Families of non-alternating knots

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結び目の数学 VIII

Waseda University, Japan

Outline

1. Notation
2. HOMFLY polynomial
   - Conway polynomial
   - Alexander polynomial
3. Non-alternating knots
4. Some properties

3-tangles

Definition

*An n-**tangle** is a pair (B^3, T) where B^3 is a 3-ball and T is a one-dimensional, embedded submanifold with non-empty boundary, which contains n arcs (i.e., n subsets homeomorphic to $[0, 1]$) and satisfies $\partial T = T \cap \partial B^3$*

3-tangle diagrams

3-braids

We shall denote a 3-braid by $\mathcal{T}(a_1, a_2, ..., a_n)$

Examples:

$\mathcal{T}(3,2,2)$

$\mathcal{T}(1,-1,1) = \mathcal{E}$

$\mathcal{T}(-1,1,-1) = \mathcal{E}^{-1}$

The set of 3-braids forms a group.

Homfly polynomial

Remember that the **HOMFLY polynomial** of an oriented link is computed by the following:

1. $v^{-1}P(L_+) - vP(L_-) = zP(L_0)$
2. $P(\bigcirc) = 1$

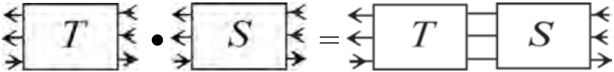
where (L_+, L_-, L_0) is a skein triple of oriented links that are the same, except in a crossing neighbourhood where they look like:

L_+ L_- L_0

$P(L_1 \cup L_2) = \delta P(L_1)P(L_2),\ \delta = \frac{v^{-1}-v}{z}$

$$P(L; v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$$

Operation:



For a given T, we have that $P(T) = \sum_{i=1}^{6} p_i \, P(\chi_i)$ where

$$p_i \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$$

χ_1 χ_2 χ_3 χ_4 χ_5 χ_6

Example: $P\Big($  $\Big) = v^{-2}\, P\Big($  $\Big) - v^{-1} z\, P\Big($  $\Big)$.

Theorem

Let T_1 and T_2 be two 3-tangles, such that

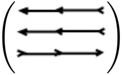
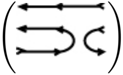
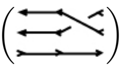
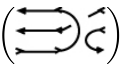
$$P(T_1) = \sum_{i=1}^{6} p_i \, P(\chi_i) \text{ and } P(T_2) = \sum_{i=1}^{6} q_i \, P(\chi_i), \text{ then}$$

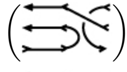
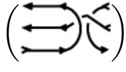
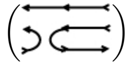
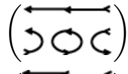
$$\begin{aligned}
P(T_1 \cdot T_2) = & \\
& [p_1 q_1 + p_3 q_3 v^2] P(\chi_1) \\
& +[p_1 q_2 + p_2(q_1 + q_4 + q_2\delta) + p_3 q_4 v^2 + p_5(q_2 + q_3 v^2 + (vz + v^2\delta) q_4)] P(\chi_2) \\
& +[p_1 q_3 + p_3 q_1 + vz p_3 q_3] P(\chi_3) \\
& +[p_1 q_4 + p_3(q_2 + vz q_4) + p_4(q_1 + q_4 + q_2\delta) + p_6(q_2 + q_3 v^2 + (vz + v^2\delta) q_4)] P(\chi_4) \\
& +[p_1 q_5 + p_2(q_3 + q_6 + q_5\delta) + p_3 q_6 v^2 + p_5(q_1 + q_5 + vz q_3 + (vz + v^2\delta) q_6)] P(\chi_5) \\
& +[p_1 q_6 + p_3(q_5 + vz q_6) + p_4(q_3 + q_6 + q_5\delta) + \\
& \quad p_6(q_1 + q_5 + vz q_3 + (vz + v^2\delta) q_6)] P(\chi_6),
\end{aligned}$$

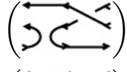
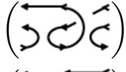
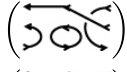
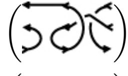
where $\delta = \dfrac{v^{-1} - v}{z}$.

