On minimal number of colors for $\mathbb{Z}$-colorable links

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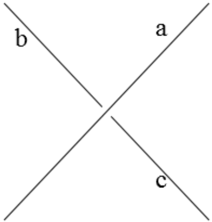
$\mathbb{Z}$-coloring

Let L be a link, and D a diagram of L.

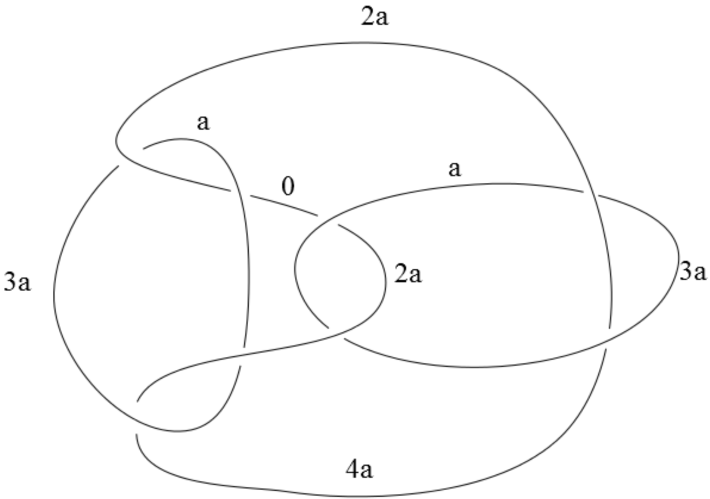
$\mathbb{Z}$-coloring

A map $\gamma : \{\text{arcs of } D\} \to \mathbb{Z}$ is called a $\mathbb{Z}$-coloring on D if it satisfies the condition $2\gamma(a) = \gamma(b) + \gamma(c)$ at each crossing of D with the over arc a and the under arcs b and c.

A $\mathbb{Z}$-coloring which assigns the same color to all the arcs of the diagram is called the trivial $\mathbb{Z}$-coloring.



Example



$\mathbb{Z}$-colorable link

$\mathbb{Z}$-colorable link

L is a $\mathbb{Z}$-colorable link if a diagram of L with a $\mathbb{Z}$-coloring exists.

Remark

L is $\mathbb{Z}$-colorable $\Longleftrightarrow \det(L) = 0$
Any knot K is non-$\mathbb{Z}$-colorable since $\det(K)$ is an odd number.

$mincol_{\mathbb{Z}}$(L)

Let us consider the cardinality of the image of a non-trivial $\mathbb{Z}$-coloring on a diagram of L.

Minimal number of colors

We call the minimum of such cardinalities among all non-trivial $\mathbb{Z}$-colorings on diagrams of L the minimal number of colors of L, and denote it by $mincol_{\mathbb{Z}}(L)$.

Known results

Theorem [Nakanishi-Nakamura-Satoh, 2014]

Let n be an odd number.
For any n-colorable knot K, let $mincol_n^*(K)$ be the minimal number of the colors on effectively n-colored diagrams of K.
Then the following holds;

$$mincol_n^*(K) > 1 + \log_2 n$$

Known results

Theorem [IM. 2015]

Let n be a natural number.
For any n-colorable link L let $mincol_n^*(L)$ be the minimal number of the colors on effectively n-colored diagrams of L.
If $\det(L) \neq 0$, then the following holds;

$$mincol_n^*(L) \geq 1 + \log_2 n$$

Question

Theorem

Let L be a non-splittable non-trivial $\mathbb{Z}$-colorable link. Then, $mincol_{\mathbb{Z}}(L) \geq 4$.

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Proposition

If the crossing number of a $\mathbb{Z}$-colorable link L is at most 9, then $mincol_{\mathbb{Z}}(L) = 4$, i.e., L can be colored by four colors.

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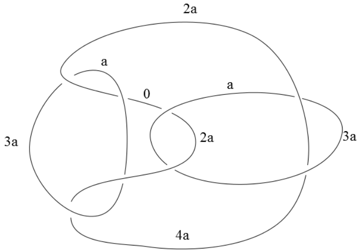
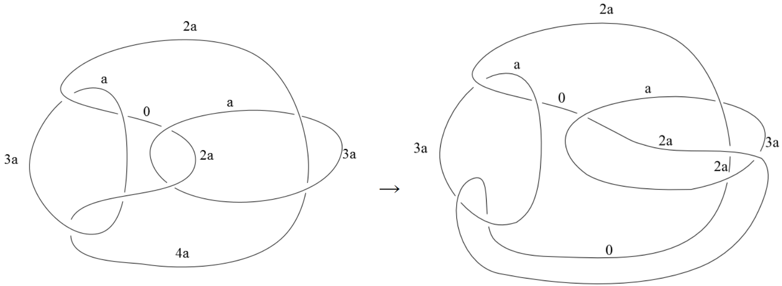
Question

How many colors are enough to color?

