Prime knots whose arc index is smaller than the crossing number

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Arc presentation and arc index

An *arc presentation* of a knot or a link $L$ is an ambient isotopic image of $L$ contained in the union of finitely many half planes, called *pages*, with a common boundary line in such a way that each half plane contains a properly embedded single arc.

![Diagram of arc presentations](image)

**Figure 1:** An arc presentation of the figure eight knot

The minimal number of pages among all arc presentations of a link $L$ is called the *arc index* of $L$ and is denoted by $\alpha(L)$.
Methods of describing arc presentation

Figure 2: Representations of arc presentation
Brief History

• (Cromwell, 1995) Every link admits an arc presentation.

• (Nutt, 1999) All knots up to arc index 9 are identified.

• (Bae-Park, 2000)
  \[ \alpha(L) = c(L) + 2 \text{ if and only if a non-split link } L \text{ is alternating.} \]
  (Knot-spoke diagrams are used for the proof.)

• (Beltrami, 2002) Arc index for prime knots up to 10 crossings are determined.

• (Jin et al., 2006) All prime knots up to arc index 10 are identified.

• (Ng, 2006) Arc index for prime knots up to 11 crossings are determined.

• (Jin-Park, 2007) All prime knots up to arc index 11 are identified.
  A prime link \( L \) is \textit{nonalternating} if and only if \( \alpha(L) \leq c(L) \).
A wheel diagram is a finite plane graph of straight edges which are incident to a single vertex. The projection of an arc presentation of a knot or a link into the $xy$-plane is of this shape.

![Wheel Diagram](image)

Figure 3: Wheel Diagrams of the figure-eight knot

For a wheel diagram with $n$ edges to represent a knot or a link, each edge must be labeled with an unordered pair of distinct integers so that each of the integers, $1, 2, \ldots, n$ appear exactly twice in the wheel diagram. These numbers indicate the $z$-levels of the endpoints of the corresponding arcs.
A knot-spoke diagram $D^*$ is a finite connected plane graph satisfying

1. There are three kinds of vertices in $D^*$: a distinguished vertex $v_0$ with valency at least four, 4-valent vertices, and 1-valent vertices.

2. Every edge incident to a 1-valent vertex is also incident to $v_0$. Such an edge is called a spoke.

Figure 4: Knot-spoke diagrams
Prime knot-spoke diagrams

A knot-spoke diagram \( D^* \) is said to be \emph{prime} if no simple closed curve meeting \( D^* \) in two interior points of edges separates multi-valent vertices into two parts.

Figure 5: Prime diagram and non-prime diagram
Cut-point

A multi-valent vertex \( v \) of a knot-spoke diagram \( D^* \) is said to be a cut-point if there is a simple closed curve \( S \) meeting \( D^* \) in \( v \) and separating non-spoke edges into two parts.

![Figure 6: Cut-point](image)

- A cut-point free knot-spoke diagram with more than one non-spoke edges cannot have a loop.
- If a prime knot-spoke diagram \( D^* \) has a cut-point, then the distinguished vertex \( v_0 \) must be the cut-point with valency bigger than four.
Contracting an edge incident to $v_0$

Let $e$ be an edge of a cut-point free knot-spoke diagram $D^*$ as in the figure. The knot-spoke diagram $(D^*)_e$ is obtained by

- contracting $e$ and
- replacing any simple loop thus created by a spoke.

![Local diagram of $D^*$ near $e$](image1)

![Local diagram of $(D^*)_e$ near $v_0$](image2)

Figure 7: Contraction of an edge in $D^*$

A loop in a knot-spoke diagram is said to be *simple* if the other non-spoke edges are in one side of it.
$D^*$ and $(D^*)_e$

There are important facts to point out.

1. $D^*$ and $(D^*)_e$ represent the same knot or link.

2. The sum of the number of regions divided by the non-spoke edges and the number of spokes is unchanged.

3. $(D^*)_e$ is prime if $D^*$ is prime.
Wheel diagram with $c(D) + 2$ spoke

Starting from a knot diagram $D$, we end up with a knot-spoke diagram with $c(D)$ spokes and only one non-spoke edge which is a non-simple loop where $c(D)$ is the number of crossings in $D$.

Figure 8: Folding the last non-spoke edge

The last non-spoke edge, which is a loop, is being folded to create two extra spokes. This shows the inequality $\alpha(L) \leq c(L) + 2$. 
A process converting $4_1$ into a wheel diagram

- Choose a vertex $v_0$ and put labels on the two edges meeting at $v_0$, to assign vertical levels of the overpass and the underpass.

- Choose an edge $e$ to contract and assign the label of a new level at the edges crossing $e$ at the other end which is the lowest if the crossing is an undercrossing and the highest otherwise.

