Disk graphs and right-angled Artin subgroups of handlebody groups

Erika Kuno

Tokyo Institute of Technology (D1)

December 23, 2015 @ Waseda University

Contents

Introduction

Γ: a finite graph having neither loops nor multi-edges.

$\mathrm{V}(\Gamma)$: the vertex set of Γ.

$\mathrm{E}(\Gamma)$: the edge set of Γ.

$A(\Gamma) = \langle \mathrm{V}(\Gamma) \mid [v_i, v_j] = 1 \text{ if and only if } \{v_i, v_j\} \in \mathrm{E}(\Gamma) \rangle$

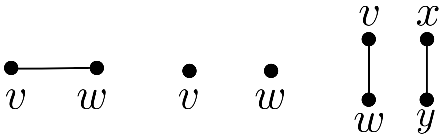
: the right-angled Artin group (RAAG) on Γ.

Example (Right-angled Artin group)

(a) $\langle v, w \mid [v, w] = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}$.

(b) $\langle v, w \mid \emptyset \rangle \cong \mathrm{F}_2$.

(c) $\langle v, w, x, y \mid [v, w] = [x, y] = 1 \rangle \cong (\mathbb{Z} \times \mathbb{Z}) * (\mathbb{Z} \times \mathbb{Z})$.



(a) (b) (c)

Introduction

- G, H: two groups.

  $H \leq G \underset{\text{def}}{\Leftrightarrow} H$ is a subgroup of G.
- Λ, Γ: two graphs.

  $\Lambda \leq \Gamma \underset{\text{def}}{\Leftrightarrow} \Lambda$ is an induced (or a full) subgraph of Γ.

Def:

For $X \subseteq V(\Gamma)$, the subgraph of Γ induced by X is the subgraph Λ of Γ defined by

$V(\Lambda) = X$ and

$E(\Lambda) = \{e \in E(\Gamma) \mid \text{the end points of } e \text{ are in } X\}$.

Introduction

$S = S_{g,n}$: an orientable surface of genus $g \geq 0$ with $n \geq 0$ marked points.

$\mathcal{C}(S)$: the curve graph of S.

$\mathrm{Mod}(S)$: the mapping class group of S.

Proposition 1 (Koberda 2012)

If $\Gamma \leq \mathcal{C}(S)$, then $A(\Gamma) \leq \mathrm{Mod}(S)$.

Proposition 2 (Kim-Koberda 2013)

S: a surface with $\xi(S) \leq 2$.

If $A(\Gamma) \leq \mathrm{Mod}(S)$, then $\Gamma \leq \mathcal{C}(S)$.

Proposition 3 (Kim-Koberda 2013)

S: a surface with $\xi(S) \geq 4$. Then there exists a finite graph Γ such that $A(\Gamma) \leq \mathrm{Mod}(S)$ but $\Gamma \not\leq \mathcal{C}(S)$.

Introduction

$H = H_{g,n}$: an orientable 3-dimensional handlebody of genus $g \geq 0$ with $n \geq 0$ marked points.

$(\partial H = S.)$

$\mathcal{D}(H)$: the disk graph of H.

$\mathrm{Mod}(H)$: the handlebody group of H.

Fact

$\diamond$ $\mathrm{Mod}(H) \leq \mathrm{Mod}(S)$ (by $\Phi \mapsto \Phi\mid_{\partial H}$.)

$\diamond$ $\mathcal{D}(H) \leq \mathcal{C}(S)$ (by $d \mapsto \partial d$.)

Question

Do similar results to Propositions 1, 2, and 3 hold for $\mathrm{Mod}(H)$ and $\mathcal{D}(H)$?

Main theorems

Theorem 1 (K.)

$\Gamma \leq \mathcal{D}(H) \Rightarrow A(\Gamma) \leq \mathrm{Mod}(H)$.

Theorem 2 (K.)

