無限ブレイド群の交換子部分群上の共役不変ノルム

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Plan of Talk

- Conjugation-invariant norms and Burago-Ivanov-Polterovich's problem
- Bavard's duality and Kawasaki's construction
- Main results and "extremal property"
- Future work

Conjugation-invariant norm

G : group

Definition [Burago-Ivanov-Polterovich]

$\nu : G \to \mathbb{R}$ is a conjugation-invariant norm if

- $\nu(1_G) = 0$
- $\nu(f) = \nu(f^{-1})$
- $\nu(fg) \leq \nu(f) + \nu(g)$
- $\nu(f) = \nu(f^g)$
- $\nu(f) > 0$ if $f \neq 1_G$

for all $f, g \in G$.

Notation : $f^g := gfg^{-1}$

Example

- **commutator length** $\mathrm{cl} : [G, G] \to \mathbb{R}$

$$\mathrm{cl}(h) = \min\{k \mid \exists f_i, \exists g_i \in G, h = [f_1, g_1] \cdots [f_k, g_k]\}$$

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- **fragmentation norm** $\nu_U : \mathrm{Diff}_0^c(M) \to \mathbb{R}$
  M: manifold, U: non-empty open subset of M

$$\nu_U(f) = \min\left\{ k \;\middle|\; \begin{array}{l} \exists g_i, \exists h_i \in \mathrm{Diff}_0^c(M) \\ \mathrm{supp}(g_i) \subset U \end{array}, \ f = g_1^{h_1} \cdots g_k^{h_k} \right\}$$

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- **Hofer norm** $\|\cdot\| : \mathrm{Ham}(M,\omega) \to \mathbb{R}$
  (M,ω): Symplectic manifold
  $$l(\{\varphi_t^H\}) = \int_0^1 \left(\max_{p\in M} H(t,p) - \min_{p \in M} H(t,p) \right) dt$$
  $$\|\psi\| = \inf_{\varphi_1^H = \psi} l(\{\varphi_t^H\})$$

Biinvariant geometry

d : biinvariant metric $\overset{\text{def}}{\Leftrightarrow} d(fg_1h, fg_2h) = d(g_1, g_2)(\forall f, \forall g_i, \forall h \in G)$

biinbariant metric $\overset{1:1}{\longleftrightarrow}$ conjugation-invariant norm

$$\begin{array}{ccc} d & \mapsto & \nu(f) := d(1_G, f) \\ d(f,g) := \nu(fg^{-1}) & \leftarrow\!\shortmid & \nu \end{array}$$

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Definition

- G is **bounded** if every conj-inv norms on G is bounded
- otherwise, G is called **unbounded**. i.e. there exists an unbounded conj-inv norm on G

Remark : For every group, there exists a bounded conj-inv norm.

Some results on (un)boundedness

Example

$G = SL(n;\mathbb{Z})$ is bounded if $n \geq 3$, unbounded if $n = 2$

Theorem [BIP($n = 3$)] [Tsuboi]

M^n : n-dim compact manifold
$G = \mathrm{Diff}_0^\infty(M)$ is bounded if $n \neq 2, 4$

On the other hand, $\mathrm{Ham}(M,\omega)$ is typically unbounded (Gambaudo-Ghys, Entov-Polterovich, etc.)

Stably (un)boundedness

$\nu : G \to \mathbb{R}$ conj-inv norm

Notation : $\bar{\nu}(g) := \lim_{n\to\infty} \frac{\nu(g^n)}{n}$ (stabilization)

ν is **stably bounded** $\overset{\text{def}}{\Leftrightarrow}$ $\bar{\nu} \equiv 0$ (otherwise ν is called **stably unbounded**)

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Definition

- G is **stably bounded** $\overset{\text{def}}{\Leftrightarrow}$ every conj-inv norm on G is stably unbounded
- G is **stably unbounded** $\overset{\text{def}}{\Leftrightarrow}$ there exists a stably unbounded norm

Burago-Ivanov-Polterovich's problem

Example

$G = SL(2;\mathbb{Z})$ is (not only bounded but also) stably unbounded.

