On deformations of isolated singularities of polar weighted homogeneous mixed polynomials

Kazumasa INABA

Tohoku University

24 December, 2015

Plan of this talk

§**1. Deformation of $f\bar{g}$**

§**2. Main Results**

§**3. Applications**

§1. Deformation of $f\bar{g}$

Let $f, g : (\mathbb{C}^2, o) \to (\mathbb{C}, 0)$ be convenient complex polynomials. Assume f, g have no-common branches.

Theorem [Pichon 2005].

There exists $\varepsilon_0 > 0$ such that $\frac{f(z)\overline{g(z)}}{|f(z)\overline{g(z)})|} : S^3_\varepsilon \setminus K_{f\bar{g}} \to S^1$ is a locally trivial fibration where $\varepsilon \leq \varepsilon_0$. The fibered link $K_{f\bar{g}} = S^3_\varepsilon \cap (f\bar{g})^{-1}(0)$ is $K_f \cup -K_g$.

Assume f, g are weighted homogeneous. Then $f(z)\overline{g(z)}$ is a polar weighted homogeneous mixed polynomial, i.e.,

$f(s \circ z)\overline{g(s \circ z)} = s^{pq(m-n)} f(z)\overline{g(z)} \quad (m > n)$, where

$s \circ z = (s^q z_1, s^p z_2), \quad s \in S^1, \quad \gcd(p, q) = 1.$

A connected component of $K_{f\bar{g}}$ is a (p, q)-torus knot.

We define a deformation F_t of $f(\mathrm{z})\overline{g(\mathrm{z})}$ as follows:

$$F_t(\mathrm{z}) = f(\mathrm{z})\overline{g(\mathrm{z})} + th(\mathrm{z}) \qquad (0 \leq t << 1),$$

$$h(\mathrm{z}) = \begin{cases} \gamma_1 z_1^{p(m-n)} + \gamma_2 z_2^{q(m-n)} & (g(\mathrm{z}) \neq \beta_1 z_1 + \beta_2 z_2) \\ z_1^m \bar{z}_1 + z_1^{m-1} + \gamma z_2^{m-1} & (g(\mathrm{z}) = \beta_1 z_1 + \beta_2 z_2). \end{cases}$$

F_t satisfies

$$F_t(s \circ \mathrm{z}) = s^{pq(m-n)} f(\mathrm{z})\overline{g(\mathrm{z})} + ts^{pq(m-n)} h(\mathrm{z}) = s^{pq(m-n)} F_t(\mathrm{z}).$$

So F_t is a polar weighted homogeneous mixed polynomial.

§2. Main results

Set $S_k(f) = \{z \in U \mid \text{rank } df(z) = 2 - k\} \qquad (k = 0, 1, 2).$

$S_1(F_t)$ **and** $S_2(F_t)$ **satisfy the following properties:**

- $S_j(F_t)$ **is the set of orbits the** S^1**-action,**
- $S_2(F_t) = \{o\}$ **or** $\emptyset$**,**
- $F_t(S_1(F_t))$ **are circles centered at the origin.**

Theorem 1 [I 2014].

There exist γ_1, γ_2 **such that** $S_1(F_t)$ **is the set of indefinite fold singularities and the link** $F_t^{-1}(0) \cap S^3$ **is a** $(p(m-n), q(m-n))$**-torus link in** $U \setminus \{o\}$ **for any** $0 < t << 1$**.**

Let $F_t(\mathrm{z})$ be a deformation of $f(\mathrm{z})\overline{g(\mathrm{z})}$ in Theorem 1.

$$F_{t,s}(\mathrm{z}) := f(\mathrm{z})\bar{g}(\mathrm{z}) + th(\mathrm{z}) + s\ell(\mathrm{z}),$$

where $\ell(\mathrm{z}) = c_1 z_1 + c_2 z_2, c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $0 < s << t << 1$.

Theorem 2 [I 2014].

There exist c_1, c_2 such that $F_{t,s}$ satisfies the following properties:

- $S_1(F_{t,s})$ is the set of indefinite fold singularities,
- $S_2(F_{t,s})$ is the set of mixed Morse singularities.

If the link $F_{t,s}^{-1}(F_{t,s}(\mathrm{w})) \cap S_{\mathrm{w}}^{2n-1}$ is isotopic to a positive Hopf link, $\mathrm{w} \in S_2(F_{t,s})$ is called a mixed Morse singularity, where S_{w}^{2n-1} is the 3-sphere centered at w.

Definition.

$f : X^4 \to Y^2$ with only indefinite fold singularities and Morse singularities is called a broken Lefschetz fibration.

Sketch of the proof

$p \in S_1(f)$ is a fold singularity if and only if w satisfies the following conditions:

1. $$\begin{aligned} j^1 f : \mathbb{R}^4 &\longrightarrow J^1(\mathbb{R}^4, \mathbb{R}^2) \\ p &\longmapsto \left(p, f(p), \frac{\partial f_k}{\partial x_j}\right)_{1 \leq k \leq 2,\ 1 \leq j \leq 4} \end{aligned}$$

   is transversal to
   $S_1(\mathbb{R}^4, \mathbb{R}^2) = \{j^1 f(p) \in J^1(\mathbb{R}^4, \mathbb{R}^2) \mid \text{rank } df_p = 1\}$,
2. rank $d(f \mid S_1(f))(p) = 1$.