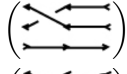
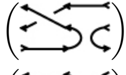
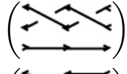
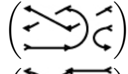
Proof.

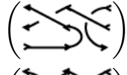
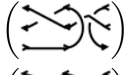
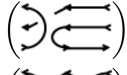
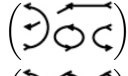
$$P(T_1 \cdot T_2) =$$

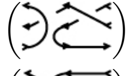
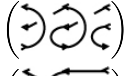
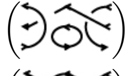
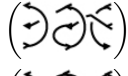
p_1q_1P  $+\, p_1q_2P$  $+\, p_1q_3P$  $+\, p_1q_4P$ 

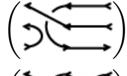
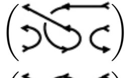
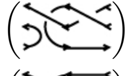
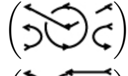
$+p_1q_5P$  $+\, p_1q_6P$  $+\, p_2q_1P$  $+\, p_2q_2P$ 

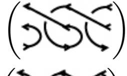
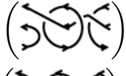
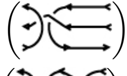
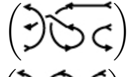
$+p_2q_3P$  $+\, p_2q_4P$  $+\, p_2q_5P$  $+\, p_2q_6P$ 

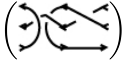
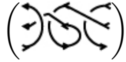
$+p_3q_1P$  $+\, p_3q_2P$  $+\, p_3q_3P$  $+\, p_3q_4P$ 

$+p_3q_5P$  $+\, p_3q_6P$  $+\, p_4q_1P$  $+\, p_4q_2P$ 

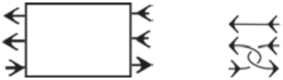
$+p_4q_3P$  $+\, p_4q_4P$  $+\, p_4q_5P$  $+\, p_4q_6P$ 

$+p_5q_1P$  $+\, p_5q_2P$  $+\, p_5q_3P$  $+\, p_5q_4P$ 

$+p_5q_5P$  $+\, p_5q_6P$  $+\, p_6q_1P$  $+\, p_6q_2P$ 

$+p_6q_3P$  $+\, p_6q_4P$  $+\, p_6q_5P$  $+\, p_6q_6P$  .

Closures of T



N_1 N_2 N_3 N_4 N_5 N_6

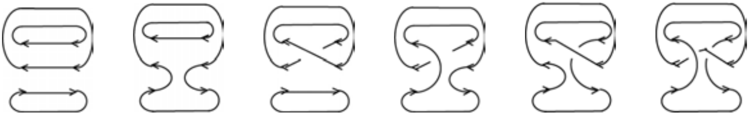
Lemma

Let T be a 3-tangle. If $P(T) = \sum_{i=1}^{6} p_i \, P(\chi_i)$ *then*

$$
\begin{aligned}
P(N_1(T)) &= \delta^2 p_1 + \delta p_2 + \delta p_3 + p_4 + p_5 + (v^2\delta + vz)p_6,\\
P(N_2(T)) &= \delta p_1 + \delta^2 p_2 + p_3 + \delta p_4 + \delta p_5 + p_6,\\
P(N_3(T)) &= \delta p_1 + p_2 + (v^2\delta^2 + vz\delta)p_3 + (v^2\delta + vz)p_4 +\\
&\quad (v^2\delta + vz)p_5 + (v^2 + vz(v^2\delta + vz))p_6,\\
P(N_4(T)) &= p_1 + \delta p_2 + (v^2\delta + vz)p_3 + p_4 + (v^2\delta^2 + vz\delta)p_5 +\\
&\quad (v^2\delta + vz)p_6,\\
P(N_5(T)) &= p_1 + \delta p_2 + (v^2\delta + vz)p_3 + (v^2\delta^2 + vz\delta)p_4 + p_5 +\\
&\quad (v^2\delta + vz)p_6,\\
P(N_6(T)) &= (v^2\delta + vz)p_1 + p_2 + (v^2 + vz(v^2\delta + vz))p_3 + (v^2\delta + vz)p_4 +\\
&\quad (v^2\delta + vz)p_5 + (v^2(v^2\delta^2 + vz\delta) + vz(v^2\delta + vz))p_6.
\end{aligned}
$$