- $\mathrm{cl} : [G, G] \to \mathbb{R}$ is stably unbounded
- $G/[G, G] \cong \mathbb{Z}/12\mathbb{Z}$ is finite

$\rightsquigarrow$ cl extends to the stably unbounded norm on G (typical case)

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Problem [BIP]

There exists a group G such that

- $G/[G, G]$ is finite
- $\mathrm{cl} : [G, G] \to \mathbb{R}$ is stably bounded
- G admits a stably unbounded norm?

Burago-Ivanov-Polterovich's problem

The answer is Yes.

Theorem [Kawasaki]

$\mathrm{Ham}(\mathbb{R}^{2n}, \omega_0)$ is stably unbounded.

Theorem [Brandenbursky-Kędra] [K.]

$[B_\infty, B_\infty]$ is stably unbounded.

In this talk, we prove the stably unboundedness of $[B_\infty, B_\infty]$ by using Kawasaki's method.

Bavard's duality theorem

Definition

A **quasimorphism** is a map $\phi : G \to \mathbb{R}$ s.t. there exist $C \geq 0$ with

$$|\phi(gh) - \phi(g) - \phi(h)| \leq C$$

for all $g, h \in G$. The smallest such C is called the **defect** of ϕ and denoted by $D(\phi)$.
A quasimorphism ϕ is said to be **homogeneous** if $\phi(g^n) = n\phi(g)$ $(\forall g \in G, \forall n \in \mathbb{Z})$

Example

- rotation number $\mathrm{rot} : \widetilde{\mathrm{Homeo}}(S^1) \to \mathbb{R}$
- signature of braids $\sigma : B_n \to \mathbb{R}$

Bavard's duality theorem

Notation : $\mathrm{scl}(g) := \lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n}$ (stable commutator length)

Theorem[Bavard]

For $g \in [G, G]$,

$$\mathrm{scl}(g) = \sup_{\phi \in Q} \frac{|\phi(g)|}{2D(\phi)}$$

where Q is the set of homogeneous quasimorphisms with $D(\phi) \neq 0$.

Corollary

$\exists \phi : G \to \mathbb{R}$ a quasimorphism s.t. $\bar{\phi} \not\equiv 0$ on $[G, G]$
$\Leftrightarrow$ $\mathrm{cl} : [G, G] \to \mathbb{R}$ is stably unbounded

Remark : If ϕ is a q.m., $\bar{\phi}$ is a homogeneous q.m.

Quasimorphisms on braid groups

Notation : $Q(G) = \{$homogeneous q.m. on $G\}$: $\mathbb{R}$-vector space

- If $G = \mathrm{MCG}(\Sigma^p_{g,b})$, G admits non-trivial quasimorphisms. (Moreover, $\dim_{\mathbb{R}} \left(Q(G)/\mathrm{Hom}(G,\mathbb{R})\right) = \infty$ [Bestvina-Fujiwara])

  In particular, $Q(B_n)$ admits (many) non-trivial quasimorphisms.
- On the other hand, the following result is known:

Proposition [Kotschick]

$Q(G) = \mathrm{Hom}(G;\mathbb{R})$ if $G = B_\infty, \mathrm{MCG}(\Sigma_\infty), \mathrm{Ham}(\mathbb{R}^{2n}, \omega_0)$, etc.

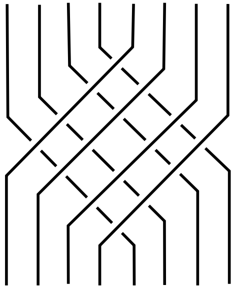
Quasimorphisms on braid groups

Why there does not exists non-trivial quasimorphisms on B_∞?
$\rightsquigarrow$ Because B_∞ is "displaceable":

Let $\alpha \in B_\infty$ be a braid with $\alpha \in B_n \in B_\infty$. Then $\alpha = \alpha^{\Delta_1}, \alpha^{\Delta_2}, \ldots, \alpha^{\Delta_m}, \ldots$ are pairwise commutative, where