We define the matrix $H(P)$ as follows:

$$H(P) := \begin{pmatrix} \left(\frac{\partial^2 P}{\partial z_j \partial z_k}\right) & \left(\frac{\partial^2 P}{\partial z_j \partial \bar{z}_k}\right) \\ \left(\frac{\partial^2 P}{\partial \bar{z}_j \partial z_k}\right) & \left(\frac{\partial^2 P}{\partial \bar{z}_j \partial \bar{z}_k}\right) \end{pmatrix},$$

where $P(\mathrm{z}, \bar{\mathrm{z}})$ is a mixed polynomial.

Lemma 1.

Let $\mathrm{w} \in S_1(P)$. $\det H(P) \neq 0$, w satisfies condition (1).

There exist γ_1, γ_2 such that $\det H(F_t) \neq 0$ on $S_1(F_t)$.

A connected component of $S_1(F_t)$ can be represented by $\{(e^{iq\theta}z_1, e^{ip\theta}z_2) \mid 0 \leq \theta \leq 2\pi\}$.

$$\begin{aligned} F_t \mid_{S_1(F_t)} &= F_t(e^{iq\theta}z_1, e^{ip\theta}z_2) \\ &= e^{ipq(m-n)\theta}F_t(z_1, z_2) \quad (F_t(z_1, z_2) \neq 0). \end{aligned}$$

$$\frac{d}{d\theta}F_t \mid_{S_1(F_t)} = ipq(m-n)e^{ipq(m-n)\theta}F_t(z_1, z_2) \neq 0.$$

Thus F_t satisfies condition (2).

Let $\ell kn(K)$ denote the number of connected components of the link K.

$$\begin{aligned}\ell kn(F_t^{-1}(0) \cap S_\varepsilon^3) &= \ell kn(h^{-1}(0) \cap S_\varepsilon^3) \\ &= m - n,\end{aligned}$$

where $\gamma_1 z_1^{p(m-n)} + \gamma_2 z_2^{q(m-n)}$.
$h^{-1}(0)$ is an invariant set of the S^1-action. The connected component of $h^{-1}(0) \cap S_{\varepsilon_t}^3$ is isotopic to a (p, q)-torus knot.

§3. Applications

Set $\tilde{M}_0 = D^4_\varepsilon \cap F_t^{-1}(\partial D^2_\tau)$ and $\tilde{M}_i = \tilde{M}_0 \bigcup (\bigcup_{j=1}^{i} \partial N_j)$ where $D^2_\tau \cap F_t(S_1(F_t)) = \emptyset$ and N_j is a neighborhood of a connected component of $S_1(F_t)$. By using the S^1-action, $\tilde{M}_i$ is a locally trivial fibration over S^1 for $i = 0, \ldots, \ell$.

Lemma 2.

Let S_0 be the fiber surface of the fibration of $\tilde{M}_0$. Then $S_0 = S^0_0 \cup S^1_0 \cup \cdots \cup S^k_0$, where S^0_0 is the fiber surface of the fibration of a $(p(m-n), q(m-n))$-torus link and S^j_0 is an annulus for $1, \ldots, k$.

Lemma 3.

Let S_i be the fiber surface of the fibration of $\tilde{M}_i$. Then $S_i = \left(S_{i-1} \setminus (\bigcup_{j=1}^{2d_p} D^2_j)\right) \bigcup \left(\bigcup_{j=1}^{d_p} A_j\right)$, where $D^2_j \subset S_{i-1}$ is a disk and A_j is an annulus, $d_p = pq(m-n)$.

Let $h : S \to S$ be a homeomorphism of a surface S. We define

$$\Delta_*(h) = \frac{\Delta_1(h)}{\Delta_0(h)},$$

where $\Delta_k(h)$ is the characteristic polynomial of $h_* : H_k(S, \mathbb{Z}) \to H_k(S, \mathbb{Z})$ for $k = 0, 1$.

The monodromy h_i of the fibration of $\tilde{M}_i$ satisfies

- $h_i(S_{i-1}) \subset S_{i-1}$ and $h_i |_{S_{i-1}} = h_{i-1}$,
- $h_i(\bigcup_{j=1}^{2d_p} D_j^2) \subset \bigcup_{j=1}^{2d_p} D_j^2$ and $h_i(\bigcup_{j=1}^{d_p} A_j) \subset \bigcup_{j=1}^{d_p} A_j$.

Theorem 3 [I 2015].

$$\Delta_*(h_i) = \Delta_*(h_{i-1})(t^{d_p} - 1)^2.$$

Since the fiber surface S_ℓ is diffeomorphic to the fiber surface of $F_t : D^4_\varepsilon \cap F_t^{-1}(\partial D^2_\delta) \to \partial D^2_\delta$,

$$\Delta_*(h_\ell) = \frac{(t^{d_p} - 1)^{m+n}}{\left(t^{d_p/p} - 1\right)\left(t^{d_p/q} - 1\right)}.$$

By lemma 2, S_0 is diffeomorphic to $S_0^0 \cup S_0^1 \cup \cdots \cup S_0^k$ and S_0^j is an annulus for $j \neq 0$. Thus

$$\Delta_*(h_0) = \prod_{j=0}^{k} \Delta_*(h_0|_{S_0^j}) = \Delta_*(h_0|_{S_0^0}) = \frac{(t^{d_p} - 1)^{m-n}}{\left(t^{d_p/p} - 1\right)\left(t^{d_p/q} - 1\right)}.$$

So we obtain $\Delta_*(h_\ell) = \Delta_*(h_0)(t^{d_p} - 1)^{2\ell} = \Delta_*(h_0)(t^{d_p} - 1)^{2n}$.

Corollary.

Let ℓ be the number of connected components of $S_1(F_t)$. Then $\ell = n$.

Thank you for your attention!