Proof. Suppose that $P(T) = \sum_{i=1}^{6} p_i \, P(\chi_i)$, then

$P(N_j(T)) = \sum_{i=1}^{6} p_i \, P(N_j(\chi_i))$. We prove for $N_1(T)$, the other cases are analogous.



$N_1(\chi_1)$ $N_1(\chi_2)$ $N_1(\chi_3)$ $N_1(\chi_4)$ $N_1(\chi_5)$ $N_1(\chi_6)$

Furthermore, $P(N_1(T)) = \delta^2 p_1 + \delta p_2 + \delta p_3 + p_4 + p_5 + (v^2\delta + vz)p_6$. □

Theorem

Let T_1 and T_2 be two 3-tangles. If

$$P(T_1) = \sum_{i=1}^{6} p_i \, P(\chi_i) \quad \text{and} \quad P(T_2) = \sum_{i=1}^{6} q_i \, P(\chi_i), \text{ then,}$$

$$P(N_1(T_1 \cdot T_2)) = \sum_{i=1}^{6} p_i \, P(N_i(T_2)).$$

Proof. Previous results. □

Theorem

Let $D = \mathcal{E}^{2k}$ *be a 3-tangle with* $k \in \mathbb{N}$. *Then* $P(D) = \sum_{i=1}^{6} A_{i_k} P(\chi_i)$, *where:*

$$\begin{array}{rcl} A_{1_k} & = & A_{1_{k-1}} v^{-2} + A_{3_{k-1}} v^{-1} z, \\ A_{2_k} & = & [-A_{3_k} - (\delta + z v^{-1})(1 - A_{1_k})]/((1-\delta^2)), \\ A_{3_k} & = & A_{1_{k-1}} v^{-3} z + (1 + z^2) v^{-2} A_{3_{k-1}}, \\ A_{4_k} & = & [\delta A_{3_k} + v^{-2}(1 - A_{1_k})]/(1-\delta^2), \\ A_{5_k} & = & [\delta A_{3_k} + v^{-2}(1 - A_{1_k})]/(1-\delta^2), \\ A_{6_k} & = & -A_{3_k} - \delta[\delta A_{3_k} + v^{-2}(1 - A_{1_k})]/(1-\delta^2), \end{array}$$

with $A_{3_0} = 0$, $A_{1_0} = 1$ *and* $\delta = \dfrac{v^{-1} - v}{z}$,

Proof. Induction. □

$$
\begin{array}{rcl}
A_{1_k} & = & A_{1_{k-1}} v^{-2} + A_{3_{k-1}} v^{-1} z, \\
A_{3_k} & = & A_{1_{k-1}} v^{-3} z + (1+z^2) v^{-2} A_{3_{k-1}},
\end{array}
$$

$$
\begin{pmatrix} A_{1_k} \\ A_{3_k} \end{pmatrix} = \begin{pmatrix} v^{-2} & v^{-1}z \\ v^{-3}z & v^{-2}(1+z^2) \end{pmatrix}^k \begin{pmatrix} A_{1_0} \\ A_{3_0} \end{pmatrix}
$$

Lemma

If $A_{1_0} = 1$ and $A_{3_0} = 0$ then $A_{1_k} = v^{-2k}\alpha_{1_k}$ and $A_{3_k} = v^{-(2k+1)}\alpha_{3_k}$.

