Identifiable projections of spatial graphs

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Abstract A generic map from a finite graph to the 2-space is called *identifiable* if any two embeddings of the graph into the 3-space obtained by lifting the map with respect to the natural projection from the 3-space to the 2-space are ambient isotopic in the 3-space. We show that only planar graphs have identifiable maps. We characterize the identifiable maps for some planar graphs.

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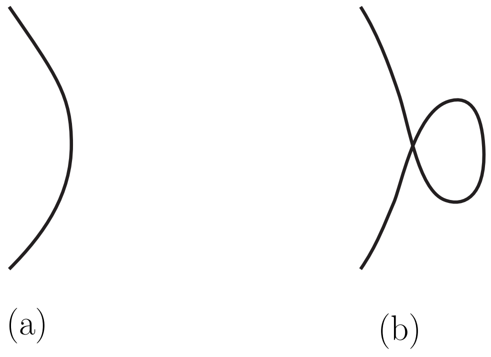
§1. Introduction

Throughout this paper we work in the piecewise linear category. Let G be a graph consisting of finitely many vertices and edges. We consider G as a topological space in the usual way. A continuous map $\varphi : G \to S^2$ from G to the unit 2-sphere S^2 is called a *regular projection* if the multiple points of φ are finitely many transversal double points away from the vertices of G. Let S^3 be the unit 3-sphere in the 4-space centered at the origin and $\pi : S^3 \setminus \{(0,0,0,1),(0,0,0,-1)\} \to S^2$ the natural projection. Let $f : G \to S^3$ be an embedding. We say that *φ is a regular projection of f* if there is an embedding $f' : G \to S^3$ ambient isotopic to f such that $f'(G) \subset S^3 \setminus \{(0,0,0,1),(0,0,0,-1)\}$ and $\varphi = \pi \circ f'$. Then we also say that *f projects on φ*. We say that a regular projection φ is *identifiable* if any two embeddings of G to S^3 each of which projects on φ are ambient isotopic.

Let C be a graph homeomorphic to a circle. It is shown in [1] [7] [9] that the identifiable projections of C are exactly the projections obtained from an embedding of C to S^2 by a finite number of local replacement from Fig 1.1 (a) to Fig. 1.1 (b). As an example we illustrate the image of an identifiable projection of C in Fig. 1.2.



(a) (b)

Fig. 1.1

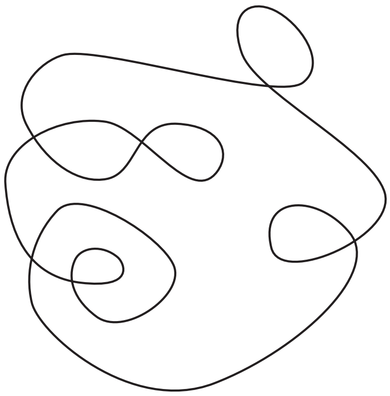
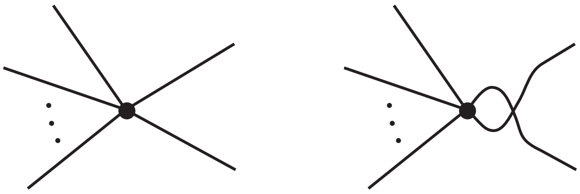


Fig. 1.2

It is actually shown in [7] [9] that every non-identifiable projection of C is a regular projection of both an embedding whose image is a trivial knot and an embedding whose image is a trefoil knot.

We say that a regular projection $\varphi : G \to S^2$ is *reduced* if the image $\varphi(G)$ has no local parts as illustrated in Fig. 1.1 (b) and Fig. 1.3 (b).



(a) (b)

Fig. 1.3

Suppose that $\varphi : G \to S^2$ and $\psi : G \to S^2$ differs just as Fig. 1.1 (a) and (b), or just as Fig. 1.3 (a) and (b). Then it is clear that $f : G \to S^3$ projects on φ if and only if f projects on ψ. Therefore φ is identifiable if and only if ψ is identifiable. Thus the characterization problem of the identifiable projections of G boils down to the characterization of the reduced identifiable projections of G. The result stated above is rephrased that only embeddings from C to S^2 are the reduced identifiable projections of C. It is also shown that only embeddings from G to S^2 are the reduced identifiable projections of G when

G is a θ-curve [4], when G is a θ_n-curve [2] and when G is homeomorphic to two circles with one point in common [12].