Example



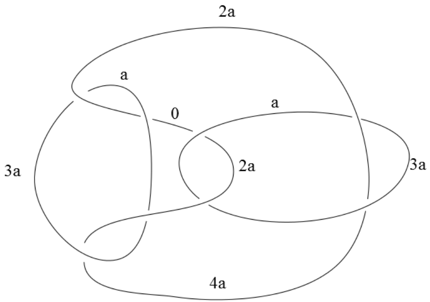
Example



Uniform $\mathbb{Z}$-coloring

Uniform $\mathbb{Z}$-coloring

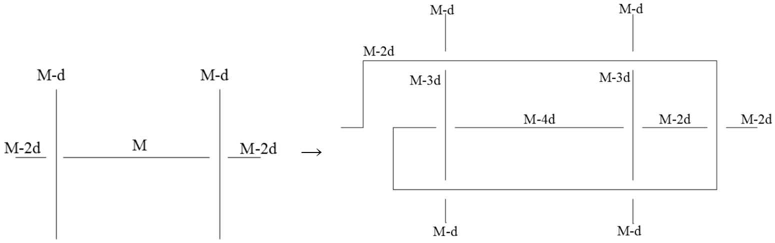
Let L be a $\mathbb{Z}$-colorable link, γ a $\mathbb{Z}$-coloring on a diagram D of L. Suppose that there exists a natural number d such that for all the crossings in D, the differences between the colors of the over arcs and the under arcs are d. Then we call γ a uniform $\mathbb{Z}$-coloring.



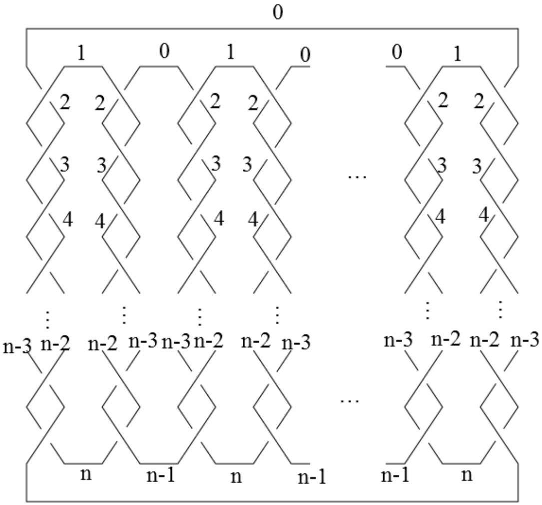
Theorem

Theorem

Let L be a non-splittable $\mathbb{Z}$-colorable link. If there exists a uniform $\mathbb{Z}$-coloring on a diagram of L, then $mincol_{\mathbb{Z}}(L) = 4$.



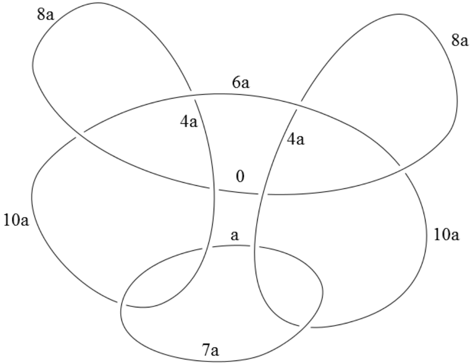
Example


0
1 0 1 0 0 1
2 2 2 2 2 2
3 3 3 3 3 3
4 4 4 4 4 4
...
n-3 n-2 n-2 n-3 n-3 n-2 n-2 n-3 n-3 n-2 n-2 n-3
...
n n-1 n n-1 n-1 n

$\mathbb{Z}$-colorable link with 10 crossings

Proposition

If a number of the crossing of a $\mathbb{Z}$-colorable link L is at most 10, L can be colored by four colors.



