Topological Symmetry Groups of Complete Graphs

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Topological Symmetry Groups

The topological symmetry group was introduced by Jon Simon (1986) to study the symmetries of molecules whose structure is not rigid. It can be used to study the symmetries of any graph embedded in $S^3$.

**Definition**

Let $\gamma$ be an abstract graph with automorphism group $\text{Aut}(\gamma)$. Let $\Gamma$ be an embedding of $\gamma$ in $S^3$. The (orientation preserving) topological symmetry group of $\Gamma$, denoted $\text{TSG}_+(\Gamma)$, is the subgroup of $\text{Aut}(\gamma)$ induced by orientation preserving homeomorphisms of the pair $(S^3, \Gamma)$.
Below are embeddings $\Gamma_1$ of $K_{2,2}$ and $\Gamma_2$ of $K_4$ where $\text{TSG}_+(\Gamma_1) = D_4$ and $\text{TSG}_+(\Gamma_2) = S_4$ (with involutions $f$ and $h$).
Which groups can be topological symmetry groups for a complete graph?

**Complete Graph Theorem (Flapan, Naimi, Tamvakis)**

A finite group $H$ is $\text{TSG}_+(\Gamma)$ for some embedding $\Gamma$ of a complete graph in $S^3$ if and only if $H$ is isomorphic to a finite subgroup of either $\text{SO}(3)$ or $D_m \times D_m$ for some odd $m$.

The finite subgroups of $\text{SO}(3)$ are the cyclic and dihedral groups, and the polyhedral groups $A_4$, $S_4$ and $A_5$.

The Complete Graph Theorem does not specify which complete graphs can realize a particular group $H$. 
Our Question

Question

For which \( m \) does \( K_m \) have an embedding \( \Gamma \) such that \( \text{TSG}^+_\Gamma \) is isomorphic to \( A_4 \), \( S_4 \) or \( A_5 \)?

Recall that \( A_4 \) is the group of symmetries of a solid tetrahedron, \( S_4 \) is the group of symmetries of a solid cube (or of a hollow tetrahedron), and \( A_5 \) is the group of symmetries of a solid dodecahedron (or of a 4-simplex).
Main Results

$A_4$ Theorem

A complete graph $K_m$ with $m \geq 4$ has an embedding $\Gamma$ in $S^3$ such that $\text{TSG}_+(\Gamma) = A_4$ if and only if $m \equiv 0, 1, 4, 5, 8 \pmod{12}$.

$S_4$ Theorem

A complete graph $K_m$ with $m \geq 4$ has an embedding $\Gamma$ in $S^3$ such that $\text{TSG}_+(\Gamma) = S_4$ if and only if $m \equiv 0, 4, 8, 12, 20 \pmod{24}$.

$A_5$ Theorem

A complete graph $K_m$ with $m \geq 4$ has an embedding $\Gamma$ in $S^3$ such that $\text{TSG}_+(\Gamma) = A_5$ if and only if $m \equiv 0, 1, 5, 20 \pmod{60}$.
We first observe the number of vertices fixed by an element of the symmetry group is determined by the conjugacy class of the element. In particular, in $A_4$ and $A_5$, two elements of the same order must fix the same number of vertices (in $S_4$ there are two conjugacy classes of elements of order 2).

The necessity of the conditions rests on determining the possible combinations of the number of fixed vertices for each conjugacy class.
We next recall the following result of Flapan, Naimi and Tamvakis:

**Isometry Theorem (Flapan, Naimi, Tamvakis)**

*Let \( \Omega \) be an embedding of \( K_m \) in \( S^3 \) such that \( \text{TSG}_+(\Omega) \) is not a cyclic group of odd order. Then \( K_m \) can be re-embedded in \( S^3 \) as \( \Gamma \) such that \( \text{TSG}_+(\Omega) \leq \text{TSG}_+(\Gamma) \) and \( \text{TSG}_+(\Gamma) \) is induced by an isomorphic finite subgroup of \( \text{SO}(4) \).*

As a result, we may assume that all symmetries are either rotations (which fix a circle) or glide rotations (with no fixed points). This means that no symmetry can fix more than 3 vertices, since \( K_4 \) cannot be embedded in a circle.
We now have a fairly small number of possible combinations for the number of vertices fixed by each element of the group; careful analysis allows us to reduce the combinations even further. For example, no involution in $A_4$ or $A_5$ can fix more than one vertex.

This leaves only a few possibilities. For example, if $A_4 \leq \text{TSG}_+(\Gamma)$, let $n_2$ be the number of vertices fixed by the elements of order 2, and $n_3$ be the number of vertices fixed by the elements of order 3. Then the possibilities for $(n_2, n_3)$ are just $(0, 0), (0, 1), (0, 2), (0, 3), (1, 1)$ and $(1, 2)$.
Finally, we take these remaining possibilities and apply Burnside’s Lemma:

**Burnside’s Lemma**

Suppose a group $G$ acts on the vertices of an embedded graph $\Gamma$. Let $\text{fix}(g)$ be the set of vertices fixed by $g \in G$. Then the number of vertex orbits is given by:

$$
\# \text{ orbits} = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|
$$

Knowing the number of orbits must be an integer, and that the identity fixes all the vertices, allows us to determine the possible number of vertices in the complete graph, modulo the size of the group.
Recall that $|A_4| = 12$, and consider the graph $K_m$. For each combination of $n_2$ and $n_3$, the value of $m \pmod{12}$ is determined by knowing that $\frac{1}{12}(m + 3n_2 + 8n_3)$ is an integer (the identity fixes $m$ vertices).