$$
\begin{pmatrix} \alpha_{1_k} \\ \alpha_{3_k} \end{pmatrix} = \begin{pmatrix} 1 & z \\ z & 1+z^2 \end{pmatrix}^k \begin{pmatrix} \alpha_{1_0} \\ \alpha_{3_0} \end{pmatrix}
$$

Lemma

i) $P(\mathcal{T}(2l)) = A_{1_l} v^{4l}\, P(\chi_1) + A_{3_l} v^{4l}\, P(\chi_3)$.

ii) $P(\mathcal{T}(2l+1)) = A_{3_l} v^{4l+2}\, P(\chi_1) + A_{1_{l+1}} v^{4l+2}\, P(\chi_3)$.

iii) $P(\mathcal{T}(-2l)) = A_{l_{l+1}} v^{2}\, P(\chi_1) - A_{3_l}\, P(\chi_3)$.

iv) $P(\mathcal{T}(-(2l+1))) = -A_{3_l} v^{2}\, P(\chi_1) + A_{1_{l+1}}\, P(\chi_3)$

v) $P(\mathcal{T}(0,2l)) = v^{2l}\, P(\chi_1) + z \sum_{i=1}^{l} v^{2i-1}\, P(\chi_2)$.

vi) $P(\mathcal{T}(0,-2l)) = v^{-2l}\, P(\chi_1) - z \sum_{i=1}^{l} v^{-(2i-1)}\, P(\chi_2)$,

where $A_{1_0} = 1, A_{3_0} = 0$.

Proof. Induction. □

Corollary

Let T be a 3-tangle and $k \in \mathbb{N} \cup \{0\}$. If

$$P(T) = \sum_{i=1}^{6} p_i P(\chi_i) \quad \textit{and} \quad P(\mathcal{E}^{2k}) = \sum_{i=1}^{6} A_{i_k} P(\chi_i),$$

then

$$P(N_1(T \cdot \mathcal{E}^{2k})) = \\ p_1 P(N_1(\mathcal{E}^{2k})) + p_2\delta + p_3 P(N_3(\mathcal{E}^{2k})) + p_4 + p_5 + p_6(v^2\delta + vz).$$

Proof.

$$P(N_1(T \cdot \mathcal{E}^{2k})) = \sum_{i=1}^{6} p_i \, P(N_i(\mathcal{E}^{2k})).$$

□

Conway-Alexander polynomial

$$P(L; v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$$

$$\nabla(L; z) \in \mathbb{Z}[z]$$

$$\Delta(L; t) \in \mathbb{Z}[t^{\pm \frac{1}{2}}]$$

$$\nabla(L; z) = P(L; 1, z)$$

$$\Delta(L; t) = P(L; 1, t^{\frac{1}{2}} - t^{-\frac{1}{2}})$$

$$\delta = \frac{v^{-1} - v}{z}$$
$$A_{1_k} = v^{-2k}\alpha_{1_k}$$
$$A_{3_k} = v^{-(2k+1)}\alpha_{3_k}$$

$$\begin{pmatrix}\alpha_{1_k}\\ \alpha_{3_k}\end{pmatrix} = \begin{pmatrix}1 & z\\ z & 1+z^2\end{pmatrix}^k \begin{pmatrix}\alpha_{1_0}\\ \alpha_{3_0}\end{pmatrix}$$

$$P(N_1(T \cdot \mathcal{E}^{2k})) =$$
$$p_1 P(N_1(\mathcal{E}^{2k})) + p_2\delta + p_3 P(N_3(\mathcal{E}^{2k})) + p_4 + p_5 + p_6(v^2\delta + vz).$$

$$\nabla(N_1(T \cdot \mathcal{E}^{2k})) = p_1\nabla(N_1(\mathcal{E}^{2k})) + p_3\nabla(N_3(\mathcal{E}^{2k})) + p_4 + p_5 + zp_6.$$

$$\nabla(N_1(T \cdot \mathcal{E}^{2k})) =$$
$$p_1(2(1-\alpha_{1k}) - z\alpha_{3k}) + p_3(-(2+z^2)\alpha_{3k} + z(1-\alpha_{1k})) + \nabla(N_1(T)).$$