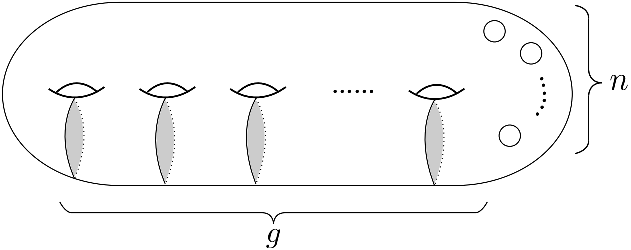
- H: *a handlebody with* $\xi(H) = 0$ *or* $\xi(H) = 1$.
  $A(\Gamma) \leq \mathrm{Mod}(H) \Rightarrow \Gamma \leq \mathcal{D}(H)$.
- H: *a handlebody with* $\xi(H) = 2$.
  $\exists f : A(\Gamma) \to \mathrm{Mod}(H)$: *a "standard" embedding* $\Rightarrow \Gamma \leq \mathcal{D}(H)$.

Theorem 3 (K.)

$H = H_{0,7}$ *or* $H = H_{1,5}$.
$\Rightarrow \exists \Gamma$: *a finite graph s.t.* $A(\Gamma) \leq \mathrm{Mod}(H)$ *but* $\Gamma \not\leq \mathcal{D}(H)$.

Handlebodies

$H = H_{g,n}$: an orientable 3-dimensional **handlebody** of genus $g \geq 0$ with $n \geq 0$ marked points.
($\partial H = S$.)



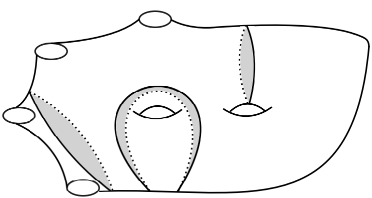
$\diamond$ $\mathrm{Mod}(H)$: the **handlebody group** of H
$\underset{\text{def}}{\Leftrightarrow}$ the group of orientation preserving homeomorphisms of H, fixing the marked points pointwise, up to ambient isotopy.

Disks

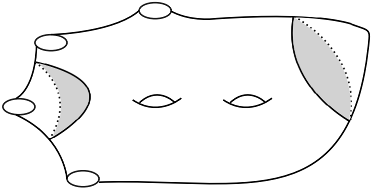
◇ A **disk** on H:

properly embedded and **essential**,

i.e. its boundary circle does not bound a disk or is not isotopic to a marked point on ∂H.



(a) essential disks



(b) non-essential disks

Multi-disks

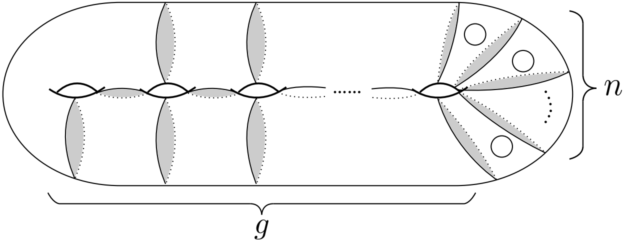
- A **multi-disk** on H

  $\underset{\text{def}}{\Leftrightarrow}$ the union of a finite collection of disjoint pairwise-non-isotopic disks in H.

- $\xi(H) = \max\{3g - 3 + n, 0\}$: the **complexity** of H.

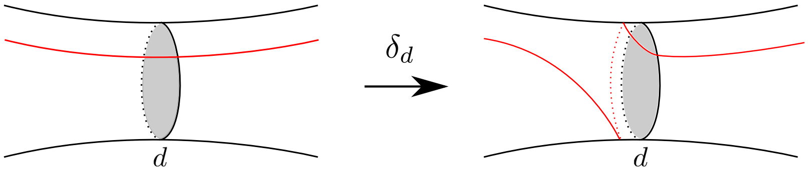
- A **maximal multi-disk** on H

  $\underset{\text{def}}{\Leftrightarrow}$ the number of its components is $3g - 3 + n$.



Multi-disk twists

◇ A **disk twist** δ_d along a disk d in H:



◇ A **multi-disk twist** in H

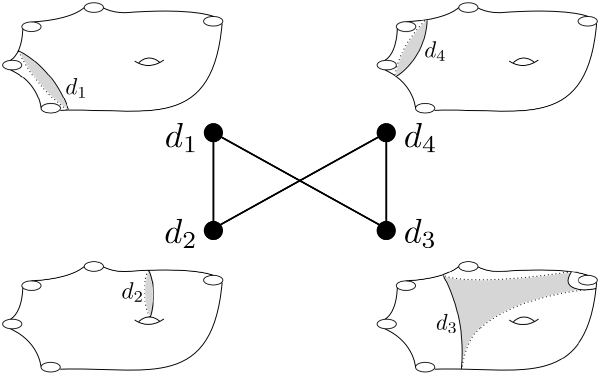
$\underset{\text{def}}{\Leftrightarrow}$ a composition of powers of disk twists along disjoint pairwise-non-isotopic disks.

