On certain $L$-functions for deformations of knot group representations

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Abstract. We study the twisted knot module for the universal deformation of an $SL_2$-representation of a knot group, and introduce an associated $L$-function, which may be seen as an analogue of the algebraic $p$-adic $L$-function associated to the Selmer module for the universal deformation of a Galois representation. We then investigate two problems proposed by Mazur: Firstly, we show the torsion property of the twisted knot module over the universal deformation ring under certain conditions. Secondly, we verify the simplicity of the zeroes of the $L$-function by some concrete examples for 2-bridge knots.

This is joint work with T. Kitayama, M. Morishita and Y. Terashima.

Introduction

Arithmetic Topology is concerned with the relation between number theory and 3-dimensional topology. Here is the dictionary of basic analogies (cf. [Mo]):

<table>
<thead>
<tr>
<th>Number theory</th>
<th>Knot theory</th>
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<tbody>
<tr>
<td>prime ideal $(p)$</td>
<td>knot $K$</td>
</tr>
<tr>
<td>Spec($\mathbb{F}_p$) $= K(\hat{\mathbb{Z}}, 1) \hookrightarrow \text{Spec}(\mathbb{Z}) \cup {\infty}$</td>
<td>$S^1 = K(\mathbb{Z}, 1) \hookrightarrow S^3 = \mathbb{R}^3 \cup {\infty}$</td>
</tr>
<tr>
<td>$p$-adic integers</td>
<td>tubular neighborhood</td>
</tr>
<tr>
<td>Spec($\mathbb{Z}_p$)</td>
<td>$V_K$</td>
</tr>
<tr>
<td>$p$-adic numbers</td>
<td>boundary torus</td>
</tr>
<tr>
<td>Spec($\mathbb{Q}_p$) $= \text{Spec}(\mathbb{Z}_p) \setminus \text{Spec}(\mathbb{F}_p)$</td>
<td>$\partial V_K = V_K \setminus K$</td>
</tr>
<tr>
<td>prime complement</td>
<td>knot complement</td>
</tr>
<tr>
<td>$X_p = \text{Spec}(\mathbb{Z}[1/p])$</td>
<td>$X_K = S^3 \setminus \text{int}(V_K)$</td>
</tr>
<tr>
<td>prime group</td>
<td>knot group</td>
</tr>
<tr>
<td>$G_p = \pi_1^{\text{et}}(X_p)$</td>
<td>$G_K = \pi_1(X_K)$</td>
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Based on the above analogies, there are close parallels between Iwasawa theory and Alexander theory (cf. [Mo; Ch.9-13]):

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Iwasawa theory | Alexander theory
--- | ---
\(G_p^\infty = \text{Gal}(X_p^\infty / \bar{X}_p)\) | \(G_K^\infty = \text{Gal}(X_K^\infty / \bar{X}_K)\)
1-dim. universal representation | 1-dim. universal representation
Iwasawa module | knot module
\(H_1(X_p^\infty, \mathbb{Z}_p) = H_1(X_p, \chi_p)\) | \(H_1(X_K^\infty, \mathbb{Z}) = H_1(X_K, \chi_K)\)
Iwasawa polynomial | Alexander polynomial
(algebraic \(p\)-adic \(L\)-function) | \(\Delta_0(H_1(X_p, \chi_p))\)

Here \(\Delta_0(H_1(X_p, \chi_p))\) (respectively \(\Delta_0(H_1(X_K, \chi_K))\)) means the order ideal of \(H_1(X_p, \chi_p)\) (resp. \(H_1(X_K, \chi_K)\)) over the Iwasawa algebra \(\mathbb{Z}_p[[T]]\) (resp. the Laurent polynomial ring \(\mathbb{Z}[t^{\pm 1}]\)). By using these analogies, many works have been done (cf. [HMM], [KM], [Su], [U1-U3]), and still in progress. In this report, we consider the non-commutative generalization.

The background of this report is the following. Motivated by the work of Hida, Mazur initiated to study the deformation theory for \(p\)-adic Galois representations: for a given representation \(\tilde{\rho} : G_p \rightarrow \text{GL}_2(\mathbb{F}_p)\), Mazur produced the universal deformation \(\rho : G_p \rightarrow \text{GL}_2(\mathbb{R}_{\tilde{\rho}})\), where \(\mathbb{R}_{\tilde{\rho}}\) is a “big” local complete ring (e.g. \(\mathbb{Z}_p[[T]]\)) and proposed a \(\text{GL}_2\)-analogue of Iwasawa polynomial (algebraic \(p\)-adic \(L\)-function) \(\Delta_0(H_1(G_p, \rho))\).