Alexander polynomial

With the change $z = t^{\frac{1}{2}} - t^{-\frac{1}{2}}$, we have:

Proposition

For all $k \in \mathbb{N}$ we have

$$\alpha_{1_k} = \left[\sum_{i=1}^{k} (t^{-(i-1)} + t^{i-1})(-1)^{k-i}\right] + (-1)^k, \tag{1}$$

$$\alpha_{3_k} = -\sum_{i=1}^{k} (t^{-(i-\frac{1}{2})} - t^{i-\frac{1}{2}})(-1)^{k-i}. \tag{2}$$

$$\alpha_{1_k} = \alpha_{1_{k-1}} + z\alpha_{3_{k-1}}, \tag{3}$$

$$\alpha_{3_k} = z\alpha_{1_{k-1}} + (1+z^2)\alpha_{3_{k-1}}. \tag{4}$$

with $\alpha_{1_0} = 1$ and $\alpha_{3_0} = 0$.

Non-alternating knots

Theorem

[Murasugi, 1958] Suppose K is an alternating knot and

$$\Delta_K(t) = a_{-m}t^{-m} + a_{-m+1}t^{-m+1} + \ldots + a_m t^m \text{ with } a_m \neq 0 \neq a_{-m}.$$

Then

1. $a_{-m}, a_{-m+1}, \ldots, a_m$ *are never equal to zero;*
2. *the sign of two consecutive coefficients alternates, i.e.,*

$$a_i a_{i+1} < 0 \quad (i = -m, -m+1, \ldots, m-1).$$

Note that if K is a knot such that its Alexander polynomial does not satisfy (1) or (2), then K is non-alternating.

Lemma

$\Delta(N_1(\mathcal{E}^{2k})) = -(t^{-k} + t^k) + 2$ *and*
$\Delta(N_3(\mathcal{E}^{2k})) = (t^{-(k+\frac{1}{2})} - t^{(k+\frac{1}{2})}) + t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ *for* $k \in \mathbb{N} \cup \{0\}$.

Theorem

Let T be a 3-tangle such that $N_1(T)$ is a knot. Then there exists $k \in \mathbb{N}$ such that for all $L \geq k, L \in \mathbb{N}$ the family $\{N_1(T \cdot \mathcal{E}^{2L})\}$ is non-alternating.

Proof.

$$\nabla(N_1(T \cdot \mathcal{E}^{2k})) = p_1 \nabla(N_1(\mathcal{E}^{2k})) + p_3 \nabla(N_3(\mathcal{E}^{2k})) + \nabla(N_1(T)).$$

$$\Delta(N_1(T \cdot \mathcal{E}^{2k})) =$$
$$p_1(-(t^{-k} + t^k) + 2) + p_3(t^{-(k+\frac{1}{2})} - t^{(k+\frac{1}{2})}) + t^{\frac{1}{2}} - t^{-\frac{1}{2}}) + \Delta(N_1(T)).$$

□

Lemma

For all $l, k, r \in \mathbb{N}$ *with* $n_i, m_i, \in \mathbb{N}, i = 1, ..., r$, *if* $r \geq 1$ *the* $span(\Delta(N_1(\mathcal{T}(2l+1, 2n_1, 2m_1, ..., 2n_r, 2m_r) \cdot c))) = 2g_r + 2$ *and* $span(\Delta(N_1(\mathcal{T}(2l+1) \cdot c))) = 2l$ *if* $r = 0$, *where* $g_r = l + \sum_{i=1}^{r} m_i$.

