Symmetries of Graphs in Homology Spheres

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Let $M$ be an orientable 3-manifold and $G$ be a graph.

**Definition**

An automorphism $\sigma$ of $G$ is said to be *realizable* in $M$ if there is an embedding $\Gamma$ of $G$ into $M$ and a homeomorphism $h : (M, \Gamma) \to (M, \Gamma)$ that induces $\sigma$ on $\Gamma$. In this case, we say that $h$ realizes $\sigma$ in $M$.

**Definition**

We say that $G$ is *intrinsically chiral* in $M$ if for every embedding $\Gamma$ of $G$ in $M$, there is no orientation reversing homeomorphism of $M$ that leaves $\Gamma$ setwise invariant. An embedding $\Gamma$ of $G$ in $M$ is *achiral* if there is an orientation reversing homeomorphism of $M$ that setwise fixes $\Gamma$. 
Motivating questions

How special is $S^3$ among orientable 3-manifolds?

**Question**

Is it true that an automorphism of a graph is realizable in $S^3$ if and only if it is realizable in every orientable 3-manifold?

**Question**

Is it true that a graph is intrinsically chiral in $S^3$ if and only if it is intrinsically chiral in every orientable 3-manifold?
The easy direction

Proposition (Y.)

An automorphism of a graph that is realizable in $S^3$ by an orientation preserving homeomorphism is realizable in every orientable 3-manifold. An automorphism of a graph that is realizable in $S^3$ by an orientation reversing homeomorphism is realizable in every orientable 3-manifold that possesses an orientation reversing homeomorphism.

Corollary (Y.)

If a graph $G$ is intrinsically chiral in an orientable 3-manifold $M$ that possesses an orientation reversing homeomorphism, then $G$ is also intrinsically chiral in $S^3$. 
The converse

The converses are not true!

**Proposition (Y.)**

For any automorphism $\sigma$ of a graph $G$, there exists an orientable 3-manifold $M$, an embedding $\Gamma$ of $G$ in $M$, and an orientation preserving homeomorphism $h$ of $(M, \Gamma)$ such that $h$ realizes $\sigma$.

**Theorem (Flapan-Howards, 2015)**

For any graph $G$, there are infinitely many orientable and irreducible 3-manifolds $M$ such that some embedding of $G$ is pointwise fixed by an orientation reversing involution of $M$.

No graph is intrinsically chiral in every 3-manifold.
Central objects: homology spheres

Need to restrict our attention
Natural generalization of $S^3$:

**Homology sphere**

An integral homology 3-sphere (abbreviated as a homology sphere) is a 3-manifold whose homology groups with $\mathbb{Z}$ coefficients are the same as those of $S^3$.

There are homology spheres which have no orientation reversing homeomorphisms, such as Poincaré’s dodecahedron space.
Rigidity of symmetries in homology spheres

Rigidity Theorem (Flapan, 1995)

Let $G$ be a 3-connected graph. Suppose $\sigma$ is an automorphism of $G$ that is realized in $S^3$ by a homeomorphism $h$. Then $\sigma$ is realizable in $S^3$ by a homeomorphism $f$ of finite order. Moreover, $f$ can be chosen such that $f$ is orientation reversing if and only if $h$ is orientation reversing.

Rigidity Theorem (Y.)

Let $G$ be a 3-connected graph and $M$ be a homology sphere. Suppose $\sigma$ is an automorphism of $G$ that is realized in $M$ by a homeomorphism $h$. Then $\sigma$ is realizable in a homology sphere $M'$ by a homeomorphism $f$ of finite order. Moreover, $f$ can be chosen such that $f$ is orientation reversing if and only if $h$ is orientation reversing.
Smith Theory

\text{fix}(h) - the fixed point set of a map } h

\textbf{Theorem (Smith, 1939)}

Let \( M \) be a homology sphere and \( h : M \to M \) be a homeomorphism of finite order. If \( h \) is orientation preserving, then \( \text{fix}(h) \) is homeomorphic to one of the following: \( M \) (in this case \( h \) is the identity map), \( S^1, \emptyset \); if \( h \) is orientation reversing, then \( \text{fix}(h) \) is homeomorphic to one of the following: \( S^2, S^0 \) (two points).
Realizable automorphisms of complete graphs

Proposition (Y.)

If an automorphism $\sigma$ of the complete graph $K_n$ is realizable in a homology sphere $M$ by a homeomorphism $h$ if and only if $\sigma$ is also realizable in $S^3$ by a homeomorphism $g$. Moreover, $g$ can be chosen so that $g$ is orientation reversing if and only if $h$ is orientation reversing.
Recall that a graph $G$ is intrinsic chiral in an orientable 3-manifold $M$ if no embedding of $G$ is setwise fixed by an orientation reversing homeomorphism of $M$.