From a viewpoint of Arithmetic Topology, our project is to fill “?” in the following dictionary:

| GL1 | Iwasawa theory | Alexander theory |
| GL2 | Hida-Mazur theory | ? |

In order to achieve this project, we introduce a new invariant (\(L\)-function) in knot theory, which may be a topological analogue of the algebraic \(p\)-adic \(L\)-function associated to Selmer module for the universal Galois deformation in number theory. We consider the twisted knot module for the universal deformation of knot group representation, as an analogue of Selmer module.

<table>
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<th>Number theory</th>
<th>Knot theory</th>
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<td>Selmer module for deformation of Galois representation</td>
<td>twisted knot module for deformation of knot group representation</td>
</tr>
<tr>
<td>(p)-adic (L)-function associated to the Selmer module</td>
<td>(L)-function associated to the twisted knot module</td>
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Also, Mazur proposed a number of problems related in his famous article [Ma1]. In this report, the author will report some results which solve Mazur’s problem. This report is based on [KMTT].
Notation. We denote by $\mathbb{Z}_p$ the $p$-adic integer and $F_p$ the finite field of order $p$. For a local ring $R$, we denote by $m_R$ the maximal ideal of $R$. For an integral domain $A$, we denote by $\text{char}(A)$ the characteristic of $A$ and by $Q(A)$ the field of fractions of $A$. For $a, b$ in a commutative ring $A$, $a \div b$ means $a = bu$ for some unit $u \in A^\times$.

1 Universal deformations

Let $G$ be a finitely generated group, $k$ a field with $\text{char}(k) \neq 2$, and $\mathcal{O}$ a complete DVR with $\mathcal{O}/m_\mathcal{O} = k$. Let $\overline{\rho} : G \to \text{SL}_2(k)$ be a representation.

The pair $(R, \rho)$ is called a deformation of $\overline{\rho}$ when $R$ is a complete local $\mathcal{O}$-algebra with $R/m_R = k$, and $\rho : G \to \text{SL}_2(R)$ is a representation such that $\rho \mod m_R = \overline{\rho}$.

Moreover, the pair $(R_\overline{\rho}, \rho)$ is called a universal deformation of $\overline{\rho}$ if $(R, \rho)$ is a deformation of $\overline{\rho}$, and for all deformation $(R, \rho)$ of $\overline{\rho}$, there exists $\psi : R_\overline{\rho} \to R$ such that $\psi \circ \rho \approx \rho$. Here $\rho_1 \approx \rho_2$ means there exists $U \in M_2(m_R)$ such that $\rho_2(g) = U \rho_1(g) U^{-1}$ for all $g \in G$.

By the universal property, a universal deformation $(R_\overline{\rho}, \rho)$ of $\overline{\rho}$ is unique (if it exists) up to strict equivalence. $R_\overline{\rho}$ is called the universal deformation ring of $\overline{\rho}$. For the existence, we have the following:

**Theorem 1** ([MTTU; Theorem 2.2.2]). If $\overline{\rho} : G \to \text{SL}_2(k)$ is an absolutely irreducible representation, then there exists a universal deformation $(R_\overline{\rho}, \rho)$ of $\overline{\rho}$.

Note that $\overline{\rho} : G \to \text{SL}_2(k)$ is called an absolutely irreducible representation when $G \overset{\overline{\rho}}{\longrightarrow} \text{SL}_2(k) \hookrightarrow \text{SL}_2(\overline{k})$ is irreducible, where $\overline{k}$ is an algebraic closure of $k$.

Even though we have one such general theorem for the existence, it is still a difficult problem to obtain an universal deformation with explicit form. However for the special case, it is known, due to Mazur, that the universal deformation ring can be determined. Let $\text{Ad}(\overline{\rho})$ be the $k$-vector space $\mathfrak{sl}_2(k)$ on which $G$ acts by $g \cdot X := \overline{\rho}(g) X \overline{\rho}(g)^{-1}$.

