], [53], [54], [59]. We remark that van der Holst and Pendavingh [58] introduced a topological graph parameter $\sigma(G)$ and showed that $\mu(G) \leq \sigma(G) + 2$. This can be compared that Pendavingh [49] proved $\mu(G) \leq \lambda(G) + 2$.

7.2. **The representativity of spatial graphs.**

Definition 7.9. The *representativity* of a non-planar graph G is defined as

$$r(G) = \max_{F \in \mathcal{F}} \min_{C \in \mathcal{C}_F} |C \cap G|,$$

where $\mathcal{F}$ is the set of all closed surfaces containing G and $\mathcal{C}_F$ is the set of all essential loops in F.

Conjecture 7.10 (The strong embedding conjecture)**.** *For a 2-connected non-planar graph G, $r(G) \geq 2$.*

Definition 7.11 ([47])**.** The *representativity* of a non-trivial spatial graph Γ is defined as

$$r(\Gamma) = \max_{F \in \mathcal{F}} \min_{D \in \mathcal{D}_F} |\partial D \cap \Gamma|,$$

where $\mathcal{F}$ is the set of all closed surfaces containing Γ and $\mathcal{D}_F$ is the set of all compressing disks for F in S^3.

Conjecture 7.12 (The strong spatial embedding conjecture)**.** *For a non-trivial spatial graph Γ of a 2-connected graph G, $r(\Gamma) \geq 2$.*

8. Youngsik Huh and Ryo Nikkuni

8.1. **A conjecture for knotted projections.** Let $\mathbb{S}^3$ be the unit 3-sphere in $\mathbb{R}^4$ centered at the origin. An embedding f of a finite graph G into $\mathbb{S}^3$ is called a *spatial embedding of* G. A graph G is said to be *intrinsically nontrivial* if for every spatial embedding f of G there exists a planar subgraph F of G such that the restriction map of f on F is not trivial [42]. It is clear that intrinsically knotted graphs and

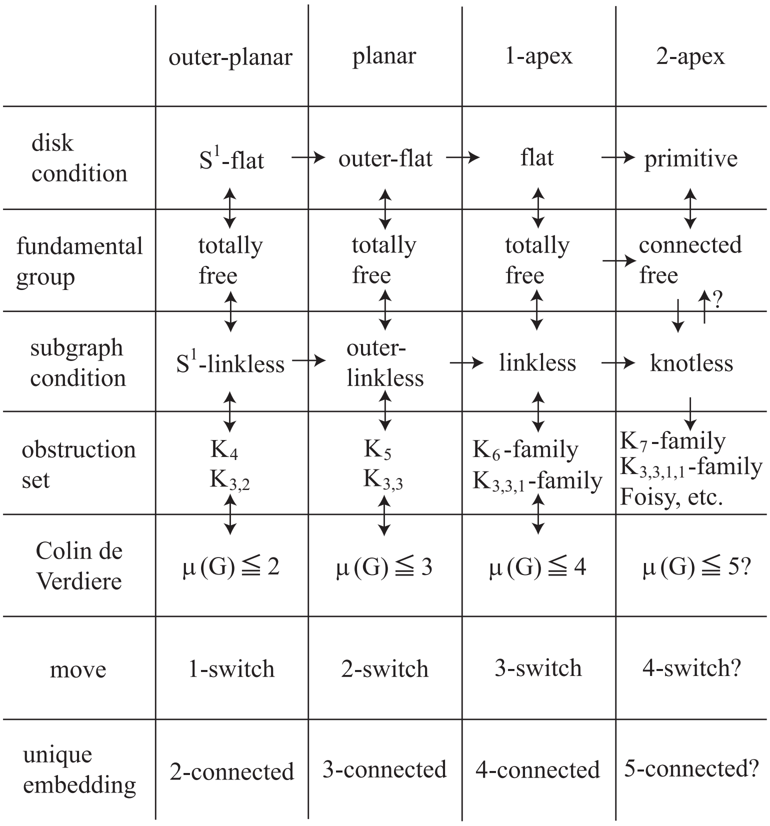

FIGURE 4. Hierarchy of graphs by spatial embeddings

intrinsically n-linked graphs are intrinsically nontrivial. See also [20] and [27] for related works. In general, every spatial embedding of an intrinsically nontrivial graph must contain a nonsplittable 2-component link. Namely, it is known the following.

Theorem 8.1 ([42]). *Intrinsically nontrivial graphs are intrinsically (2-)linked.*

On the other hand, a continuous map φ from G to the unit 2-sphere $\mathbb{S}^2$ is called a *regular projection* of G if the multiple points of φ are only finitely many transversal double points away from the vertices of G. For a spatial embedding f of G, we also say that φ is a *regular projection of f* or *f projects on φ*, if there exists an embedding f' of G into $\mathbb{S}^3 \setminus \{(0,0,0,1),(0,0,0,-1)\}$ such that f is ambient isotopic to f' and $\pi \circ f' = \varphi$, where π is the natural projection from $\mathbb{S}^3 \setminus \{(0,0,0,1),(0,0,0,-1)\}$ to $\mathbb{S}^2$. A regular projection φ of a planar graph G is said to be *knotted* if there does not exist a trivial spatial embedding of G which projects on φ [55].