<table>
<thead>
<tr>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$m \pmod{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 or 3</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>8</td>
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<tr>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>1</td>
<td>2</td>
<td>5</td>
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To prove the sufficiency of the conditions, we will construct embeddings of the complete graphs with the desired topological symmetry groups. For a given group $G$, this is done by:

1. Embedding the vertices of the graph so there is a faithful action of $G$.
2. Showing that we can extend the embedding to the edges so that $G \leq \text{TSG}_+(\Gamma)$.
3. Modifying the embedding to restrict the symmetries to $G$.

We embed the vertices using the regular polyhedra. The second step, embedding the edges, requires an Edge Embedding Lemma.
The Edge Embedding Lemma

Let $G$ be a finite subgroup of $\text{Diff}_+(S^3)$ which acts faithfully on the embedded vertices $V$ of a graph $\gamma$. Suppose that adjacent pairs of vertices in $V$ satisfy the following hypotheses:

1. If a pair is pointwise fixed by non-trivial elements $h, g \in G$, then $\text{fix}(h) = \text{fix}(g)$.

2. For each pair $\{v, w\}$ in the fixed point set $C$ of some non-trivial element of $G$, there is an arc $A_{vw} \subseteq C$ bounded by $\{v, w\}$ whose interior is disjoint from $V$ and from any other such arc $A_{v'w'}$.

3. If a point in the interior of some $A_{vw}$ or a pair $\{v, w\}$ bounding some $A_{vw}$ is setwise invariant under an $f \in G$, then $f(A_{vw}) = A_{vw}$.

4. If a pair is interchanged by some $g \in G$, then the subgraph of $\gamma$ whose vertices are pointwise fixed by $g$ can be embedded in a proper subset of a circle.

5. If a pair is interchanged by some $g \in G$, then $\text{fix}(g)$ is non-empty, and for any $h \neq g$, then $\text{fix}(h) \neq \text{fix}(g)$.

Then there is an embedding of the edges of $\gamma$ in $S^3$ such that the resulting embedding of $\gamma$ is setwise invariant under $G$. 
Using the Edge Embedding Lemma

The Edge Embedding Lemma allows us to embed a graph so that it has certain symmetries (and possibly others as well) by embedding the vertices so that certain conditions are satisfied. This is much easier than trying to embed all the edges by hand!

For complete graphs, if the desired group is $S_4$ or $A_5$, the Complete Graph Theorem says that these cannot be proper subgroups of the topological symmetry group, so we’re done. If the desired group is $A_4$, we may need to modify the embedding to ensure the symmetry group is not $S_4$ or $A_5$. 
Example: Embedding $K_{24n+20}$ so that $\text{TSG}_+ = S_4$

The gray arcs are the arcs required by the Edge Embedding Lemma. The involutions not in $A_4$ interchange $\tau_1$ and $\tau_2$. 
We apply the Subgroup Theorem to our embeddings for $S_4$ and $A_5$ to obtain embeddings for $A_4$.

**Subgroup Theorem**

Let $\Gamma$ be an embedding of a 3-connected graph in $S^3$. Suppose that $\Gamma$ contains an edge $e$ which is not pointwise fixed by any non-trivial element of $TSG_+(\Gamma)$. Then for every $H \leq TSG_+(\Gamma)$, there is an embedding $\Gamma'$ of $\Gamma$ with $H = TSG_+(\Gamma')$.

For example, in the embedding for $K_{20}$, any of the edges not marked in gray will satisfy the hypothesis.
Special Case: An Embedding of $K_4$ with $\text{TSG}_+ = A_4$

Embed the vertices of $K_4$ as the vertices of the standard tetrahedron, and embed the edges as shown (when the tetrahedron is unfolded):
Special Case: An Embedding of $K_4$ with $\text{TSG}_+ = A_4$

Then the three edges around each face form the following non-invertible knot:
If $G$ is a cyclic or dihedral group, or a subgroup of $D_m \times D_m$ with $m$ odd, for which $n$ does $K_n$ have an embedding $\Gamma$ such that $\text{TSG}_+(\Gamma) = G$? (Work in progress.)

In general, if $\gamma$ is a 3-connected graph, the topological symmetry group can only be a subgroup of $\text{SO}(4)$ (Flapan, Naimi, Pommersheim, Tamvakis). For any such subgroup, can we characterize the graphs which can realize that group as a topological symmetry group?
Thank you all for coming to this talk.
Thanks to the organizers and Waseda University for hosting this conference!
Preprint is available at arXiv:1008.1095
Any questions?