- Contract the edge and replace each simple loop with a spoke and label it with the two labels of the loop.
Let $D$ be a knot diagram. We may consider $D$ as a connected 4-valent plane graph with $c(D)$ vertices and $2c(D)$ edges.

A spanning tree of $D$ is a tree which contains all the vertices of $D$.

A filtered spanning tree of $D$ is an increasing sequence

$$T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_{c(D)-1}$$

The closure of $T_i$, denoted by $\overline{T}_i$, is the subgraph of $D$ obtained from $T_i$ by adding the edges which are incident $T_i$ at both ends.
Edges not contained in the spanning tree

An edge $e$ of $\overline{T_i} \setminus \overline{T_{i-1}} \subset D$ is said to be *good* if $e$ meets the edge $T_i \setminus T_{i-1}$ transversely at the vertex not contained in $T_{i-1}$.

An edge $e$ of $\overline{T_i} \setminus \overline{T_{i-1}} \subset D$ is said to be *bad* if $e$ meets the edge $T_i \setminus T_{i-1}$ vertically at the vertex not contained in $T_{i-1}$.

Figure 9: Good edges and a bad edge
Good filtered tree and Good filtered spanning tree

Let $T_0 \subset T_1 \subset \cdots \subset T_m$ be a filtered tree in a diagram $D$ which does not span $D$. If the knot-spoke diagram obtained by contraction of the edges $e_i = T_i \setminus T_{i-1}$, $i = 1, \ldots, m$ is cut-point free, we say that the filtered tree is good.

A filtered spanning tree $T_0 \subset T_1 \subset T_2 \subset \cdots \subset T_{c(D)-1}$ is said to be good if $T_0 \subset T_1 \subset \cdots \subset T_m$ is good filtered tree for each $m$, $1 \leq m \leq c(D) - 2$ and there is only one ‘bad’ edge in $D$ which belongs to $D \setminus \overline{T}_{c(D)-1}$.

Theorem 1 (Bae-Park, 2000) A prime link diagram $D$ admits a good filtered spanning tree and therefore we can obtain an arc presentation with $c(D) + 2$ arcs.
Good filtered tree (Cont.)

**Proposition 2**  Let $T_0 \subset T_1 \subset \cdots \subset T_m$ be a filtered tree in a diagram $D$ which does not span $D$. Then the following are equivalent.

1. Every edge of $\overline{T_m} \setminus T_m$ is a good edge, and a sufficiently small neighborhood of $\overline{T_m}$ has connected exterior in $D$.

2. The filtered tree is good.

**Corollary 3**  Let $T_0 \subset T_1 \subset \cdots \subset T_m$ be a good filtered tree in a diagram $D$ which does not span $D$. Let $e$ be an edge in $D$ such that $T_m \cap e$ is a single vertex, so that $T_m \cup e$ is a tree. If $T_0 \subset T_1 \subset \cdots \subset T_m \subset (T_m \cup e)$ is not a good filtered tree, then one of the following holds.

- $\overline{T_m \cup e}$ has a bad edge.
- A sufficiently small neighborhood of $\overline{T_m \cup e}$ has disconnected exterior in $D$. 
**Cutting arc**

Let $T$ be a filtered tree in $D$ which does not span $D$. A simple arc $\Gamma$ is called a *cutting arc* of $T$ if it satisfies the following conditions.

1. $\Gamma \cap D$ consists of the endpoints of $\Gamma$ which are two distinct vertices of $T$.
2. A proper subcollection of edges of $D \setminus \overline{T}$ is enclosed by the simple closed curve $\overline{\Gamma}$ constructed by $\Gamma$ and the path in $T$ joining the endpoints of $\Gamma$.

![Figure 10: Cutting arc of a filtered tree](image)

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Doubly good edges

A good edge $e \subset \overline{T_i} \setminus \overline{T_{i-1}}$ is said to be \textit{doubly good} if the three edges $e$, $e_i = T_i \setminus T_{i-1}$, and $e_{i-1} = T_{i-1} \setminus T_{i-2}$ together bound a nonalternating triangular region in $D/T_{i-2}$.

![Doubly good edge on a nonalternating triangular region](image)

Figure 11: Doubly good edge on a nonalternating triangular region

The doubly good edge on the filtered tree corresponds to removable spoke in the knot-spoke diagram obtained by contraction of edges in $T_i$. 
Doubly good edges (Cont.)

It is known that every prime nonalternating diagram admits a good filtered spanning tree having at least two doubly good edges.