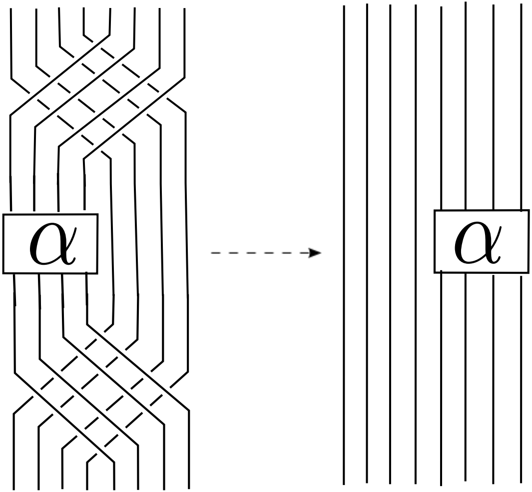
$$\Delta_m = A_{n,m-1} \cdots A_{n,2} A_{n,1} A_{n,2}^{-1} \cdots A_{n,m-1}^{-1} \text{ and}$$

$$A_{n,i} = \prod_{k=1}^{n} \prod_{j=1}^{n} \sigma_{in-k+j} \text{ (the \textbf{argyle braid})} .$$



the argyle braid $A_{4,1}$

Quasimorphisms on braid groups



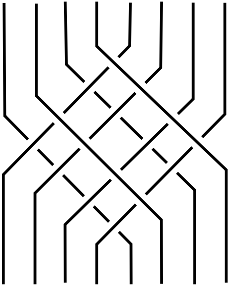
the braid α^{Δ_2} (which commute with α)

Quasimorphisms on braid groups

Similarly, we can see that $[B_\infty, B_\infty]$ is also "displaceable" by using the **commutator argyle braid** $A'_{n,i} = \prod_{k=1}^{n} \prod_{j=1}^{n} \sigma_{in-k+j}^{(-1)^j}$ instead of $A_{n,i}$ and $\Delta'_m = A'_{n,m-1} \cdots A'_{n,2} A'_{n,1} A'^{-1}_{n,2} \cdots A'^{-1}_{n,m-1}$ instead of Δ_m.

$\rightsquigarrow$ $[B_\infty, B_\infty]$ has no non-triv q.m.

$\rightsquigarrow$ $\mathrm{cl}_{[B_\infty, B_\infty]}$ is stably bounded



the commutator argyle braid $A_{4,1}$

Remark

- $[B_\infty, B_\infty]$ is a perfect group.
- $\mathrm{cl}_{[B_\infty, B_\infty]}$ is bounded ($\mathrm{cl} \leq 2$).

Kawasaki's construction

$\nu : G \to \mathbb{R}$ conj-inv norm, $p > 0$, $q > 0$
$[f, g] \in G$ is a (ν, p, q)-**commutator** $\overset{\text{def}}{\Leftrightarrow}$ $\nu(f) \leq p$ and $\nu(g) \leq q$
$[G, G]_{\nu,p,q}$: the subgroup of G generated by (ν, p, q)-commutators

Definition

(ν, p, q)-**commutator length** $\mathrm{cl}_{\nu,p,q} : [G, G]_{\nu,p,q} \to \mathbb{R}$ is defined by

$$\mathrm{cl}_{\nu,p,q}(h) = \min \left\{ k \;\middle|\; \begin{array}{l} \exists f_i, \exists g_i \in G, \\ \nu(f_i) \leq p, \nu(g_i) \leq q \end{array}, h = [f_1, g_1] \cdots [f_k, g_k] \right\}.$$

Remark : The (ν, p, q)-commutator length is a conj-inv norm.

Kawasaki's construction

$\nu : G \to \mathbb{R}$ conj-inv norm

Definition

A ν-**quasimorphism** is a map $\phi : G \to \mathbb{R}$ s.t. there exist $C \geq 0$ with

$$|\phi(gh) - \phi(g) - \phi(h)| \leq C \cdot \min\{\nu(g), \nu(h)\}$$

for all $g, h \in G$.