Before stating our results we prepare some terminology. For a set X we denote the cardinality of X by $|X|$. For a graph G we denote the set of the vertices of G by $V(G)$ and the set of the edges of G by $E(G)$. A graph G is *n-connected* if $|V(G)| \geq n+1$ and for any subset W of $V(G)$ with $|W| \leq n-1$ the graph $G-W$ is connected. Here $G-W$ means the maximal subgraph of G with $V(G-W) = V(G) - W$. A *cycle* of G is a subgraph of G that is homeomorphic to a circle. A graph G is *planar* if it is embeddable in S^2. A graph is *simple* if it has no loops and multiple edges.

In this paper we show the following results.

Proposition 1.1. *Only planar graphs have identifiable projections.*

Theorem 1.2. *Let G be a simple 2-connected planar graph. Suppose that G satisfies the following two conditions:*

(1) *if e_1, e_2 and e_3 are edges of G such that $e_1 \cup e_2 \cup e_3$ is homeomorphic to a closed interval then there is a cycle of G that contains all of them,*

(2) *if e_1 and e_2 are disjoint edges of G then there are disjoint cycles of G containing them respectively.*

Then only embeddings from G to S^2 are the reduced identifiable projections of G.

We will show in Proposition 2.1 that a 3-connected graph satisfies the condition (1) of Theorem 1.2 and a 4-connected planar graph satisfies the condition (2) of Theorem 1.2. By the definition a 4-connected graph is 3-connected. Therefore we have the following corollary.

Corollary 1.3. *Let G be a simple 4-connected planar graph. Then only embeddings from G to S^2 are the reduced identifiable projections of G.*

Remark 1.4. There are reduced identifiable projections that are not embeddings. We show two such examples in Fig. 1.4.

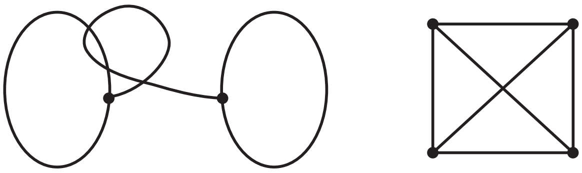


Fig. 1.4

§2. Proofs

Proof of Proposition 1.1. Let G be a non-planar graph and $\varphi : G \to S^2$ a regular projection. Let $f : G \to S^3$ be an embedding that projects on φ. Let $h : S^3 \to S^3$ be the reflection defined by $h(x,y,z,w) = (x,y,z,-w)$. Set $g = h \circ f$. Then it is clear that g also projects on φ. We will show that f and g are not ambient isotopic. By the Kuratowski graph planarity criterion [5] there is a subgraph H of G homeomorphic to the complete graph K_5 or the complete bipartite graph $K_{3,3}$. Let $\mathcal{L}(f|_H)$ and $\mathcal{L}(g|_H)$ be the Simon invariants [11] of the restriction maps $f|_H$ and $g|_H$ respectively. Then by the definition of the Simon invariant we have that $\mathcal{L}(f|_H) = -\mathcal{L}(g|_H)$. It is also shown in [11] that the value of the Simon invariant is always an odd number. In particular it is non-zero. Therefore we have $\mathcal{L}(f|_H)$ and $\mathcal{L}(g|_H)$ are not equal. Therefore $f|_H$ and $g|_H$ are not ambient isotopic. Therefore f and g are not ambient isotopic. $\square$

Proof of Theorem 1.2. Though the theme of this paper is somewhat different from that in [2] or [3], the proof here is similar to that in [2] or [3].

Let $\varphi : G \to S^2$ be a reduced identifiable projection. First suppose that there is a double point of φ such that the preimage of it is contained in two disjoint edges, say e_1 and e_2, of G. Then by the condition (2) there are disjoint cycles γ_1 and γ_2 of G with $e_1 \subset \gamma_1$ and $e_2 \subset \gamma_2$. Then we have that $\varphi(\gamma_1 \cup \gamma_2)$ is a regular projection of both a trivial link and a Hopf link [10]. Therefore $\varphi|_{\gamma_1 \cup \gamma_2}$ is not an identifiable projection. Therefore φ itself is not an identifiable projection.

Next suppose that φ has a double point whose preimage is contained in an edge, say e, of G. Then there is a sub-arc I of e such that $\varphi(I)$ is a simple closed curve on S^2. Since φ is reduced $\varphi(I)$ contains another double point of φ other than $\varphi(\partial I)$. Since G is 2-connected there is a cycle γ of G such that the double point is a double point of $\varphi|_\gamma$. Then by the result in [7] or [9] we have that the restriction map $\varphi|_\gamma$ is a regular projection of both a trivial knot and a trefoil knot. Therefore $\varphi|_\gamma$ is not identifiable. Therefore φ itself is not identifiable.