**Theorem 4 (Jin-Park, 2007)** A prime link $L$ is nonalternating if and only if $\alpha(L) \leq c(L)$.

**Theorem 5** A prime diagram $D$ of a nonalternating knot has a good filtered spanning tree which has at least two doubly good edges. Furthermore, if there are $d$ doubly good edges, then one can obtain an arc presentation with $c(D) + 2 - d$ arcs.
**Goal**

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<td>spoke</td>
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<tr>
<td>removable arc</td>
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**Goal**: To construct a good filtered tree to find as many doubly good edges as possible.
Supporting arc and String \( \overrightarrow{ve} \)

For two regions \( R \) and \( S \) in a diagram, an arc \( \Delta \) is said to be a *supporting arc* of \( R \) and \( S \) if \( \Delta \) consists of at least 3 edges and one of the end edges of \( \Delta \) is one of the boundary edges of \( R \) and the other is one of the boundary edges of \( S \).

A string from a vertex \( v \) extending \( e \) in a knot diagram \( D \) is a portion of \( D \) that goes from \( v \) passing through \( e \) along \( D \) and is denoted by \( \overrightarrow{ve} \).

![Supporting arc and String](image)

Figure 12: Supporting arc and String \( \overrightarrow{ve} \)
Three types of nonalternating diagram

Let $n \geq 2$. A nonalternating knot diagram $D$ is said to be \textit{$(n, 1)$-nonalternating} if it can be decomposed of two alternating tangles one of which is an $(n, 1)$-tangle.

Let $n \geq 1$. A nonalternating knot diagram $D$ is said to be \textit{$n$-nonalternating} if it can be decomposed of two alternating tangles one of which is an $n$-tangle.

A 1-nonalternating diagram is also called an \textit{almost alternating diagram}.

(a) The $(n, 1)$-tangle \hspace{1cm} (b) The $n$-tangle \hspace{1cm} (c) The 1-tangle

Figure 13: Tangles
Theorem A

Let $D$ be a prime $(n, 1)$-nonalternating minimal crossing knot diagram having a nonalternating triangular region with some edges and regions labeled as in Figure 14 for some integer $n \geq 2$. Then $\alpha(D) < c(D)$ if $D$ satisfies the two conditions below:

![Diagram](image)

Figure 14: $(2, 1)$-nonalternating diagram
Theorem A (Cont.)

1. The string $\overrightarrow{q_1e_4}$ and at least one of the two strings $\overrightarrow{q_1e_{51}}, \overrightarrow{q_2e_{52}}$ meet at a crossing before they become incident to the region $R_2$, or there is a supporting arc of $R_4$ and $R_5$ which does not contain any edge of $\partial R_2$ and $\partial R_3$.

2. At least one of the three strings $\overrightarrow{q_1e_4}, \overrightarrow{q_1e_{51}}, \overrightarrow{q_2e_{52}}$ is incident to $R_2$ before or at the same time to $R_1$, or there is a supporting arc of $R_4$ and $R_5$, not incident to $R_3$, whose extension is incident to $R_2$ before or at the same time to $R_1$. 
Diagrams on which Theorem A can be applied

13n3256
13n1404
13n1042

13n1974
13n1974
Theorem B

Let $D$ be a prime, $(n)$-nonalternating and minimal crossing knot diagram having a nonalternating triangular region. If $D$ satisfies the condition 1, 2, and 3 where the regions and edges near the nonalternating triangular region are labeled as in Figure 15. Then $\alpha(D) < c(D)$.

Figure 15: 2-nonalternating diagram
1. String $\overrightarrow{q_1e_4}$ and at least one of the two strings $\overrightarrow{q_1e_5_1}$, $\overrightarrow{q_2e_5_2}$ meet at a crossing before they become incident to the regions $R_1$ or $R_2$, or there is a supporting arc of $R_4$ and $R_5$ which does not contain any edge of $R_1$, $R_2$ and $R_3$.

2. At least one of the three strings $\overrightarrow{q_1e_4}$, $\overrightarrow{q_1e_5_1}$, $\overrightarrow{q_2e_5_2}$ is incident to $R_2$ before or at the same time to $R_1$, or there is a supporting arc of $R_4$ and $R_5$, not incident to $R_3$, whose extension is incident to $R_2$ before or at the same time to $R_1$.

3. $R_2$ is bounded by at least $n + 3$ edges.
Diagrams on which Theorem B can be applied
Theorem C

Let $D$ be a prime, almost alternating, and minimal crossing knot diagram having a nonalternating triangular region. If $D$ satisfies the condition 1, 2, 3 where the regions and edges near the nonalternating triangular region are labeled as in Figure 16. Then $\alpha(D) < c(D)$

![Almost alternating diagram](image)

Figure 16: Almost alternating diagram
Theorem C (Cont.)

