On the Casson-Walker invariant for 3-manifolds admitting genus one open book decompositions

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Intro

Casson-Walker inv $\lambda(M) \in \mathbb{Q}$

- for a $\mathbb{Z}$HS M (Casson, 1980s)

$$H_*(M;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$$

- for a $\mathbb{Q}$HS M (Walker, 1992)

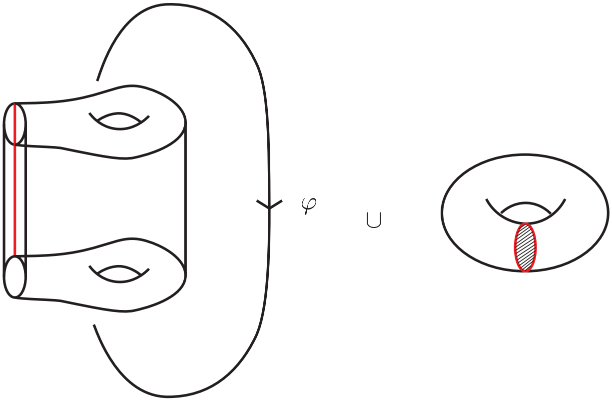
$$H_*(M;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$$

Examples of calculation

- For lens spaces, $\lambda(L(p,q)) = -\frac{1}{2}s(q,p)$
  ($s(q,p)$: Dedekind sum)
- For $\frac{p}{q}$ surgery along a knot in S^3,
  $\lambda(S^3_K(p/q)) = -\frac{1}{2}s(q,p) + \frac{q}{2p}\Delta''_K(1)$
  ($\Delta_K(t)$: Conway poly of K)

Construction in terms of mapping class groups
(Morita, $\cdots$, 1980s, 90s)

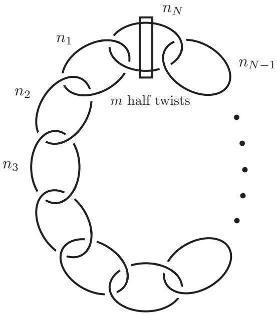
An open book decompositions (genus 1, 1 boundary comp) of a 3-manifold M

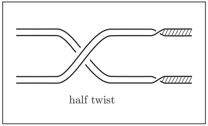
$M =$ 

where φ : a homeo of $\Sigma_{1,1}$ s.t. $\varphi|_{\partial\Sigma_{1,1}} = \mathrm{id}_{\partial\Sigma_{1,1}}$

Fact : A 3-mfd with genus one open book decomp has the following surgery presentation.

$L_{n_1,\cdots,n_N;m}$





$$S^3 \xrightarrow{\text{surgery along } L_{n_1,\cdots,n_N;m}} M_{n_1,\cdots,n_N;m}$$

$M_{n_1,\cdots,n_N;m}$: 3-mfd with genus one open book decomp
obtained by surgery along a link $L_{n_1,\cdots,n_N;m}$

Suppose that $M_{n_1,\cdots,n_N;m}$ is a $\mathbb{Q}HS$

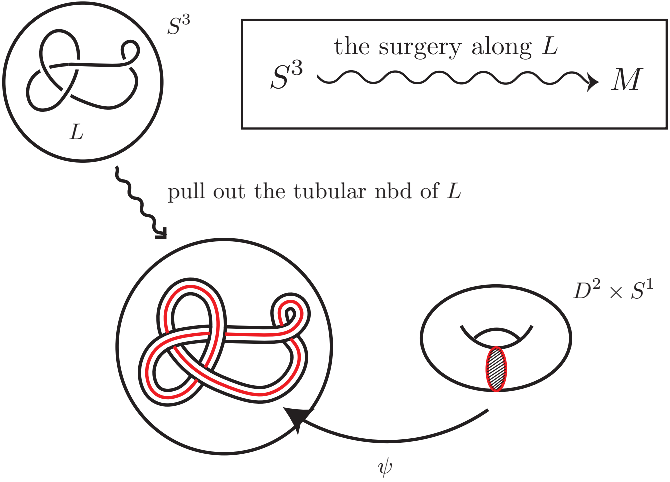
Main theorem

$$\lambda(M_{n_1,\cdots,n_N;m}) = -\frac{1}{24}\left(\sum_i n_i - 3\sigma\right) - \frac{(-1)^m}{24|H_1|}\left(2\sum_i n_i + 6N - 12m\right)$$