**Theorem 2** ([Ma2; 1.6, Proposition 2]). If $H^2(G, \text{Ad}(\overline{\rho})) = 0$, then we have $R_\overline{\rho} = \mathcal{O}[[X_1, \ldots, X_d]]$, where $d := \dim_k (R_\overline{\rho}/\mathcal{O})$. 

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When \( H^2(G, \text{Ad}(\mathfrak{p})) = 0 \), we say the deformation problem is \textit{unobstructed}. Unfortunately, we can say the deformation problem is not unobstructed in general for a representation of a knot group. In order to give one such example, we recall the following lemma due to Thurston:

**Lemma 1** ([CS; Proposition 3.2.1]). For an irreducible representation \( \rho : G_K \to \text{SL}_2(\mathbb{C}) \), the irreducible component of the character variety containing \([\rho]\) has the dimension greater than 0.

**Example 1.** Let \( K \) be a hyperbolic knot (e.g. figure-eight knot, knot 5_2, etc.), \( \rho_h : G_K \to \text{SL}_2(\mathbb{C}) \subset \text{SL}_2(\mathbb{C}) \) a holonomy representation, and \( \overline{\rho}_h : G_K \to \text{SL}_2(\mathbb{C}) \) a representation induced by \( \rho_h \), where \( F \) is a number field and \( \mathfrak{p} \) is a prime ideal of ring of integer \( \mathcal{O}_F \). By using Lemma 1, we have 
\[
H^2(G_K, \text{Ad}(\overline{\rho}_h)) \neq 0.
\]

### 2 Character schemes

Let \( G \) be a finitely generated group and \((\mathcal{A}(G), \sigma_G)\) a \textit{universal (tautological) representation} of \( G \), namely let \( \mathcal{A}(G) \) be a commutative ring with identity, and \( \sigma_G : G \to \text{SL}_2(\mathcal{A}(G)) \) a representation satisfying the following universal property: for any commutative ring \( A \) with identity and for any representation \( \rho : G \to \text{SL}_2(A) \), there exists a unique \( \psi : \mathcal{A}(G) \to A \) such that \( \psi \circ \sigma_G = \rho \).

\[
\begin{array}{ccc}
\text{SL}_2(\mathcal{A}(G)) & \xrightarrow{\sigma_G} & \mathcal{A}(G) \\
\downarrow \psi & & \downarrow \psi \\
G & \xrightarrow{\rho} & \text{SL}_2(A)
\end{array}
\]

When the group presentation of \( G \) has a form \( \langle g_1, \ldots, g_n \mid r_l \ (l \in L) \rangle \), then its universal representation is given explicitly by the following:

\[
\mathcal{A}(G) = \left\langle \mathbb{Z}[X_{ij}(g_1), \ldots, X_{ij}(g_n) \ (1 \leq i, j \leq 2)] \middle| r_l(X(g_1), \ldots, X(g_n))_{ij}, \det(X(g_h)) - 1 \ (1 \leq h \leq n) \right\rangle,
\]

\[
\sigma_G(g_h) = X(g_h),
\]

where \( X(g_h) = (X_{ij}(g_h))_{1 \leq i, j \leq 2} \) is a 2 \times 2 matrix and \( r_l(X(g_1), \ldots, X(g_n))_{ij} \) denotes the \((i,j)\)-entry of \( r_l(X(g_1), \ldots, X(g_n)) \).

We define the \textit{representation scheme} \( \mathcal{R}(G) \) of \( G \) over \( \mathbb{Z} \) as follows:

\[
\mathcal{R}(G) := \text{Spec}(\mathcal{A}(G)).
\]

Let \( \text{PGL}_2 \) be the group scheme over \( \mathbb{Z} \) whose coordinate ring \( A(\text{PGL}_2) \) is the subring of \( \mathbb{Z}[Y_{ij} \ (1 \leq i, j \leq 2)]_{\det(Y)} \) consisting of homogeneous elements of
degree 0. Then the adjoint action $\text{Ad} : \mathcal{R}(G) \times \text{PGL}_2 \to \mathcal{R}(G)$ is given by the dual action

$$\text{Ad}^* : \mathcal{A}(G) \to \mathcal{A}(G) \otimes_\mathbb{Z} \mathcal{A}(\text{PGL}_2); \quad X_i(j)(g) \mapsto (Y X(g) Y^{-1})_{ij}.$$ 