Theorem [Kawasaki]

$\exists \phi : G \to \mathbb{R}$ a ν-quasimorphism s.t. $\bar{\phi} \not\equiv 0$ on $[G, G]_{\nu,p,q}$
$\Rightarrow \mathrm{cl}_{\nu,p,q} : [G, G]_{\nu,p,q} \to \mathbb{R}$ is stably unbounded

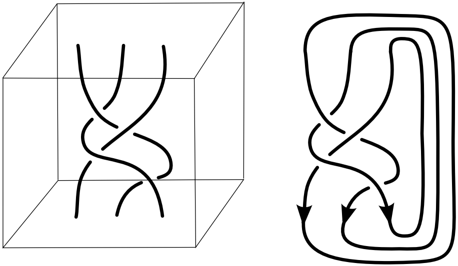
The signature of braids

B_n : the braid group on n strands

The **signature of braids** $\sigma : B_n \to \mathbb{R}$ is defined by

$$\begin{aligned} \sigma(\alpha) &= \text{the signature of the link } \hat{\alpha} \\ &= \text{the signature of } M + {}^tM \ (M : \text{the Seifert matrix of } \hat{\alpha}) \end{aligned}$$

($\hat{\alpha}$ denotes the closure of $\alpha \in B_n$)



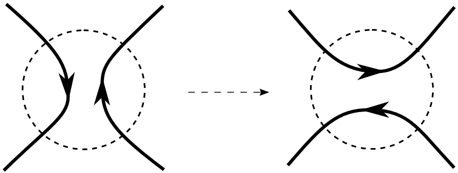
a braid and its closure

The signature of braids

Proposition

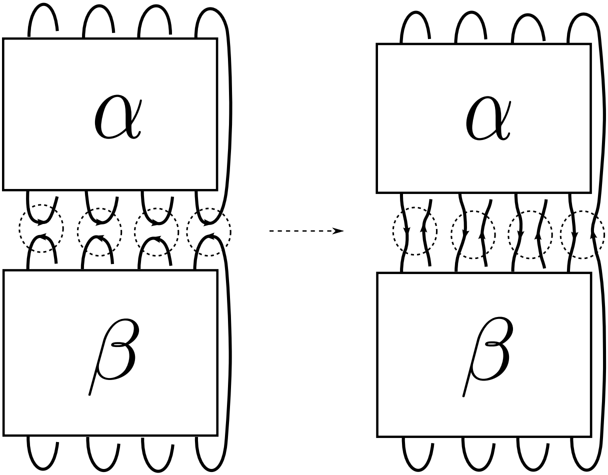
$\sigma : B_n \to \mathbb{R}$ is a quasimorphism with $D(\sigma) \leq n$.

(proof) For $\alpha, \beta \in B_n$, $\sigma(\widehat{\alpha} \sqcup \widehat{\beta}) = \sigma(\widehat{\alpha}) + \sigma(\widehat{\beta})$.
The signature changes at most ± 1 by taking a "saddle move" [Murasugi]. We can obtain the link $\widehat{\alpha\beta}$ from the link $\widehat{\alpha} \sqcup \widehat{\beta}$ by taking n saddle moves.
$|\sigma(\alpha\beta) - \sigma(\alpha) - \sigma(\beta)| = |\sigma(\widehat{\alpha\beta}) - \sigma(\widehat{\alpha} \sqcup \widehat{\beta})| \leq n$. □



saddle move

The signature of braids



$\widehat{\alpha} \sqcup \widehat{\beta}$ $\widehat{\alpha\beta}$

The signature of braids

- $\iota_n : B_n \to B_{n+1}$ the standard inclusion
  (i.e. ι_n is "adding a trivial string")
  $B_1 \subset B_2 \subset \cdots \subset B_n \subset \cdots$
  $\rightsquigarrow B_\infty := \bigcup_{n=\infty} B_n$: the infinite braid group
- the signature σ and the inclusion ι_n are compatible
  (i.e. $\sigma(\iota_n(\alpha)) = \sigma(\alpha)$ for $\alpha \in B_n$)
  $\rightsquigarrow \sigma : B_\infty \to \mathbb{R}$ is well-defined

The above argument does not imply that $\sigma : B_\infty \to \mathbb{R}$ is a q.m.
In fact, there is no non-trivial q.m. on B_∞.

Our strategy : Prove that $\sigma : B_\infty \to \mathbb{R}$ is a "ν-quasimorphism".

Main results

Definition

For $n \geq 2$, we define the **fragmentation norm** $\nu_n : B_\infty \to \mathbb{R}$ by

$$\nu_n(\alpha) = \min\{k \mid \exists \beta_i \in B_n \subset B_\infty, \exists \gamma_i \in B_\infty, \alpha = \beta_1^{\gamma_1} \cdots \beta_k^{\gamma_k}\}.$$

Proposition [K.]