Thus we have that for each double point of φ the preimage of it is contained in two adjacent edges of G.

Let P_1 be a double point of φ. Let e_1 and e_2 be the adjacent edges that contain the preimage $\varphi^{-1}(P_1)$. Let v be a vertex incident to both e_1 and e_2. Let $e_1, e_2, \ldots, e_n$ be the edges incident to v. Let $K = e_1 \cup e_2 \cup \cdots \cup e_n$ and $P_1, P_2, \ldots, P_m$ the double points of $\varphi|_K$. For each $i \in \{1, 2, \ldots, n\}$ let $p_{i,1}, p_{i,2}, \ldots, p_{i,\alpha(i)}$ be the points of $\varphi^{-1}(\{P_1, P_2, \ldots, P_m\})$ on e_i lying in this order from v to the other vertex incident to e_i. Let $\tau(i,j)$ and $\mu(i,j)$ be the functions characterized by $\{p_{i,j}, p_{\tau(i,j),\mu(i,j)}\} = \varphi^{-1}(\varphi(p_{i,j}))$. Let $I_{i,j}$ be a sub-arc of e_i with $\partial(I_{i,j}) = \{v, p_{i,j}\}$. Suppose that $\varphi(I_{i,\alpha(i)})$ contains a double point Q of φ other than $P_1, P_2, \ldots, P_m$. Let e be another edge with $\varphi(e) \ni Q$. By the condition (1) there is a cycle γ containing $e_{\tau(i,\alpha(i))} \cup e_i \cup e$. Then we have as before that $\varphi|_\gamma$ is not identifiable. Therefore we have that for each i $\varphi(I_{i,\alpha(i)})$ contains no double points of φ other than $P_1, P_2, \ldots, P_m$.

Since φ is reduced we have that $\mu(i,1) > 1$ for each $i \in \{1, 2, \ldots, n\}$. By renaming the edges we may suppose without loss of generality that $\tau(1,1) = 2$. If $\tau(1,i) = 2$ and $\mu(1,i) < \mu(1,1)$ for some i then we take the smallest such i and we stop here.

If not then we consider the point $p_{\tau(2,1),\mu(2,1)}$. By renaming the edges we may suppose that $\tau(2,1) = 3$. If $\tau(1,i) = 3$ and $\mu(1,i) < \mu(2,1)$ for some i then we take the smallest such i and we stop here. If $\tau(2,i) = 3$ and $\mu(2,i) < \mu(2,1)$ for some i then we take the smallest such i, forget e_1, rename e_2, e_3 to e_1, e_2, and we stop here.

If not then we consider the point $p_{\tau(3,1),\mu(3,1)}$. By renaming the edges we may suppose that $\tau(3,1) = 4$. If $\tau(1,i) = 4$ and $\mu(1,i) < \mu(3,1)$ for some i then we take the smallest such i and we stop here. If $\tau(2,i) = 4$ and $\mu(2,i) < \mu(3,1)$ for some i then we take the smallest such i, forget e_1, rename e_2, e_3, e_4 to e_1, e_2, e_3, and we stop here. If $\tau(3,i) = 4$ and $\mu(3,i) < \mu(3,1)$ for some i then we take the smallest such i, forget e_1, e_2, rename e_3, e_4 to e_1, e_2, and we stop here.

Repeating the arguments we finally obtain the following situation. There is a natural number k with $k < n$ such that

(a) $\tau(i,1) = i+1$ and $\tau(i+1,j) > i+1$ for each i, j with $1 \leq i < k$, $j < \mu(i,1)$,

(b) $\tau(k,1) = k+1$, and

(c) for each $j < \mu(k,1)$ $\tau(k+1,j) > k+1$ or $\tau(k+1,j) = 1$, and for some l $\tau(1,l) = k+1$ and $\mu(1,l) < \mu(k,1)$.

We take smallest such l. Now we consider $T = I_{1,l} \cup I_{2,\mu(1,1)} \cup \cdots \cup I_{k+1,\mu(k,1)}$. Let T' be a sufficiently small neighbourhood of T in G. Then we have that $\varphi|_{T'}$ has just $k+1$ double points. Therefore we have that $\varphi(T')$ looks like that illustrated in Fig. 2.1 or the mirror image of it. Since G is 2-connected there is a subgraph H of G such that $E(H)$ contains $e_1, e_2, \ldots, e_{k+1}$ and H contracts to a graph with two vertices and $k+1$ edges joining them coming from $e_1, e_2, \ldots, e_{k+1}$. We will show that $\varphi|_H$ is not identifiable. In fact $\varphi|_H$ is a regular projection of two embeddings f and g of H to S^3 that differs only near a neighbourhood of v as illustrated in Fig. 2.2.