We define the character algebra $\mathcal{B}(G)$ of $G$ over $\mathbb{Z}$ by

$$\mathcal{B}(G) := \mathcal{A}(G)^{\text{PGL}_2} := \{x \in \mathcal{A}(G) \mid \text{Ad}^*(x) = x \otimes 1\},$$

and the character scheme $\mathcal{X}(G)$ of $G$ over $\mathbb{Z}$ by

$$\mathcal{X}(G) := \text{Spec}(\mathcal{B}(G)) = \mathcal{R}(G)}/\text{PGL}_2.$$

**Example 2.** Let $K$ be a 2-bridge knot $B(m,n)$. Then the knot group $G_K$ have a presentation of the form

$$G_K = \langle g_1, g_2 \mid wg_1 = g_2w \rangle,$$

where $w := g_1^{m_1} g_2^{m_2} \cdots g_1^{m_{m-2}} g_2^{m_{m-1}}$ and $\epsilon_i := (-1)^{\frac{m_i}{m}} \epsilon_{m-i}$. Let $x := \text{tr}(\sigma_{G_K}(g_1))$ ($= \text{tr}(\sigma_{G_K}(g_2))$) and $y := \text{tr}(\sigma_{G_K}(g_1 g_2))$. Note that $x$ and $y$ are the elements in $\mathcal{B}(G_K)$. For the character scheme, we have the following Proposition due to Le:

**Proposition 1** ([L; Theorem 3.3.1]). Let $k$ be a field with $\text{char}(k) \neq 2$. Then $\mathcal{X}(G_K)_k := \mathcal{X}(G_K) \otimes_\mathbb{Z} k$ is given by the algebraic curve

$$(y - x^2 + 2)\Phi_K(x, y - x^2 + 2) = 0$$

for some polynomial $\Phi_K(x, y - x^2 + 2) \in k[x, y]$.

The equation $y - x^2 + 2 = 0$ is a locus of reducible representation whereas the equation $\Phi_K(x, y - x^2 + 2) = 0$ is a locus of absolutely irreducible representation.

See [KMTT; Section 4] for the definition of the polynomial $\Phi_K(x, y - x^2 + 2)$.

When $K := B(3,1)$, the trefoil knot, we have

$$\Phi_K(x, y - x^2 + 2) = y - 1,$$

and when $K := B(7,3)$, the knot $5_2$, we have

$$\Phi_K(x, y - x^2 + 2) = y^3 - (x^2 + 1)y^2 + (3x^2 - 2)y - 2x^2 + 1.$$ 

Next, we consider the relation between character algebras and universal deformation rings. Let $k$ be a field with $\text{char}(k) \neq 2$, $\mathcal{O}$ a complete DVR with $\mathcal{O}/\mathfrak{m}_\mathcal{O} = k$. Then we define $\mathcal{B}(G)_k$ and $\mathcal{X}(G)_k$ as follows:

$$\mathcal{B}(G)_k := \mathcal{B}(G) \otimes_\mathbb{Z} k, \quad \mathcal{X}(G)_k = \text{Spec}(\mathcal{B}(G)_k).$$

Let $\overline{\rho} : G \to \text{SL}_2(k)$ be an absolutely irreducible representation, $\mathcal{R}_\overline{\rho}$ a universal deformation ring of $\overline{\rho}$. Denote $[\overline{\rho}]$ as a maximal ideal of $\mathcal{B}(G)_k$ corresponding to $\overline{\rho}$ and let $(\mathcal{B}(G)_k)_{[\overline{\rho}]}$ be a $[\overline{\rho}]$-adic completion of $\mathcal{B}(G)_k$.

The relation between character algebras and universal deformation rings can be claimed as follows:
Theorem 3 ([KMTT; Theorem 2.2.1]). We have an isomorphism of $k$-algebras
\[ R_{\overline{\rho}} \otimes_{\mathcal{O}} k \cong (B(G)_k)_{[\overline{\rho}]}^1. \]

Namely, we can regard $R_{\overline{\rho}}$ as an infinitesimal deformation of $B(G)_k$. For the proof, we use the relations between skein algebras. The relation between skein algebras and character algebras has been studied by K. Saito (cf. [Sa]).