Disk graphs

$\diamond$ $\mathcal{D}(H)$: a **disk graph** of H:

the vertex set $\underset{\text{def}}{\Leftrightarrow}$ isotopy classes of disks in H,

an edge $\underset{\text{def}}{\Leftrightarrow}$ the corresponding two isotopy classes have pair of disjoint representatives.



Standard embeddings

Def

An embedding $f\colon A(\Gamma) \to \mathrm{Mod}(H)$ is **standard**.

$\underset{\text{def}}{\Leftrightarrow}$ f satisfies the following two conditions:

(i) $\forall v \in V(\Gamma)$, $f(v)$ is a multi-disk twist,

(ii) $\forall u, v \in V(\Gamma)$, $\mathrm{supp}(f(u)) \not\subseteq \mathrm{supp}(f(v))$.

Proof of Theorem 1

Theorem 1 (K.)

$\Gamma \leq \mathcal{D}(H) \Rightarrow A(\Gamma) \leq \mathrm{Mod}(H)$.

It is sufficient to show the following lemma.

Lemma

Γ: *a finite graph.*

H: *a handlebody.*

$i\colon \Gamma \to \mathcal{D}(H)$: *an embedding as an induced subgraph.*

$\Rightarrow \exists N > 0$, *the map*

$$i_{*,N}\colon A(\Gamma) \to \mathrm{Mod}(H)$$

given by sending $v \in V(\Gamma)$ *to the* N*th power* $\delta_{i(v)}^N$ *of a disk twist* $\delta_{i(v)}$ *is injective.*

Proof of Theorem 1

We use the following proposition and fact to prove this lemma.

Proposition (Koberda 2012)

Γ: *a finite graph.*

S: *an orientable surface.*

$i' : \Gamma \to \mathcal{C}(S)$: *an embedding as an induced subgraph.*

$\Rightarrow \exists N > 0$, *the map*

$$i'_{*,N} : A(\Gamma) \to \mathrm{Mod}(S)$$

given by sending $v \in V(\Gamma)$ *to the* N*th power* $T^N_{i'(v)}$ *of a Dehn twist* $T_{i'(v)}$ *is injective.*

Fact

A Dehn twist T_c along a s.c.c. c in S extends to a homeomorphism on H iff c is contractible in H.

Proof of Theorem 1

Proof of Lemma

$i\colon \Gamma \to \mathcal{D}(H)$: an embedding as an induced subgraph.

We define $j\colon \mathcal{D}(H) \to \mathcal{C}(S)$ by $j(d) = \partial d$.

$i' = j \circ i \Rightarrow i'\colon \Gamma \to \mathcal{C}(S)$: an embedding as an induced subgraph.

By Proposition (Koberda 2012), $\exists N > 0$ s.t. the map

$i'_{*,N}\colon A(\Gamma) \to \mathrm{Mod}(S)$ given by sending $v \in V(\Gamma)$ to $T_{i'(v)}^N$ is injective.

$i'(v) = j \circ i(v) = \partial(i(v)) \Rightarrow T_{i'(v)}$ is extended to $\delta_{i(v)}$.

Therefore, $i_{*,N}\colon A(\Gamma) \to \mathrm{Mod}(H)$ given by sending $v \in V(\Gamma)$ to $\delta_{i(v)}^N$ is injective.

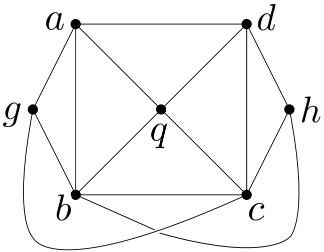
□

Proof of Theorem 3

Theorem 3 (K.)

$H = H_{0,7}$ *or* $H = H_{1,5}$.

$\Rightarrow \exists\Gamma$: *a finite graph s.t.* $A(\Gamma) \leq \mathrm{Mod}(H)$ *but* $\Gamma \not\leq \mathcal{D}(H)$.