Theorem

Let $T = \mathcal{T}(2l+1, 2n_1, 2m_1, ..., 2n_r, 2m_r) \cdot c$ *be a 3-tangle, for all* $l, k, r \in \mathbb{N}$ *with* $n_i, m_i, \in \mathbb{N}, i = 1, ..., r$ *and* $k \geq 3$. *Then the family of knots* $\{N_1(T \cdot \mathcal{E}^{2k})\}$ *is non-alternating.*

Proof.

$$\begin{aligned}
&\Delta(N_1(T \cdot \mathcal{E}^{2k})) \\
&= \Delta(N_1(T)) - (t^{-g_r} + t^{g_r}) + (t^{-(g_r+1)} + t^{(g_r+1)}) \\
&+ (t^{-(g_r+k)} + t^{(g_r+k)}) - (t^{-(g_r+k+1)} + t^{(g_r+k+1)}).
\end{aligned}$$

□

Corollary

For all $l \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$ the knots $N_1(\mathcal{T}(2l+1) \cdot c \cdot \mathcal{E}^{2k})$ are non-alternating.

Proof.

From the previous results:

$$\begin{aligned}\Delta(N_1(\mathcal{T}(2l+1) \cdot c \cdot \mathcal{E}^{2k})) &= \left[\sum_{i=1}^{l}(t^{-i}+t^{i})(-1)^{l-i}\right] + (-1)^{l} \\ &\quad +(-t^{-l} - t^{l} + t^{-(l+1)} + t^{(l+1)}) \\ &\quad +(t^{-(l+k)} + t^{(l+k)} - t^{-(l+k+1)} - t^{(l+k+1)}).\end{aligned}$$

□

Corollary

For all $l \in \mathbb{N}$ *and* $k \in \mathbb{N} \cup \{0\}$ *the knots* $N_1(\mathcal{T}(2l+1,2,2) \cdot c \cdot \mathcal{E}^{2k})$ *are non-alternating.*

Proof.

$$\Delta(N_1(\mathcal{T}(2l+1,2,2) \cdot c \cdot \mathcal{E}^{2k})) =$$

$$\left[\sum_{i=1}^{l}(t^{-(i-1)} + t^{(i-1)})(-1)^{l-i}\right] + (-1)^l$$
$$+2(-t^{-(l+1)} - t^{l+1} + t^{-(l+2)} + t^{l+2})$$
$$+(t^{-(l+k+1)} + t^{l+k+1} - t^{-(l+k+2)} - t^{l+k+2}).$$

□

Some properties

$alt(L)$ Alternation number

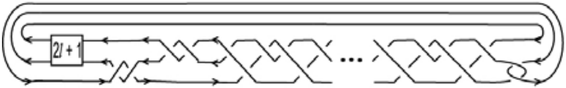
The alternation number of a link diagram D is the minimum number of crossing changes necessary to transform D into some (possibly non-alternating) diagram of an alternating link.
The alternation number of a link L, denoted $alt(L)$, is the minimum alternation number of any diagram of L.

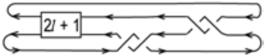
Kawauchi introduced the alternation number of a knot K [Kawauchi, 2010].

Corollary

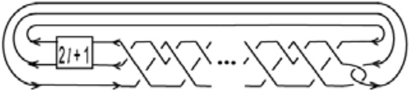
If K is, either the knot $N_1(\mathcal{T}(2l+1,2,2)\cdot c\cdot \mathcal{E}^{2k})$ for $l\in\mathbb{N}$ and $k\in\mathbb{N}\cup\{0\}$ or the knot $N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^{2k})$ for $l\in\mathbb{N}\cup\{0\}$ and $k\in\mathbb{N}$, then $alt(K)=1$.

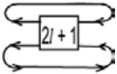
Note that:





and





$NK = \{K \in \{N_1(\mathcal{T} \cdot c \cdot \mathcal{E}^{2k})\} \mid K \text{ is non-alternating}\}.$

Lemma

If K is a knot in NK then $br(K) = 3$

Proof. The knot K has $br(K) \leq 3$ and is non-alternating. □

Theorem

If K is a knot in NK then it is a prime knot.