1. $\overrightarrow{q_1e_4}$ and at least one of $\overrightarrow{q_1e_{51}}$, $\overrightarrow{q_2e_{52}}$ meet at a crossing before they become incident to the regions $R_1$ or $R_2$, or there is a supporting arc of $R_4$ and $R_5$ which does not contain any edge of $R_1$, $R_2$ and $R_3$.

2. At least one of the three strings $\overrightarrow{q_1e_4}$, $\overrightarrow{q_1e_{51}}$, $\overrightarrow{q_2e_{52}}$ is incident to $R_2$ before or at the same time to $R_1$, or there is a supporting arc of $R_4$ and $R_5$, not incident to $R_3$, whose extension is incident to $R_2$ before or at the same time to $R_1$.

3. At least one of $\overrightarrow{q_1e_4}$, $\overrightarrow{q_1e_{51}}$ is incident to $R_1$ at $v_0$ for the first time without being incident to $R_2$. 
A diagram on which Theorem C can be applied
A sketch of proof of Theorem A

Let $D'$ be the diagram obtained from $D$ by a type 3 Reidemeister move over the region $R_3$ as in Figure 17.

Figure 17: $(2, 1)$-nonalternating diagram after a type 3 Reidemeister move

We construct a good filtered tree whose closures gradually contain $\partial R'_1, \partial R'_2$ and $\partial R'_3$. Let $\overline{v_i v_j}$ denote the edge joining $v_i$ and $v_j$. The edges $\overline{v_2 v_3}$, $\overline{v_4 v_5}$ and $\overline{v_6 v_7}$ will become doubly good edges.
A sketch of proof of Theorem B

Let \( D' \) be the diagram obtained from \( D \) by a type 3 Reidemeister move over the region \( R_3 \) as in Figure 18.

![Figure 18: 2-nonalternating diagram after a type 3 Reidemeister move](image)

We construct a good filtered tree whose closures gradually contain \( \partial R'_1, \partial R'_2 \) and \( \partial R'_3 \). The edges \( v_1v_2, v_3v_4 \) and \( v_5v_6 \) will become doubly good edges.
A sketch of proof of Theorem C

Let $D'$ be the diagram obtained from $D$ by a type 3 Reidemeister move over the region $R_3$ as in Figure 19.

![Almost alternating diagram after a type 3 Reidemeister move](image)

Figure 19: Almost alternating diagram after a type 3 Reidemeister move

We construct a good filtered tree whose closures gradually contain $\partial R'_1, \partial R'_2$ and $\partial R'_3$. The edges $v_1v_2$, $v_3v_4$ and $v_4v_5$ will become doubly good edges.
An Example of Theorem A

Figure 20: (2, 1)-nonalternating diagram: 13n2004
Examples of Theorem A (knots with arc index 12)

13n563  13n572  13n651  13n652  13n689

13n690  13n789  13n790  13n820  13n926
13n2778  13n2783  13n2786  13n2791  13n2797
13n2800  13n2803  13n2806  13n2807  13n2809
Examples related to Theorem A (knots with arc index 12)

\begin{itemize}
\item 13n2204
\item 13n3051
\item 13n3070
\item 13n3401
\item 13n3680
\item 13n3701
\end{itemize}
An Example Theorem B

Figure 21: 2-nonalternating diagram: 13n2942
Examples of Theorem B  
(knots with arc index 12)

13n1221  
13n1252  
13n1558  
13n1943  
13n2053

13n2174  
13n2433  
13n2473  
13n2942  
13n3180
13n4588  13n4797  13n4800  13n4858  13n4894

13n4895  13n4897  13n5095
Examples related to Theorem B (knots with arc index 12)

13n1824  13n1933  13n3047  13n3475  13n4308

13n4386  13n4467  13n4580
An Example of Theorem C

Figure 22: Almost alternating diagram : 13n0635
Examples of Theorem C (knots with arc index 12)

13n613  13n635  13n649  13n714  13n4031
Knots whose arc index equals crossing number (1)

Figure 23: $\alpha(9n8) = 9$, $\alpha(10n41) = 10$
Knots whose arc index equals crossing number (2)

Figure 24: $\alpha(10n42) = 10$, $\alpha(11n163) = 11$
Knots whose arc index equals crossing number (3)

Figure 25: $\alpha(10n24) = 10$, $\alpha(11n85) = 11$
Knots whose arc index equals crossing number (4)

Figure 26: $\alpha(11n113) = 11, \alpha(11n169) = 11$
Knots whose arc index equals crossing number (5)

Figure 27: $\alpha(11n93) = 11$, $\alpha(11n124) = 11$
Knots whose arc index equals crossing number (6)

Figure 28: $\alpha(11n121) = 11, \alpha(11n127) = 11$
Thank you very much.