Re-normalized 3-manifold invariants

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This is joint work with F. Costantino and B. Patureau.

This talk has foun main parts:

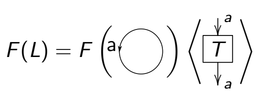
1. Why re-normalized?
2. Re-normalized graph quantum invariants.
3. 3-manifold invariant.
4. If time, idea of the proof.

Let $q \in \mathbb{C}$. The usual Reshetikhin-Turaev quantum invariants of links

$$F : \{\text{colored links}\} \to \mathbb{C}[q, q^{-1}]$$

are generalizations of the Jones polynomial. Here the colors are labelings of the edges of link which correspond to some underlying algebra.

When q is a root of unity then F can be trivial on large classes of colored links: let L be a link and T be a 1-1 tangle whose closure is L then



here $F(\bigcirc_a) = qdim(a)$ can be zero, making the invariant zero.

With Turaev and Patureau I developed an approach to re-normalize these invariants by replacing $qdim$ with a suitable function. This generalizes ADO: Akutsu, Deguchi and Ohtsuki.

Studying re-normalized invariants or more generally invariants coming from non-semi-simple representation theory one can take advantage of the useful and well developed tools of usual quantum topology while simultaneously creating invariants with new features and properties:

Usual Quantum Topology	Re-normalized Quantum Topology
Jones Polynomial	Alexander Polynomial and Reidemeister torsion
Cannot tell apart the lens spaces $L(65, 8)$ and $L(65, 18)$	Distinguish all lens spaces

New features with mapping class group representation from TQFT:

Action of the Dehn twist has finite order	Action of the Dehn twist on separating curve has infinite order

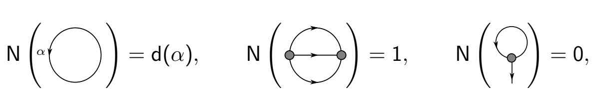
Fix an integer $r \geq 2$ and let $q = e^{i\pi/r}$.

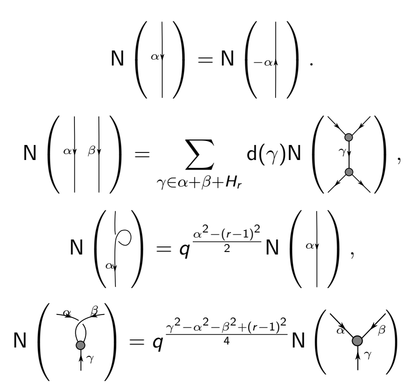
Modified dimension: for $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ let

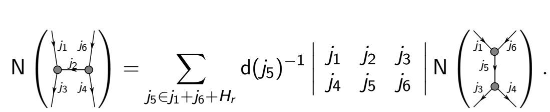
$$d(\alpha) = (-1)^{r-1} \prod_{j=1}^{r-1} \left(\frac{q^j - q^{-j}}{q^{\alpha+r-j} - q^{-\alpha-r+j}} \right).$$

All graphs and links will be oriented and framed. G be the set of trivalent graphs in S^3 whose edges are colored by element of $\mathbb{C}$.

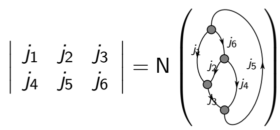
There exists an invariant $N_r = N : G \to \mathbb{C}$ which generically can be computed by the following axioms: N is zero on split graphs and has the following normalizations:

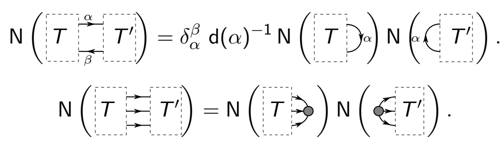






where $H_r = \{1-r, 3-r, \ldots, r-3, r-1\}$ and





For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ define the Kirby color Ω_α in $Span_{\mathbb{C}} < [x] \mid x \in \mathbb{C} >$ by

$$\Omega_\alpha = \sum_{k \in H_r} \mathrm{d}(\alpha + k)[\alpha + k]$$

where $H_r = \{1-r, 3-r, \ldots, r-3, r-1\}$.