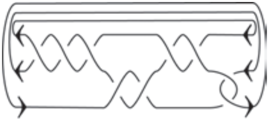
Proof. If K is the connected sum of K_1 and K_2, then the bridge index of K is one less than the sum of the bridge index of K_1 and K_2 [Schubert, 1954]. As $br(K) = 3$ then K_1 and K_2 have bridge index two and therefore they are alternating knots. Further, the connected sum of alternating knots is an alternating knot and then $K \notin NK$. Hence K is a prime knot. □

Corollary

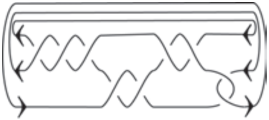
If K is a knot in NK different to $N_1(\mathcal{T}(3,2,2)\cdot c)$ or $N_1(\mathcal{T}(5,2,2)\cdot c)$ then it is a hyperbolic knot.

Proof. Let $K \in NK$. Then K is prime, it has bridge index 3 and alternation number one. The prime knots with bridge index ≤ 3, that are not torus knots, are hyperbolic [Kawauchi, 2012]. The only torus knots with alternation number one are 8_{19} and 10_{124} [Abe, 2009]. □

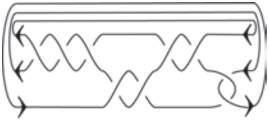
Some diagrams



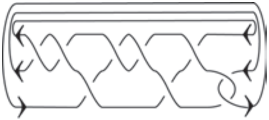
$N_1(\mathcal{T}(3,2,2)\cdot c)$



$N_1(\mathcal{T}(-3,2,2)\cdot c)$



$N_1(\mathcal{T}(3,2,-2)\cdot c)$



$N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^2)$

Homogeneous links

Let D be a diagram of an oriented link and let G be its Seifert graph. Each edge in G can be given a sign $+$ or $-$ depending on whether it passed through a positive or negative crossing. If the signs of all edges in each blocks of G is the same, then D is a homogeneous diagram. A knot K is homogeneous if K has a homogeneous diagram.

The class of homogeneous knots includes alternating knots and positive knots [Cromwell, 1989].

Theorem

[Cromwell, 1989] A link is non-homogeneous if $P(L)$ has no terms of the form $\lambda(-1)^{\frac{1}{2}(r-s)}v^s z^r$ for $\lambda \in \mathbb{N}, r = maxdeg_z P(L), s \leq r$.

Lemma

$P(N_1(E^{2k})) = -(A_{1_k} + A_{1_{k+1}})v^2 + \delta^2 A_{1_k} + \delta A_{3_k} + 2,$
$P(N_3(E^{2k})) =$
$-(A_{3_k} + A_{3_{k+1}})v^2 + \delta^2(1+z^2)A_{1_k} + (3tz\delta + z^3\delta + \delta^2 v^2)A_{3_k} - (\delta z^2 - zv^{-1})$

Theorem

1. *The links* $N_1(\mathcal{E}^{2k})$, $N_3(\mathcal{E}^{2k})$ *and* $N_1(\mathcal{E}^{2k}\cdot c)$ *are non-homogeneous.*
2. *The knots* $N_1(\mathcal{T}(2l+1)\cdot c\cdot\mathcal{E}^{2k})$, $N_1(\mathcal{T}(-(2l+1),2,2)\cdot c)$ *and* $N_1(\mathcal{T}(3,2,-2l)\cdot c))$ *are non-homogeneous knots.*

Proof.
$-v^{-2k}z^{2k}$

$P(N(T\cdot\mathcal{E}^{2k})) =$
$p_1P((N_1(\mathcal{E}^{2k}))) + p_2\delta + p_3P((N_3(\mathcal{E}^{2k}))) + p_4 + p_5 + p_6(v^2\delta + vz).$
$-v^{2l-2k+2}z^{2l+2k+2}$ □

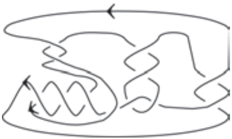
Theorem

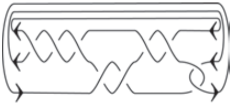
[Cromwell, 1989] Let L be a homogeneous link. Then L is fibred if and only if the leading coefficient of $\nabla(L)$ is ± 1.