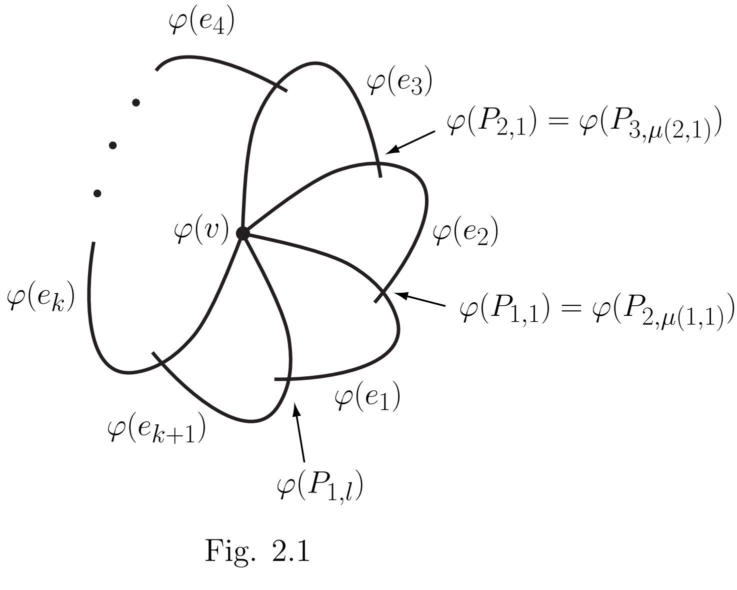


Fig. 2.1

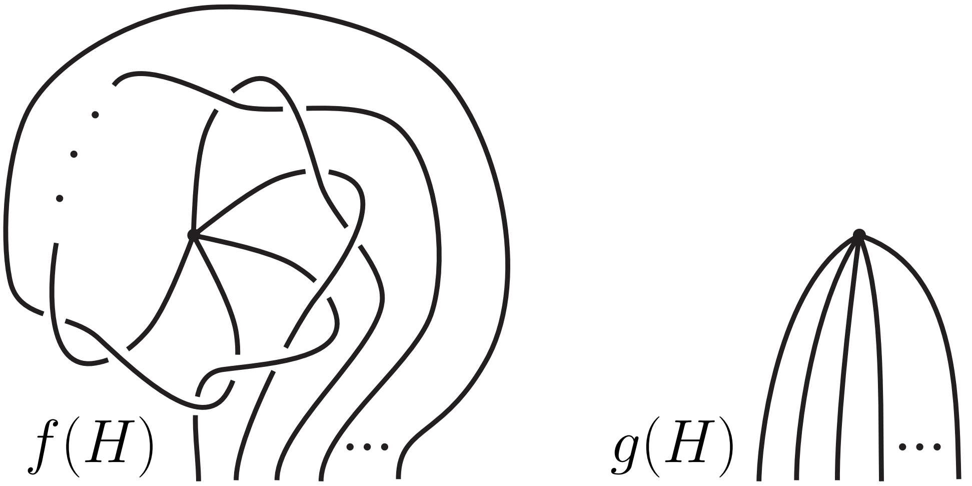


Fig. 2.2

By contracting the edges of $f(H)$ and $g(H)$ other than $f(e_1), f(e_2), \ldots, f(e_{k+1})$ and $g(e_1), g(e_2), \ldots, g(e_{k+1})$ in S^3 we have two graphs embedded in S^3. Each of them is a graph with two vertices and $k+1$ edges joining them. Note that the graph obtained from $f(H)$ is a vertex-connected sum of Suzuki's nontrivial θ_{k+1}-curve [8] and the graph obtained from $g(H)$. By the uniqueness of the prime decomposition of such graphs in S^3 [6] we have that they are not ambient isotopic. Since edge contraction in S^3 is well-defined up to ambient isotopy we have that f and g are not ambient isotopic. Thus we have that $\varphi|_H$ is not identifiable.