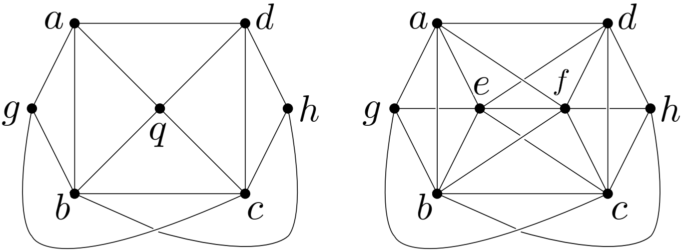


Γ_0

Proof of Theorem 3

Lemma (Kim-Koberda 2013)

The map $\phi\colon A(\Gamma_0) \to A(\Gamma_1)$ defined by $\phi(q) = ef$ and $\phi(v) = v$ $(v \in V(\Gamma_0) - \{q\})$ is injective.



(a) Γ_0 (b) Γ_1

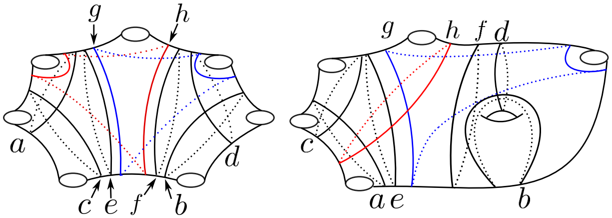
Proof of Theorem 3

Next we show the following.

Lemma

The graph Γ_1 is embedded into $\mathcal{D}(H_{0,7})$ and $\mathcal{D}(H_{1,5})$ as an induced subgraph.

Proof of Lemma



(a) $H_{0,7}$ (b) $H_{1,5}$

□

Proof of Theorem 3

Theorem 3 (K.)

$H = H_{0,7}$ or $H = H_{1,5}$.
$\Rightarrow \exists\Gamma$: a finite graph s.t. $A(\Gamma) \leq \mathrm{Mod}(H)$ but $\Gamma \not\leq \mathcal{D}(H)$.

Proof of Theorem 3

[First half]

Since $\phi\colon A(\Gamma_0) \to A(\Gamma_1)$ is injective, $A(\Gamma_0) \leq A(\Gamma_1)$.

Since Γ_1 is embedded into $\mathcal{D}(H)$ and Theorem 1, we obtain

$A(\Gamma_1) \leq \mathrm{Mod}(H)$.

$\Rightarrow A(\Gamma_0) \leq \mathrm{Mod}(H)$.

Proof of Theorem 3

[Second half]

Assume $\Gamma_0 \leq \mathcal{D}(H)$.

C_4: the 4-cycle spanned by $\{a, b, c, d\}$.

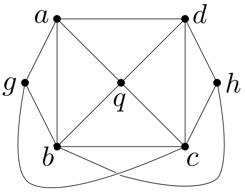
S_1: the regular neighborhood of ∂a and ∂c in ∂H.

S_2: the regular neighborhood of ∂b and ∂d in ∂H.

$S_0 = \overline{\partial H - (S_1 \cup S_2)}$.

$\xi(H) = 4 \Rightarrow$ we have five cases for (S_0, S_1, S_2).

$\xi(H) = 5 \Rightarrow$ we have 17 cases for (S_0, S_1, S_2).



Γ_0

Proof of Theorem 3

Γ_0 is not embedded into $\mathcal{D}(H)$ in any cases for $H = H_{0,7}$ and $H = H_{1,5}$.

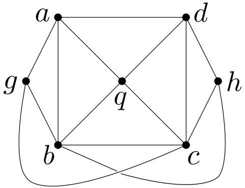
Example $(\xi(H) = 5)$

$(S_0, S_1, S_2) = (S_{0,3}, S_{0,4}, S_{0,4})$, $S_0 \cap S_1 \approx S^1 \coprod S^1$, and $S_0 \cap S_2 \approx S^1$.

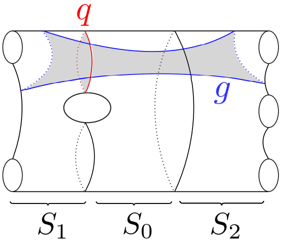
Note that $\partial q \subseteq S_0$. Let $\partial q \subseteq S_0 \cap S_1$.

By $g \cap q \neq \emptyset$, ∂g intersects ∂q. Then it follows that ∂g intersects S_1.

This contradicts the assumption that $g \cap S_1 = \emptyset$.



Γ_0



Therefore $\Gamma_0 \not\leq \mathcal{D}(H)$.

□

Thank you for your attention!