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Proposition [K.]

$\sigma : B_\infty \to \mathbb{R}$ is a ν_n-quasimorphism

Lemma

$\phi : B_\infty \to \mathbb{R}$ is a ν_n-quasimorphism
$\Leftrightarrow \exists C > 0$ s.t. $|\phi(\alpha\beta) - \phi(\alpha) - \phi(\beta)| \leq C$ holds for all $\alpha \in B_\infty$ and all $\beta \in B_\infty$ with $\nu_n(h) = 1$.

Remark : This lemma essentially appears in [Entov-Polterovich].

Main results

(proof of Proposition)

- From the above lemma, it is sufficient to show that $\exists C$, $|\sigma(\alpha\beta) - \sigma(\alpha) - \sigma(\beta)| \leq C$ $(\forall \alpha, \beta \in B_\infty$ with $\nu_n(\beta) = 1)$

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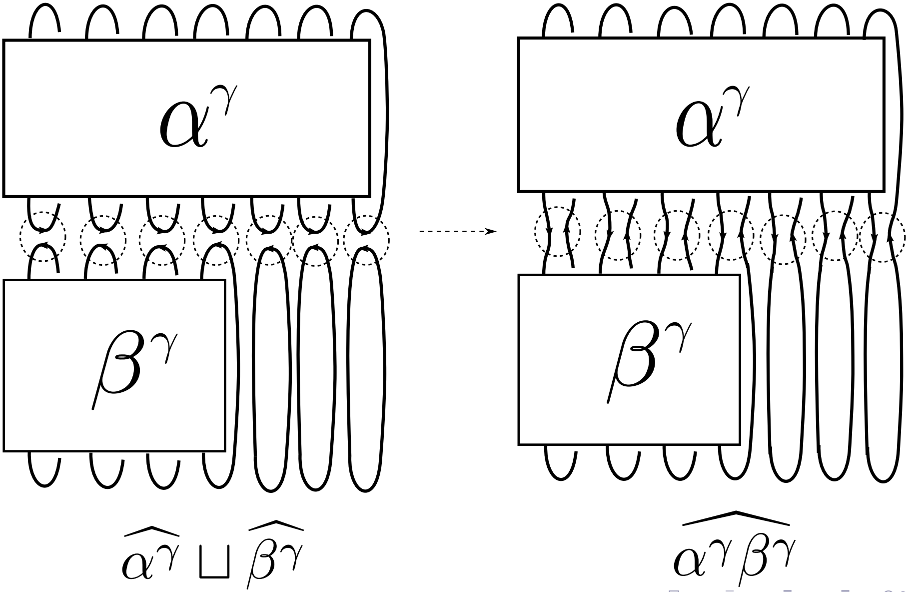
- $\nu_n(\beta) = 1 \Leftrightarrow \exists \gamma \in B_\infty$ s.t. $\beta^\gamma \in B_n \subset B_\infty$

(proof of Proposition)

- From the above lemma, it is sufficient to show that $\exists C$, $|\sigma(\alpha\beta) - \sigma(\alpha) - \sigma(\beta)| \leq C$ ($\forall \alpha, \beta \in B_\infty$ with $\nu_n(\beta) = 1$)
- $\nu_n(\beta) = 1 \Leftrightarrow \exists \gamma \in B_\infty$ s.t. $\beta^\gamma \in B_n \subset B_\infty$
- Let $m \in \mathbb{N}$ s.t. $\alpha^\gamma \in B_m$ $(m > n)$. We obtain the link $\widehat{\alpha^\gamma \beta^\gamma}$ from $\widehat{\alpha^\gamma} \sqcup \widehat{\beta^\gamma}$ by taking saddle moves m times. The link $\widehat{\beta^\gamma}$ has $m - n$ unknot components since β^γ has trivial strings after $(n+1)$-th one.
  Since the signature does not changed by taking connected sum to an unknot, the signature changes at most n by the m times saddle moves. Hence

$$|\sigma(\alpha\beta) - \sigma(\alpha) - \sigma(\beta)| = |\sigma(\alpha^\gamma \beta^\gamma) - \sigma(\alpha^\gamma) - \sigma(\beta^\gamma)| \leq n. \square$$

Main results



Main results

- $[B_\infty, B_\infty]_{\nu_n,p,q} = [B_\infty, B_\infty]$? $\rightsquigarrow$ Yes if $n \geq 2, p, q \geq 1$
- $\bar{\sigma}$ is nontrivial on $[B_\infty, B_\infty]$? $\rightsquigarrow$ Yes $(\bar{\sigma}(\sigma_1^4\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}) = 2)$

Theorem [K.]