where σ : signature of linking matrix of $L_{n_1,\cdots,n_N;m}$

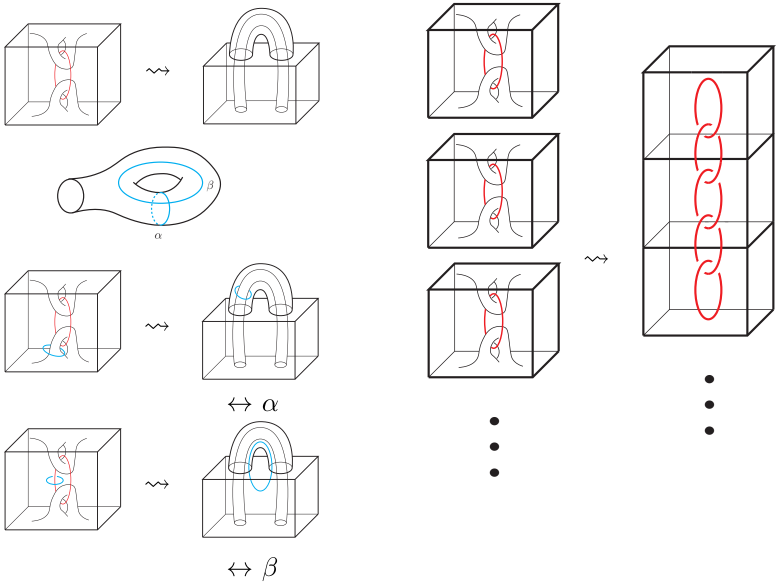
$|H_1|$: order of $H_1(M_{n_1,\cdots,n_N;m};\mathbb{Z})$

Surgery

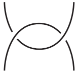


$$M = (S^3 \setminus \text{the nbd of } L) \cup_\psi (\sqcup^\ell D^2 \times S^1)$$

Surgery presentations and monodromies



$$\lambda(M) = \underset{Z_1^{\mathrm{LMO}}(M)}{\text{deg=1 LMO inv}} \overset{\iota}{\leftsquigarrow} \underset{Z(L)}{\text{Kontsevich inv of } L}$$

Caluculate $Z\Big($  $\Big)$

glue them

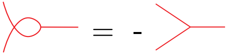
$$Z_1^{\mathrm{LMO}}(M_{n_1,\cdots,n_N;m}) \overset{\iota}{\leftsquigarrow} Z(L_{n_1,\cdots,n_N;m})$$

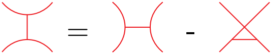
Jacobi diagrams

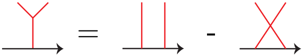
A Jacobi diag on $\sqcup^{\ell} S^1$

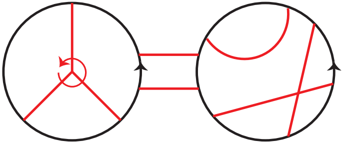
deg of a Jacobi diag $= \frac{1}{2}$ # vertices

$$\mathcal{A}(\sqcup^{\ell} S^1) = \mathrm{span}_{\mathbb{C}} \{\text{Jacobi diags on } \sqcup^{\ell} S^1\}/\text{AS, IHX, STU rel}$$

AS : 

IHX : 

STU : 

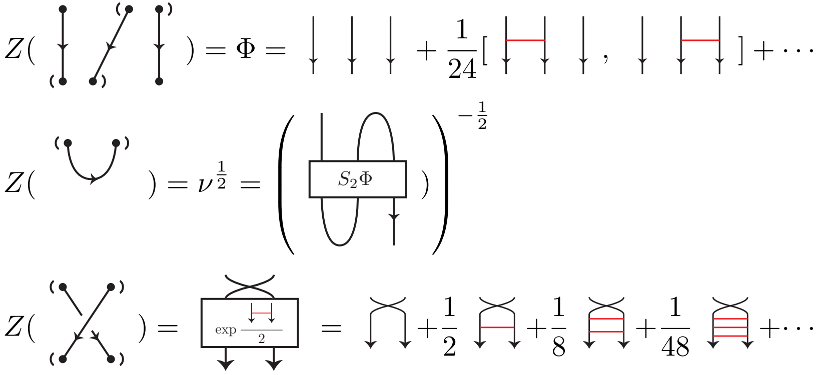


The Kontsevich inv is an inv of a link L

$$Z(L) = Z(T_1) \circ Z(T_2) \circ \cdots \circ Z(T_k) \in \mathcal{A}(\sqcup^{\ell} S^1)$$

(T_i : elementary q-tangle)