**Observation**

Let $M$ be an orientable 3-manifold that does not possess an orientation reversing homeomorphism. Then every graph is intrinsically chiral in $M$. 
Chirality and planarity

Observation (Y.)

In an orientable 3-manifold that possesses an orientation reversing homeomorphism, planar graphs are achiral.

Proposition (Y.)

Any non-planar graph that has no order two automorphism is intrinsically chiral in any homology sphere.

With a slightly stronger requirement:

Proposition (Y.)

Let $P$ be a connected simplicial complex embedded in a homology sphere $M$. If there is an orientation reversing homeomorphism $h$ of $M$ such that $P \subseteq \text{fix}(h)$, then $P$ can be embedded into $\mathbb{S}^2$.

First established for $\mathbb{S}^3$ by Jiang and Wang in 2000
Intrinsic chirality of 3-connected graphs

Flapan-Weaver, 1996: Intrinsic chirality is related to not possessing certain types of automorphisms

**Proposition (Y.)**

An automorphism of a 3-connected graph is realizable in $S^3$ by an orientation reversing homeomorphism if and only if it is realizable in every homology sphere that possesses an orientation reversing homeomorphism by an orientation reversing homeomorphism.

**Example**

Let $M$ be a homology sphere that possesses an orientation reversing homeomorphism. Then the complete graph $K_n$ is intrinsically chiral in $M$ if and only if $n \equiv 3 \mod 4$ and $n \neq 3$.

First established for $S^3$ by Flapan and Weaver in 1992
Intrinsic chirality of Möbius ladders

The Möbius ladder $M_n$ consists of a loop of $2n$ vertices and $n$ rungs connecting antipodal vertices on the loop.

Example

Let $M$ be a homology sphere that possesses an orientation reversing homeomorphism. Then $M_n$ is intrinsically chiral in $M$ if and only if $n$ is odd and $n > 3$. Moreover, no orientation reversing homeomorphism of $M$ can setwise fix an embedded $M_3$ and its loop.

First established for $S^3$ by Flapan in 1989
Petersen graphs

\[
\begin{align*}
K_6 & \xrightarrow{\Delta Y} G_7 & \xrightarrow{\Delta Y} K^-_{4,4} \\
K_{3,3,1} & \xrightarrow{\Delta Y} G_8 & \xrightarrow{\Delta Y} G_9 & \xrightarrow{\Delta Y} PG
\end{align*}
\]
Linking of Petersen graphs

The only graphs that are intrinsically linked and minor minimal with respect to this property

\[ \omega \] - modulo 2 sum of linking numbers of all disjoint pairs of loops of an embedded graph in a homology sphere

**Theorem (Sachs, 1984; Flapan-Howards-Lawrence-Mellor, 2006; Nikkuni-Taniyama, 2012)**

Let \( \Gamma \) be an embedding of a Petersen graph in \( \mathbb{S}^3 \). Then \( \omega(\Gamma) = 1 \).

**Proposition (Y.)**

Let \( \Gamma \) be an embedding of a graph in the Petersen family in a homology sphere \( M \). Then \( \omega(\Gamma) = 1 \).
Unrealizable automorphisms of $K_6$, $G_8$ and $PG$

**Proposition (Y.)**

The automorphism (1234) of $K_6$ is not realizable in any homology sphere.

First established for $S^3$ by Flapan in 1989

Orbits of pairs of loops under the action of (1234):

\[
\{156, 256, 356, 456\}, \{125, 235, 345, 145\}, \{135, 245\}
\]

Same method applies to find unrealizable automorphisms of $G_8$ and $PG$
Unrealizable automorphism of $G_7$, $K_{4,4}$ and $K_{3,3,1}$

\[\begin{align*}
1 & \rightarrow 4 \\
2 & \rightarrow 5 \\
3 & \rightarrow 6 \\
\end{align*}\]

$G_7$

\[\begin{align*}
1 & \rightarrow 4 \\
2 & \rightarrow 5 \\
3 & \rightarrow 6 \\
8 & \rightarrow 7 \\
\end{align*}\]

$K_{4,4}$

\[\begin{align*}
1 & \rightarrow 2 \\
2 & \rightarrow 5 \\
4 & \rightarrow 3 \\
5 & \rightarrow 6 \\
7 & \rightarrow 3 \\
\end{align*}\]

$K_{3,3,1}$

**Proposition (Y.)**

With the labeling above, the automorphism (123) of each of $G_7$, $K_{4,4}$ and $K_{3,3,1}$ is not realizable in any homology sphere.

Any homeomorphism $h$ realizing (123) must fix edges $4\overrightarrow{7}, 5\overrightarrow{7}, 6\overrightarrow{7} \Rightarrow \text{fix}(h)$ is $\mathbb{S}^2 \Rightarrow h^2$ is the identity map
Proposition (Y.)

Every automorphism of $G_9$ is realizable in $S^3$. 
ありがとうございます！
Thank you!