The norms $\mathrm{cl}_{\nu_n,p,q}$ are stably unbounded on $[B_\infty, B_\infty]$.
$(n \geq 2,\ p, q \geq 1)$

Extremal property for $\mathrm{cl}_{\nu_n,p,q}$

Proposition A [K.]

Let $n \geq 2$, $p, q \geq 1$. For any conj-inv norm ν on $[B_\infty, B_\infty]$, there exists a positive number $\lambda > 0$ such that $\nu \leq \lambda\, \mathrm{cl}_{\nu_n,p,q}$.

Corollary

For any integer $n \geq 2$ and any real numbers $p, q \geq 1$, the norms $\mathrm{cl}_{\nu_n,p,q}$ are equivalent to each other.

Extremal property for $\mathrm{cl}_{\nu_{n,p,q}}$

Lemma

For $x, y, z \in G$, assume that x and y^z commute. Then $[x, y]$ is written as the products of 4 conjugates of z or z^{-1}.

(proof) By assumption,
$[x,[y,z]] = xy(y^{-1})^z x^{-1} y^z y^{-1} = xyx^{-1}y^{-1} = [x,y]$. Thus
$[x,y] = [x,[y,z]] = xz^y z^{-1} x^{-1} z (z^{-1})^y = z^{xy}(z^{-1})^x z (z^{-1})^y$. □

Comment : Such a deformation appears in a argument about diffeomorphism groups, for example.
$(\mathrm{supp}(f), \mathrm{supp}(g) \subset U$ and $h(U) \cap U = \rightsquigarrow f$ and g^h commute)

Extremal property for $\mathrm{cl}_{\nu_n,p,q}$

(proof of Proposition A)
First we prove the inequality $\nu \leq \lambda \, \mathrm{cl}_{\nu_n,1,1}$. Let $\alpha, \beta \in B_\infty$ satisfy $\nu_n(\alpha) = \nu_n(\beta) = 1$. Then there exists $\gamma, \gamma' \in B_\infty$ such that $\widetilde{\alpha} = \alpha^\gamma \in B_n$, $\widetilde{\beta} = \beta^{\gamma'} \in B_n$. If m is sufficiently large, $\widetilde{\alpha}^{\gamma'\gamma^{-1}}$ and $\widetilde{\beta}^{\Delta'_m}$ commute.
By above Lemma, $[\alpha, \beta]^{\gamma'} = [\alpha^{\gamma'}, \beta^{\gamma'}] = [\widetilde{\alpha}^{\gamma'\gamma^{-1}}, \widetilde{\beta}]$ is written as the products of 4 conjugates of $\Delta'^{\pm 1}_m$. Thus $\nu([\alpha, \beta]) \leq 4\nu(\Delta'_m)$.
Since Δ'_m is conjugate to $A'_{n,1}$, $\nu(\Delta'_m) = \nu(A'_{n,1})$.
Then we obtain $\nu \leq 4\nu(A'_{n,1}) \, \mathrm{cl}_{\nu_n,1,1}$.
Since $\mathrm{cl}_{\nu_n,1,1} \leq pq \, \mathrm{cl}_{\nu_n,p,q}$, the proposition follows. □

Extremal property for $\|\cdot\|$

The **biinvariant word norm** $\|\cdot\| : B_\infty \to \mathbb{R}$ is defined by

$$\|\alpha\| = \min\{k \mid \exists \alpha_i \in B_\infty, \exists \varepsilon_i \in \{\pm 1\}, \alpha = (\sigma_1^{\varepsilon_1})^{\alpha_1} \cdots (\sigma_1^{\varepsilon_k})^{\alpha_k}\}.$$

Theorem [Brandenbursky-Kędra]

The biinvariant word norm $\|\cdot\|$ is stably unbounded on $[B_\infty, B_\infty]$

Extremal property for $\|\cdot\|$

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Theorem [Brandenbursky-Kędra]

The biinvariant word norm $\|\cdot\|$ is stably unbounded on $[B_\infty, B_\infty]$

Proposition B [K.]