Using Theorem 3, we have the following useful theorem to determine $R_{\overline{\rho}}$:

Theorem 4 ([KMTT; Theorem 2.2.4]). Suppose

(1) $[\overline{\rho}]$ is a regular point of $X(G)_k$,
(2) We can take $g \in G$ such that $\text{tr}(\sigma_G(g)) - \text{tr}(\overline{\rho}(g))$ is a local parameter at $[\overline{\rho}]$,
(3) There is $\alpha \in \mathcal{O}$ such that $\alpha \mod m_{\mathcal{O}} = \text{tr}(\overline{\rho}(g))$,
(4) There is a deformation $(\mathcal{O}[[x - \alpha]], \rho)$: of $\overline{\rho}$ such that $\text{tr}(\rho(g)) = x$.

Then $(\mathcal{O}[[x - \alpha]], \rho)$ is the universal deformation of $\overline{\rho}$.

3 Twisted knot modules

Let $K \subset S^3$ be a knot. Denote $X_K := S^3 \setminus \text{int}(V_K)$ as a knot complement and $G_K := \pi_1(X_K)$ as a knot group. Let $k$ be a field with $\text{char}(k) \neq 2$, and $\overline{\rho} : G_K \to \text{SL}_2(k)$ an absolutely irreducible representation. As before, let $(R_{\overline{\rho}}, \rho)$ a universal deformation of $\overline{\rho}$, and $V_\rho := (R_{\overline{\rho}})^{\oplus 2}$ a representation space of $\rho$.

We define the twisted knot module $H_1(\rho)$ for the universal deformation $(R_{\overline{\rho}}, \rho)$ by
\[ H_1(\rho) := H_1(X_K; V_\rho). \]

Mazur proposed a following problem about $H_1(\rho)$:

Problem 1 (cf. [Ma1]). What is the structure of $H_1(\rho)$ as an $R_{\overline{\rho}}$-module? In particular, is $H_1(\rho)$ a finitely generated torsion $R_{\overline{\rho}}$-module?

In order to solve this problem, we consider a criterion for $H_1(\rho)$ to be a finitely generated torsion $R_{\overline{\rho}}$-module by using twisted Alexander invariants $\Delta_K(\rho; t)$. The following Theorem is the criterion:

Theorem 5 ([KMTT; Theorem 3.2.4]). Suppose that the following conditions are satisfied:

(1) $R_{\overline{\rho}}$ is a Noetherian integral domain.
(2) There is a deformation $(R, \rho)$ of $\overline{\rho}$, and there exists $g \in G_K$ such that
(2-1) $\det(\rho(g) - I) \neq 0$,
(2-2) $\Delta_K(\rho; 1) \neq 0$.

Then $H_1(\rho)$ is a finitely generated torsion $R_{\overline{\rho}}$-module.

As a special case, we have the following:

Corollary ([KMTT; Corollary 3.2.5]). If we can take $\overline{\rho}$ for $\rho$ in the condition (2), then $H_1(\rho) = 0$. 
4 \textit{L}-functions

Assume that the conditions on Theorem satisfy, namely assume $R_\pi$ is a Noetherian UFD, and $H_1(\rho)$ is a finitely generated torsion $R_\pi$-module.

We define the \textit{L}-function $L_K(\rho)$ of $K$ associated to $\rho$ by

$$L_K(\rho) := \Delta_0(H_1(\rho)).$$

Note that $L_K(\rho)$ is seen as a section of the coherent sheaf $H_1(\rho)$ associated to $H_1(\rho)$ on the universal deformation space Spec($R_\pi$).

The $L$-function $L_K(\rho)$ is a computable invariant. Regard the boundary map $\partial_2$ of the chain complex as following (big) $(n - 1) \times n$ matrix:

$$\partial_2 := (\rho \circ \pi \frac{\partial r_j}{\partial g_i}) : (V_\rho)^{\oplus (n-1)} \to (V_\rho)^{\oplus n},$$

where $\frac{\partial}{\partial g_i}$ is the Fox derivative over $R_\pi$, extended from $\mathbb{Z}$.

\textbf{Proposition 2 (}[KMTT; Proposition 3.2.7]).

$$L_K(\rho) = \Delta_2(\text{Coker}(\partial_2)),$$

where $\Delta_2(\text{Coker}(\partial_2))$ is the greatest common divisor of generators of the 2nd elementary ideal $E_2(\text{Coker}(\partial_2))$.