Theorem

1. *The knots $N_1(\mathcal{T}(2l+1) \cdot c \cdot \mathcal{E}^2)$ are fibered knots (these knots are non-homogeneous).*
2. *The knots $N_1(\mathcal{T}(2l+1, 2, 2) \cdot c)$ and $N_1(\mathcal{T}(2l+1, 2, 2) \cdot c \cdot \mathcal{E}^2)$ are fibered knots.*

Proof.





□

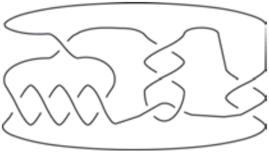
Corollary

The genus of the knots $N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^2)$,$N_1(\mathcal{T}(2l+1,2,2)\cdot c)$ *and* $N_1(\mathcal{T}(2l+1,2,2)\cdot c\cdot \mathcal{E}^2)$ *is* g_r+1+k.

Theorem

The knots $N_1(\mathcal{T}(2l+1)\cdot c\cdot \mathcal{E}^2)$, $N_1(\mathcal{T}(2l+1,2,2)\cdot c)$ *and* $N_1(\mathcal{T}(2l+1,2,2)\cdot c\cdot \mathcal{E}^2)$ *are Montesinos knots.*

Proof.



□

Corollary

The $det(N_1(T \cdot \mathcal{E}^{2k})) = \mid \Delta(T;-1) + 4((-1)^k - 1)(-1)^{g_r} \mid$ *where* $g_r = l + \sum_{i=1}^{r} m_i$ *if* $r \geq 1$ *and* $g_r = l$ *if* $r = 0$.

In particular,
$det(N_1(\mathcal{T}(2l+1) \cdot \mathcal{E}^{2k})) = \mid 2l + 1 + 4((-1)^k - 1) \mid$ and
$det(N_1(\mathcal{T}(2l+1,2,2) \cdot \mathcal{E}^{2k})) = \mid -2l + 9 + 4(-1)^k \mid$.

Corollary

Let $K = N_1(T \cdot \mathcal{E}^{2k})$ *then* $Arf(K) = 0$ *if and only if* $\Delta(N_1(T);-1) \cong \pm 1 mod\ 8$.

$Arf(N_1(\mathcal{T}(2l+1) \cdot \mathcal{E}^{2k})) = 0 \Leftrightarrow l \cong 0\ mod\ 4$ or $l \cong\ 3\ mod\ 4$,
$Arf(N_1(\mathcal{T}(2l+1,2,2) \cdot \mathcal{E}^{2k})) = 0 \Leftrightarrow l \cong 2\ mod\ 4$ or $l \cong\ 3\ mod\ 4$.

Proposition

$$\sigma(N_1(\mathcal{T}(2l+1)\cdot\mathcal{E}^{2k})) = \begin{cases} -2(l+1) & \text{if } k \text{ is odd and } l \leq 3 \\ -2l & \text{otherwise} \end{cases}$$

$$\sigma(N_1(\mathcal{T}(2l+1,2,2)\cdot\mathcal{E}^{2k})) = \begin{cases} -2(l+2) & \text{if } k \text{ is odd and } l \leq 6 \text{ or} \\ & \text{if } k \text{ is even and } l \leq 2 \\ -2(l+1) & \text{otherwise} \end{cases}.$$

Proposition

$u(N_1(\mathcal{T}(2l+1)\cdot\mathcal{E}^{2k})) = l+1$ *if k is odd and $l \leq 3$.*
$u(N_1(\mathcal{T}(2l+1,2,2)\cdot\mathcal{E}^{2k})) = l+2$ *if k is odd and $l \leq 6$ or if k is even and $l \leq 2$.*

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