Let ν be a conj-inv norm on $[B_\infty, B_\infty]$. If ν satisfies $\nu(\alpha^\beta) = \nu(\alpha)$ for all $\alpha \in [B_\infty, B_\infty]$ and $\beta \in B_\infty$ (not only $[B_\infty, B_\infty]$), then there exists a positive number $\lambda > 0$ such that $\nu \leq \lambda \|\cdot\|$.

Corollary

For any $n \geq 2$ and any real numbers $p, q \geq 1$, the norm $\mathrm{cl}_{\nu_{n,p,q}}$ is equivalent to the biinvariant word norm $\|\cdot\|$.

Extremal property for $\| \cdot \|$

(proof of Proposition B)
Let $\alpha \in [B_\infty, B_\infty]$ with $\|\alpha\| = k$. Then α is written as follows:

$$\begin{aligned}\alpha =& (\sigma_1^{\varepsilon_1})^{\alpha_1}(\sigma_1^{\varepsilon_2})^{\alpha_2}\cdots(\sigma_1^{\varepsilon_k})^{\alpha_k} \\ =& [\alpha_1, \sigma_1^{\varepsilon_1}]\sigma_1^{\varepsilon_1}[\alpha_2, \sigma_1^{\varepsilon_2}]\sigma_1^{\varepsilon_2}\cdots[\alpha_k, \sigma_1^{\varepsilon_k}]\sigma_1^{\varepsilon_k} \\ =& [\alpha_1, \sigma_1^{\varepsilon_1}][\alpha_2, \sigma_1^{\varepsilon_2}]^{\sigma_1^{\varepsilon_1}}\sigma_1^{\varepsilon_1+\varepsilon_2}\cdots[\alpha_k, \sigma_1^{\varepsilon_k}]\sigma_1^{\varepsilon_k} \\ = \cdots =& [\alpha_1, \sigma_1^{\varepsilon_1}][\alpha_2, \sigma_1^{\varepsilon_2}]^*\cdots[\alpha_k, \sigma_1^{\varepsilon_k}]^*\sigma_1^{\varepsilon_1+\cdots+\varepsilon_k},\end{aligned}$$

where $\varepsilon_i \in \{\pm 1\}$ and $\alpha_i \in B_\infty$. Since $\alpha \in [B_\infty, B_\infty]$, $\varepsilon_1 + \cdots + \varepsilon_k = 0$. Thus α is written as a products of k commutators of the form $[*, \sigma_1^{\pm 1}]^*$.
For a sufficient large m, α_i and $(\sigma_1^{\varepsilon_i})^{\Delta'_m}$ are commutative in B_∞.
Therefore, by the assumption of ν, $\nu([\alpha_i, \sigma_1^{\varepsilon_i}]) \leq 4\nu(\Delta'_m) = 4\nu(A'_{2,1})$ and it follows that $\nu(\alpha) \leq 4\nu(A'_{2,1})k = 4\nu(A'_{2,1})\|\alpha\|$. □

Future work

Problem

$\mathrm{MCG}(\Sigma_\infty)$ is stably unbounded?

- $\iota_g : \mathrm{MCG}(\Sigma_{g,1}) \to \mathrm{MCG}(\Sigma_{g+1,1})$ the standard inclusion
  $\mathrm{MCG}(\Sigma_{1,1}) \subset \mathrm{MCG}(\Sigma_{2,1}) \subset \cdots \subset \mathrm{MCG}(\Sigma_{g,1}) \subset \cdots$
  $\mathrm{MCG}(\Sigma_\infty) := \bigcup_{g=\infty} \mathrm{MCG}(\Sigma_{g,1})$: the stable mapping class group
- $\mathrm{MCG}(\Sigma_{g,1})$ has many non-triv q.m., but $\mathrm{MCG}(\Sigma_\infty)$ does not.
- There exists a family of "good" quasimorphisms on $\mathrm{MCG}(\Sigma_{g,1})$ like the signature of braids?