Thus we have shown that φ has no double points. $\square$

Proposition 2.1. (1) *A 3-connected graph G satisfies the condition* (1) *of Theorem 1.2.*

(2) *A 4-connected planar graph G satisfies the condition* (2) *of Theorem 1.2.*

Proof of Proposition 2.1 (1). Let e_1, e_2 and e_3 be edges of G such that $e_1 \cup e_2 \cup e_3$ is homeomorphic to a closed interval. Let v_1, v_2, v_3 and v_4 be the vertices on $e_1 \cup e_2 \cup e_3$ lying in this order. Since G is 3-connected $G - \{v_2, v_3\}$ is connected. Hence there is a path W in $G - \{v_2, v_3\}$ joining v_1 and v_4 so that $W \cup e_1 \cup e_2 \cup e_3$ is the desired cycle. $\square$

For the proof of Proposition 2.1 (2) we prepare the followings. Let G be a simple

3-connected graph and $\psi : G \to S^2$ an embedding. Then it is easy to check that the closure of each component of $S^2 - \psi(G)$ is homeomorphic to a 2-disk. A cycle γ of G is called a *region cycle* with respect to ψ if $\psi(\gamma)$ is the boundary of some component of $S^2 - \psi(G)$.

Proposition 2.2. *Let G be a simple 3-connected graph and $\psi : G \to S^2$ an embedding. Let γ_1 and γ_2 be region cycles of G with respect to ψ. Then $\gamma_1 \cap \gamma_2$ is an empty set, a vertex of G or an edge of G.*

Let G be a graph and F a subset of $E(G)$. By $G - F$ we denote the maximal subgraph of G with $V(G - F) = V(G)$ and $E(G - F) = E(G) - F$.

Proposition 2.3. *Let G be an n-connected graph, v a vertex of G and e an edge of G. Then both $G - \{v\}$ and $G - \{e\}$ are $(n-1)$-connected.*

The proofs of Propositions 2.2 and 2.3 are easy and we omit them.

Proof of Proposition 2.1 (2). Let e_1 and e_2 be disjoint edges of G.

First suppose that one of them, say e_1, is a loop. Let v be the vertex incident to e_1. Then $G - \{v\}$ is 3-connected. Therefore $G - \{v\}$ is 2-connected. Therefore there is a cycle γ of $G - \{v\}$ containing e_2. Then e_1 and γ are the desired disjoint cycles.

Next suppose that one of e_1 and e_2, say e_1, is a multiple edge of G. Namely there is an edge e_3 such that $e_1 \cup e_3$ is a cycle. Let u and v be the vertices incident to e_1. Since $G - \{u, v\}$ is 2-connected there is a cycle γ of $G - \{u, v\}$ containing e_2. Then $e_1 \cup e_3$ and γ are the desired disjoint cycles.

Therefore we may suppose that each of e_1 and e_2 is neither a loop nor a multiple edge. Let G' be a maximal simple subgraph of G. Then it is clear that G' is still 4-connected. Let $\psi : G' \to S^2$ be an embedding. Let γ_{i1} and γ_{i2} be the region cycles of G' with respect to ψ containing e_i for $i = 1, 2$. Note that $\gamma_{i1} \cap \gamma_{i2} = e_i$ for $i = 1, 2$. Let v_{i1} and v_{i2} be the vertices incident to e_i for $i = 1, 2$.

First suppose that $\gamma_{1i} = \gamma_{2j}$ for some $i, j \in \{1, 2\}$. We may suppose without loss of gerenrality that $\gamma_{12} = \gamma_{21}$. Then we have $\gamma_{11} \neq \gamma_{22}$. Suppose that $\gamma_{11} \cap \gamma_{22}$ is a vertex, say v_3, or an edge incident to vertices, say v_3 and v_4. Then we have that $G' - \{v_{1i}, v_{2j}, v_3\}$ is not connected for some $i, j \in \{1, 2\}$. Then G' is not 4-connected. This is a contradiction. Thus we have that γ_{11} and γ_{22} are disjoint.

Next suppose that $\gamma_{1i} \neq \gamma_{2j}$ for any $i, j \in \{1, 2\}$. Suppose that $\gamma_{1i} \cap \gamma_{2j} \neq \emptyset$ for any $i, j \in \{1, 2\}$. Let γ be the cycle obtained from $\gamma_{11} \cup \gamma_{12}$ by removing the interior of e_1. Since $G' - \{e_1\}$ is a simple 3-connected graph we have that $\gamma \cap \gamma_{2i}$ is a vertex or an edge for $i = 1, 2$. If $\gamma \cap \gamma_{2i}$ is a vertex then it must be v_{11} or v_{12} for $i = 1, 2$. If $\gamma \cap \gamma_{2i}$ is an edge then one of the vertices incident to it must be v_{11} or v_{12} for $i = 1, 2$. Then we have that $(G - \{e_2\}) - \{v_{11}, v_{12}\}$ is not connected. By Proposition 2.3 $G - \{e_2\}$ is 3-connected. This is a contradiction. Thus we have that $\gamma_{1i} \cap \gamma_{2j} = \emptyset$ for some $i, j \in \{1, 2\}$. $\square$

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