Following Mazur’s problem in [Ma1], we ask the following:

\textbf{Problem 2 (cf. [Ma1]). Is the order of $L_K(\rho)$ on Spec($R_\pi$) at any prime divisor zero or one?}

We verify the problem affirmatively by some concrete examples in the next section.

5 Examples

\textbf{Example 3.} Let $K := B(3, 1)$, the trefoil knot, whose group is given by

$$G_K = \langle g_1, g_2 \mid g_1g_2g_1 = g_2g_1g_2 \rangle.$$

Let $k = \mathbb{F}_3$ and $\mathcal{O} = \mathbb{Z}_3$, and consider the following absolutely irreducible representation:

$$\varphi_1 : G_K \to \text{SL}_2(\mathbb{F}_3); \quad \varphi_1(g_1) = \begin{pmatrix} 0 & 2 \\ 1 & 2 \end{pmatrix}, \quad \varphi_1(g_2) = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}.$$

Let $\rho_1 : G_K \to \text{SL}_2(\mathbb{Z}_3[[x - 2]])$ be the representation defined by

$$\rho_1(g_1) = \begin{pmatrix} \frac{x + \sqrt{x^2 - 3}}{2} & -1 \\ \frac{1}{3} & \frac{x - \sqrt{x^2 - 3}}{2} \end{pmatrix},$$

$$\rho_1(g_2) = \begin{pmatrix} \frac{x - \sqrt{x^2 - 3}}{2} & -1 \\ \frac{1}{3} & \frac{x + \sqrt{x^2 - 3}}{2} \end{pmatrix}.$$
We see by the straightforward computation that $\rho_1$ is indeed a representation of $G_K$ and a deformation of $\bar{\rho}_1$. Moreover, we have $\text{tr}(\rho_1(g_1)) = x$, hence $\rho_1$ satisfies the conditions of Theorem 4. Therefore $(R_{\bar{\rho}_1} = \mathbb{Z}_3[[x-2]], \rho_1)$ is the universal deformation of $\bar{\rho}_1$.

We have the following necessary condition for the $L$-function $L_K(\rho)$ to be non-trivial under a mild condition. The proof uses the functorial property of the twisted Alexander invariants.

**Lemma 2** ([KMTT; Proposition 3.2.9]). Assume that $\Delta_0(H_0(\rho)) = 1$. If $L_K(\rho) \neq 1$, we have $\Delta_K(\bar{\rho}; 1) = 0$.

In the present case, we easily see that $\Delta_0(H_0(\rho_1)) = 1$ and $\Delta_K(\bar{\rho}_1; t) = 1 + t^2$, hence, $\Delta_K(\bar{\rho}_1; 1) = 2 \neq 0$. Therefore, by Lemma 2, we have $H_1(\rho_1) = 0$, $L_K(\rho_1) \neq 1$.

**Example 4.** Let $K := B(7,3)$, the knot $5_2$, whose group is given by $G_K = \langle g_1, g_2 \mid g_1g_2g_1^{-1}g_2^{-1}g_1g_2g_1 = g_2g_1g_2g_1^{-1}g_2^{-1}g_1g_2 \rangle$.

Let $k = \mathbb{F}_19$ and $O = \mathbb{Z}_{19}$, and consider the following absolutely irreducible representation:

$\rho_2 : G_K \to \text{SL}_2(\mathbb{F}_{19}); \rho_2(g_1) = \begin{pmatrix} 14 & 1 \\ 1 & 11 \end{pmatrix}, \rho_2(g_2) = \begin{pmatrix} 11 & 1 \\ 1 & 14 \end{pmatrix}$.

Let $\beta := \frac{3 + \sqrt{5}}{2}$ so that $\beta \mod 19 = 6$. Let $y = y(x)$ be the unique solution in $\mathbb{Z}_{19}[[x - \beta]]$ satisfying the equation

$y^3 - (x^2 + 1)y^2 + (3x^2 - 2)y - 2x^2 + 1 = 0$ (★)

and

$y(\beta) = \beta$. (♦)

Such a $y(x)$ is proved, by Hensel’s lemma ([Se; II, §4, Proposition 7]), to exist uniquely. Now, let $\rho_2 : G_K \to \text{SL}_2(\mathbb{Z}_{19}[[x - \beta]])$ be the representation defined by

$\rho_2(g_1) = \begin{pmatrix} \frac{x + \sqrt{x^2 - y(x)^2}}{4} & 1 \\ \frac{y(x)^2 - 2}{4} & \frac{x - \sqrt{x^2 - y(x)^2}}{2} \end{pmatrix},$

$\rho_2(g_2) = \begin{pmatrix} \frac{x - \sqrt{x^2 - y(x)^2}}{4} & 1 \\ \frac{y(x)^2 - 2}{4} & \frac{x + \sqrt{x^2 - y(x)^2}}{2} \end{pmatrix}$.