Example 8.2 ([55]). Let φ be the regular projection of the octahedron graph as illustrated in Figure 5. Then we can see that every spatial embedding of the graph which projects on φ contains a Hopf link, namely φ is knotted.

Motivating by Theorem 8.1, we ask the following question.

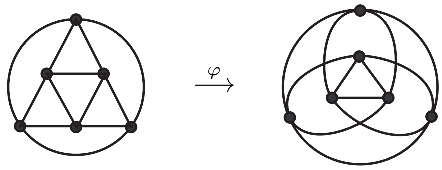

FIGURE 5

Question 8.3. *Let φ be a knotted regular projection of a planar graph G. Then, does a spatial embedding of G which projects on φ always contain a nonsplittable 2-component link?*

In particular, we present the following conjecture.

Conjecture 8.4. *Let φ be a knotted regular projection of a planar graph G. Then, every spatial embedding of G which projects on φ contains a 2-component link with odd linking number.*

Note that if Conjecture 8.4 is true, then it also gives an affirmative answer to the following question asked by Nikkuni-Ozawa-Taniyama-Tsutsumi.

Question 8.5 ([44]). *Let φ be a knotted regular projection of a planar graph G and f a spatial embedding of G which projects on φ. Then, does there exist a proper subgraph H of G such that the restriction map of f on H is nontrivial?*

It is known that a knotted regular projection of a planar graph has at least three double points [30]. In particular for knotted projections with exactly three double points, Conjecture 8.4 is true as follows.

Theorem 8.6 ([31]). *Let φ be a knotted regular projection of a planar graph G with exactly three double points. Then, every spatial embedding of G which projects on φ contains a Hopf link.*

9. Ryo Nikkuni and Akira Yasuhara

9.1. **Problems about C_k-equivalence on spatial theta curves.** An embedding f of a finite graph G into the 3-sphere is called a *spatial embedding* of G or simply a *spatial graph*. A C_1*-move* is a crossing change and a C_k*-move* is a local move on spatial graphs as illustrated in Figure 6 for $k \geq 2$ [25, 24]. Two spatial embeddings of G are said to be C_k*-equivalent* if they are transformed into each other by C_k-moves and ambient isotopies. By the definition of a C_k-move, it is easy to see that C_k-equivalence implies C_{k-1}-equivalence.

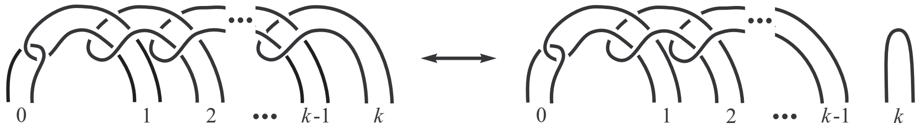

FIGURE 6

In particular for oriented knots, it is known the following.

Theorem 9.1 ([24, 25])**.** *Two oriented knots J and K are C_k-equivalent if and only if $v(J) = v(K)$ for any finite type invariant v of order less than or equal to $k-1$.*

Let Θ be the graph with two vertices u and v and three edges e_1, e_2 and e_3 each of which joins u and v. A spatial embedding of Θ is called a *spatial theta curve*.

Problem 9.2. *For a positive integer k, classify all spatial theta curves up to C_k-equivalence.*

Moreover, we ask the following question.

Question 9.3. *Are C_k-equivalence classes of spatial theta curves classified by only finite type invariants of order less than or equal to $k-1$?*

Some affirmative partial answers for Question 9.3 have already given by Motohashi-Taniyama for $k = 2$ [56, 38], Taniyama-Yasuhara for $k = 3$ [57], Yasuhara for $k = 4$ [60], Nikkuni-Yasuhara for $k = 5$ [46] and Meilhan-Yasuhara for $k = 6$ [37]. The case of $k \geq 7$ is still open.

It is known that the set of C_k-equivalence classes of oriented knots forms an abelian group under the connected sum [25]. It is also known that the set of C_k-equivalence classes of spatial theta curves forms a group G_k under the vertex connected sum [60], but these groups behave quite differently than the case of oriented knots as follows.

Theorem 9.4. *G_k is abelian if $k \leq 6$, and nonabelian if $k \geq 12$.*

Actually, Yasuhara showed that G_k is abelian if $k \leq 4$, and nonabelian if $k \geq 12$ [60]. It is also known that the set of C_k-equivalence classes of *m-string links* forms a group $S_k(m)$ under the composition [25]. Then there exists an epimorphism from $S_k(2)$ to G_k, see Figure 7. Thus if $S_k(2)$ is abelian then G_k is also abelian. Meilhan-Yasuhara showed that $S_k(2)$ is abelian if $k \leq 6$ [37]. Namely G_k is also abelian if $k \leq 6$.

Problem 9.5. *Find the minimum number k ($7 \leq k \leq 12$) such that G_k is nonabelian.*

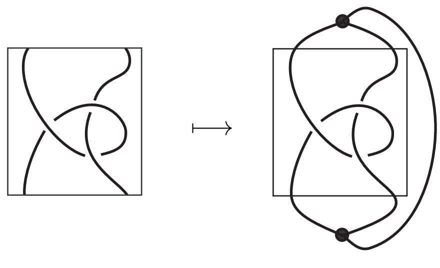

FIGURE 7

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