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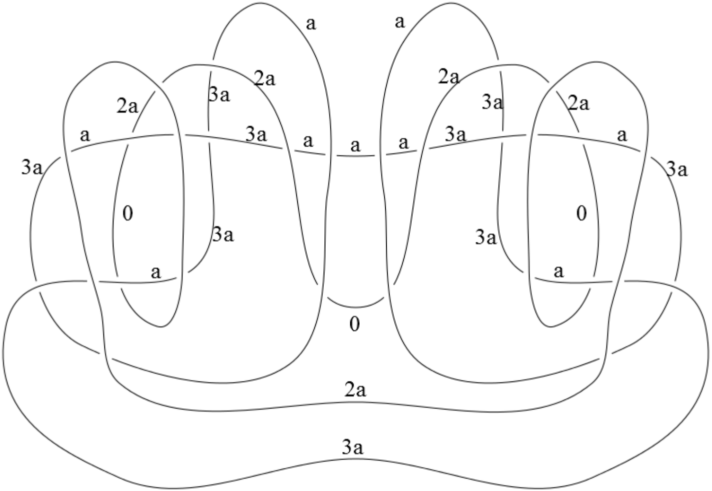
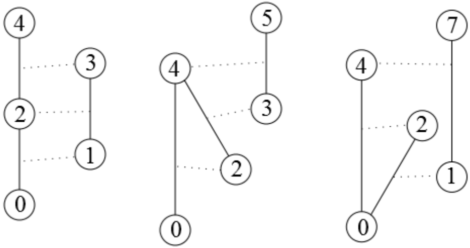


$\mathbb{Z}$-coloring by 5 colors

Theorem

For a $\mathbb{Z}$-coloring γ that the cardinality of the image of γ is 5 and the minimum of the image of γ is 0, the image of γ is shown by $\{0,1,2,3,4\}$,$\{0,1,2,3,5\}$,$\{0,1,2,3,6\}$,$\{0,1,2,4,7\}$, $\{0,2,3,4,5\}$,$\{0,3,4,5,6\}$ and $\{0,3,5,6,7\}$.



$\mathbb{Z}$-coloring by 5 colors

Theorem

Suppose that a non-splittable link L admits a $\mathbb{Z}$-coloring γ. If $\mathrm{Im}(\gamma) = \{0, 1, 2, 3, 4\}$, $\{0, 1, 2, 3, 5\}$ or $\{0, 2, 3, 4, 5\}$, then $mincol_{\mathbb{Z}}(L) = 4$.

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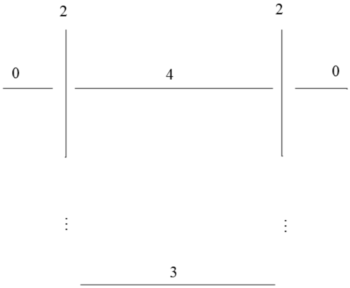

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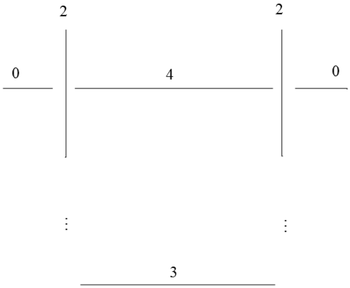

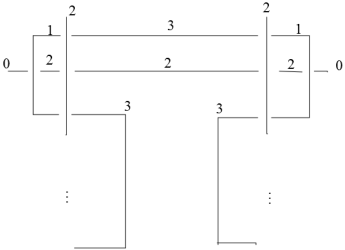

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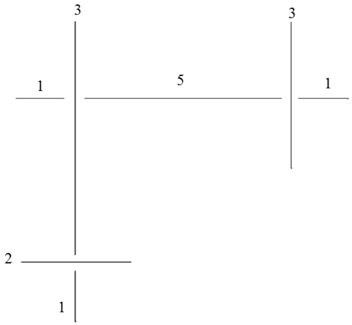

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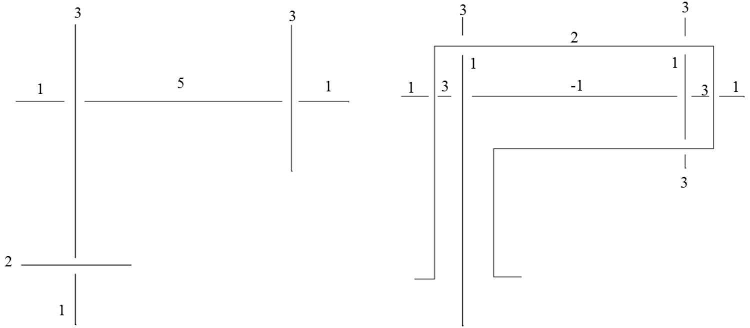

Question

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For any $\mathbb{Z}$-colorable link L, $mincol_{\mathbb{Z}}(L) = 4$?

Thank you

for your attention.