We can verify by (★) that $\rho_2$ is indeed a representation of $G_K$; and by (♦) that $\rho_2$ is a deformation of $\bar{\rho}_2$. Moreover, we have $\text{tr}(\rho_2(g_1)) = x$, hence $\rho_2$ satisfies the conditions of Theorem 4. Therefore $(R_{\bar{\rho}_2} = \mathbb{Z}_{19}[[x - \beta]], \rho_2)$ is the universal deformation of $\bar{\rho}_2$. 
Consider the 19-adic lifting $\rho_2 : G_K \to \SL_2(\mathbb{Z}_{19})$ of $\mathfrak{p}_2$ defined by $\rho_2|_{x=6}$:

$$\rho_2(g_1) = \left(\frac{6 + \sqrt{34 - \gamma}}{2}, \frac{1}{6 - \sqrt{34 + \gamma}}\right),$$

$$\rho_2(g_2) = \left(\frac{6 - \sqrt{34 - \gamma}}{2}, \frac{1}{6 + \sqrt{34 + \gamma}}\right),$$

where $\gamma$ is the unique solution in $\mathbb{Z}_{19}$ satisfying $(\star)$ with $x = 6$ and $\gamma \mod 19 = 6$. Then we easily see that $\det(\rho_2(g_2) - I) = -4 \neq 0$, and that $\Delta_K(\rho_2; t) = \gamma^2 - 36\gamma + 72 - 12t + (\gamma^2 - 36\gamma + 72)t^2$, hence, $\Delta_K(\rho_2; 1) = 2(\gamma^2 - 36\gamma + 66) \neq 0$. Therefore, by Theorem 5, $H_1(\rho_2)$ is a finitely generated torsion $\mathbb{Z}_{19}[[x - \beta]]$-module.

We let $r := g_1g_2^{-1}g_1^{-1}g_1^{-1}g_1g_2^{-1}g_1^{-1}g_1g_2^{-1},$ and set

$$\partial_2 = \left(\rho_2 \circ \pi \left(\frac{\partial \rho}{\partial g_1}\right), \rho_2 \circ \pi \left(\frac{\partial \rho}{\partial g_2}\right)\right) =: (a_1, a_2, a_3, a_4).$$

By the computer calculation, all 2-minors of $\partial_2$ are given by

$$\det(a_1, a_2) = 2(x - 2)[(y - 2)x^2 + x - y^2],$$

$$\det(a_1, a_3) = -\frac{1}{2}[(y - 2)x^2 - (y - 2)x - (y - 1)^2] \sqrt{x^2 - y - 2},$$

$$\det(a_1, a_4) = (y - 2)x^4 - (2y - 5)x^3 - (y^2 - y + 4)x^2 + 4(2y^2 - y + 2)x - (y - 1)^2$$

$$(\star)$$

$$\det(a_2, a_3) = -\frac{1}{2}[(y - 2)x^4 - (2y - 5)x^3 - (y^2 - y + 4)x^2 + 4(2y^2 - y + 2)x - (y - 1)^2]$$

$$(\star)$$

$$\det(a_2, a_4) = 2\{y - 2\}x^2 + x - y^2 \sqrt{x^2 - y - 2},$$

$$\det(a_3, a_4) = 2\{y - 2\}x^2 + x - y^2.$$

By (\star) and the computer calculation, we find that $x = \beta$ ($y(\beta) = \beta$) gives a common zero of all 2-minors of $\partial_2$ and that the greatest common divisor of all 2-minors is $x - \beta$. Therefore, by Proposition 2, we have

$$H_1(\rho_2) \simeq \mathbb{Z}_{19}, \ L_K(\rho_2) \simeq x - \beta.$$

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