For example,



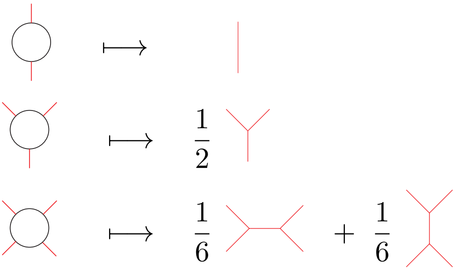
The deg=1 part of LMO inv

$$Z_1^{LMO}(M) = \frac{\iota(Z(L))}{\iota(Z(\infty))^{\sigma_+}\iota(Z(\infty))^{\sigma_-}} \in \mathrm{span}_{\mathbb{C}}\{\emptyset, \ominus\}$$

$$\iota : \mathcal{A}(\sqcup^{\ell} S^1) \to \mathrm{span}_{\mathbb{C}}\{\emptyset, \ominus\}$$

where

ι :



The value of the deg=1 LMO invariant of 3-manifolds concerned

$M_{n_1,\cdots,n_N;m}$: 3-mfd with genus one open book decomp obtained by surgery along a link $L_{n_1,\cdots,n_N;m}$

A_N : linking matrix of $L_{n_1,\cdots,n_N;m}$

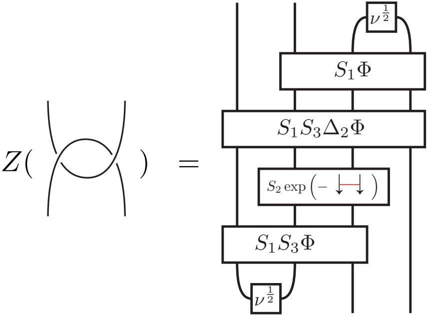
Suppose $\det A_N \neq 0$

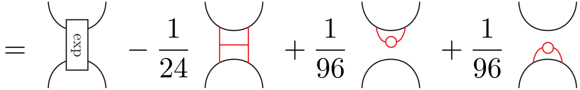
$$Z_1^{\mathrm{LMO}}(M) = c_0(M) + c_1(M)\theta$$

Prop

$$\begin{aligned} c_1\left(M_{n_1,\cdots,n_N;m}\right) = &-\frac{1}{48}(-1)^{N+\sigma_+}\det A_N(\mathrm{tr}A_N - 3\sigma) \\ &-\frac{1}{48}(-1)^{m+N+\sigma_+}\left(2\mathrm{tr}A_N + 6N - 12m\right) \end{aligned}$$

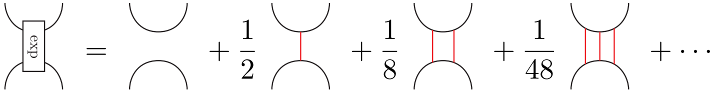
The value of a clasp





(Ohtsuki, 2007)

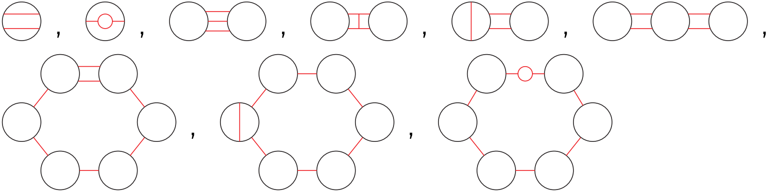
where



Proof

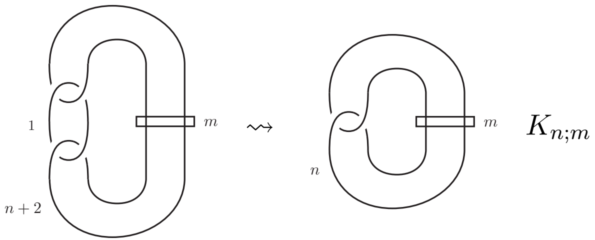
Fact : $\lambda(M) = \dfrac{2c_1(M)}{|H_1|}$ when $b_1(M) = 0$

Jacobi diags which become  through ι



As for non circular case, it can be calculated as LMO inv of lens spaces.
As for circular case, we have only to claculate above three diags. □

The value of twist knots



(When $m = 2k$, a Brieskorn homology sphere $\Sigma(2, 5, 6k + 1)$)

$$\lambda(M_{n;m}) = \begin{cases} -\frac{1}{24}\frac{((n+2)-1)((n+2)-2)}{n+2} - \frac{m}{2(n+2)} & (m : \text{even}) \\ -\frac{1}{24}\frac{((n-2)-1)((n-2)-2)}{n-2} - \frac{m+1}{2(n-2)} & (m : \text{odd}) \end{cases}$$

Summary and Future directions

Summary

We calculated the Casson-Walker invariant of 3-manifolds with genus one open book decompositions through the calculation of the deg=1 part of the LMO invariant.

Future directions

- relation to MCG
- general cases