We can "color" a knot K with a Kirby color Ω_α: let $K(\Omega_\alpha)$ be the formal linear combination of knots $\sum_{k \in H_r} \mathrm{d}(\alpha+k)K(\alpha+k)$ where $K(\alpha+k)$ is the knot K colored with $\alpha + k$.

Extend the invariant N to knots colored with Kirby colors:

$$\mathrm{N}(K(\Omega_\alpha)) = \sum_{k \in H_r} \mathrm{d}(\alpha+k)\mathrm{N}(K(\alpha+k)))$$

and similarly for links.

Let M be a compact, connected oriented 3–manifold and T a trivalent graph in M whose edges are colored by elements of $\mathbb{C}$.

Let ω be a cohomology class in $H^1(M \setminus T, \mathbb{C}/2\mathbb{Z})$ which is compatible with T in the sense that if e is an edge of T colored by α then $\alpha \equiv \omega(m) \bmod 2\mathbb{Z}$ where m is a meridian of e.

A surgery presentation via $L \subset S^3$ for a compatible tuple (M, T, ω) is computable if one of the following two conditions holds:

(1) $\omega(m) \in \mathbb{C}/2\mathbb{Z} \setminus \mathbb{Z}/2\mathbb{Z}$ where m is a meridian of any edge of L,

or

(2) $L = \emptyset$ and $T \neq \emptyset$.

Theorem (Costantino, G, Patureau)

If ω is not integral then there exists a surgery presentation of (M, T, ω) which is computable.

Let L be a computable surgery presentation of (M, T, ω).

Color each component L_i of L with a Kirby color Ω_{α_i} where $\alpha_i \in \mathbb{C}$ such that $\alpha_i \equiv \omega(m_i) \bmod 2\mathbb{Z}$ where m_i is a meridian of L_i.

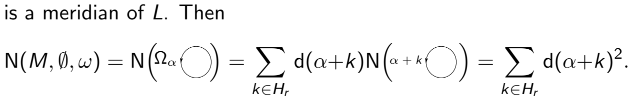
Theorem (Costantino, G, Patureau)

The assignment

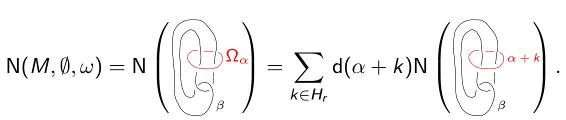
$$\mathrm{N}(M, T, \omega) = \frac{\mathrm{N}(L \cup T)}{C_{lk(L)}}$$

is a well defined topological invariant (i.e. depends only of the homeomorphism class of the triple (M, T, ω)) where $C_{lk(L)}$ is a constant that depends on the linking matrix of L.

Example: Let $M = S^2 \times S^1$, $T = \emptyset$ and $\omega \in H^1(M, \mathbb{C}/2\mathbb{Z})$. Here L is the unknot. Let $\alpha \in \mathbb{C}$ such that $\omega(m) \equiv \alpha \bmod 2\mathbb{Z}$ where m is a meridian of L. Then



Example: Let $M = S^2 \times S^1$, T is the "Whitehead double of the core" colored with $\beta \in \mathbb{C}$ and $\omega \in H^1(M \setminus T, \mathbb{C}/2\mathbb{Z})$. Then



Property: Distinguish the homotopically equivalent manifolds $L(65, 8)$ and $L(65, 18)$ which the standard Reshetikhin-Turaev-Witten quantum invariants do not.

A generalization of the Murakami-Murakami Volume Conjecture for links in 3-manifolds

Let L be a 0-colored link in a compact, oriented 3-manifold M such that $M \setminus L$ has a complete hyperbolic metric with volume $\mathrm{Vol}(M \setminus L)$. For each odd integer r, let $\omega_r \in H^1(M \setminus L; \mathbb{C}/2\mathbb{Z})$ be the zero cohomology class. Then

$$\lim_{r \to \infty} \frac{2\pi}{r} \log |\mathrm{N}_r(M, L, \omega_r)| = \mathrm{Vol}(M \setminus L).$$

When $M = S^3$ this is the usual volume conjecture.

We prove this for an infinite set of hyperbolic links in a connected sum of $k \geq 2$ copies of $S^2 \times S^1$.