Periodicity Property of the colored Jones polynomial and the volume of the root polytope

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Motivation 1

Let K be a link. For the ordinary Jones polynomial $J_{K,2}(q)$, it is known that

- $J_{K,2}(-1) = \Delta_K(-1) = \text{determinant of } K.$
- $J_{K,2}\left(e^{\frac{2}{3}\pi\sqrt{-1}}\right) = 1.$

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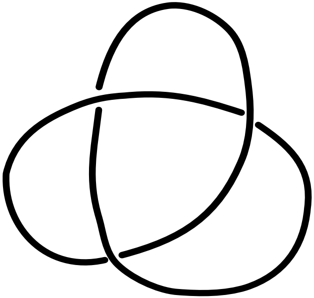
Question 1.1

How about substituting these primitive roots of unity to the colored Jones polynomial?

Motivation 2

We can obtain a bipartite graph G_D from a link diagram D.



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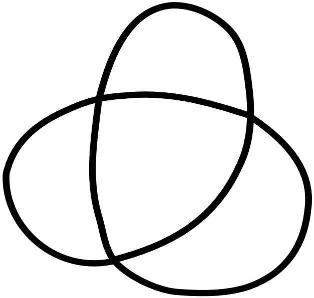

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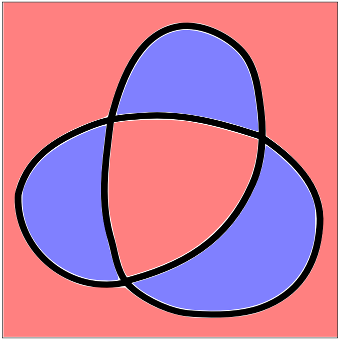

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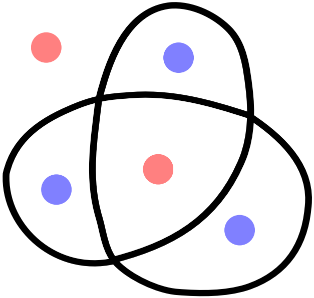

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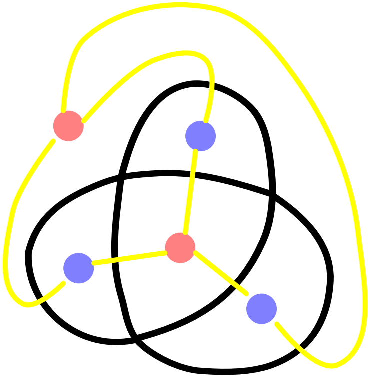

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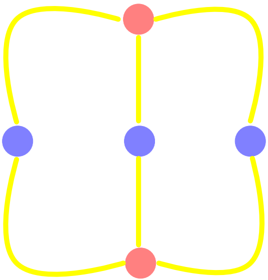
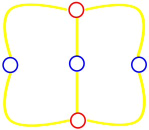
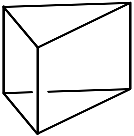


Motivation 2

$$\text{bipartite graph} \mapsto \text{root polytope}$$





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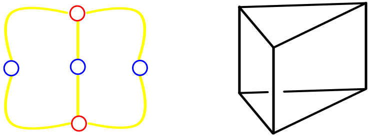

Question 1.2

Is there any relationship between the link and the root polytope?

Colored Jones Polynomial

The colored Jones polynomial is a sequence of Laurent polynomials in one variable with integer coefficients. It is indexed by a positive integer, called color.
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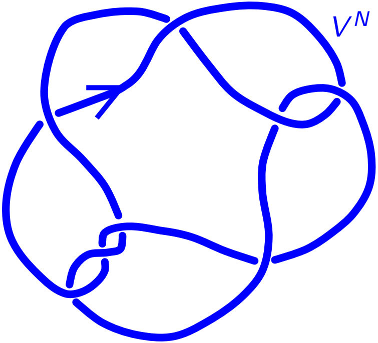
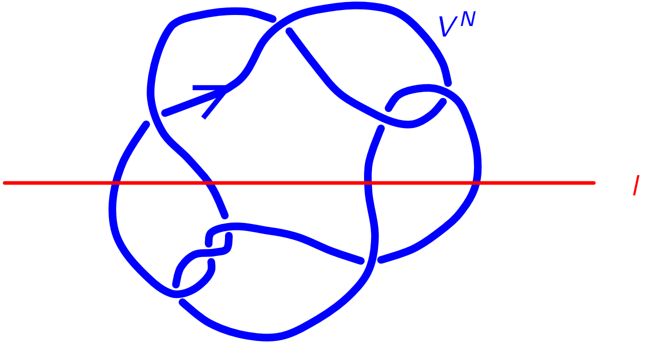
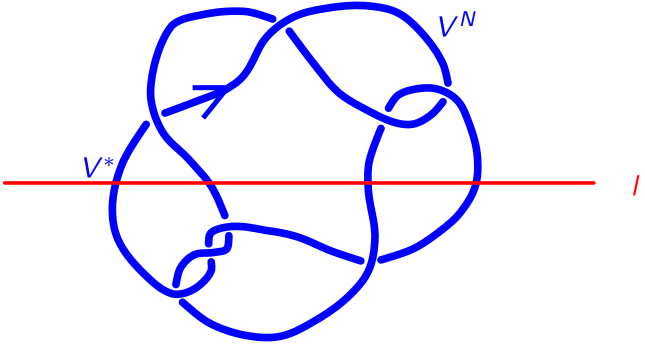
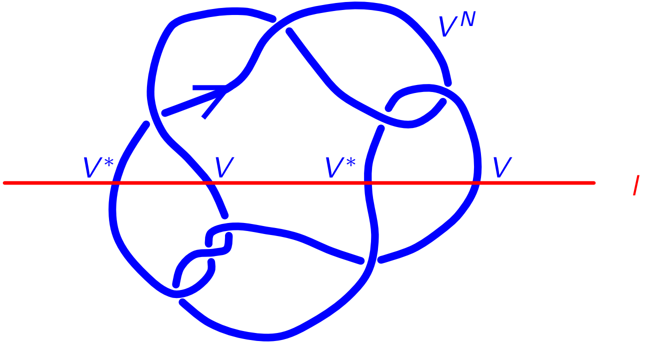
Remark 2.1

It is known that the 2-colored Jones polynomial coincides with the ordinary Jones polynomial.


V^N


V^N
l


V^N
V^*
l


V^N
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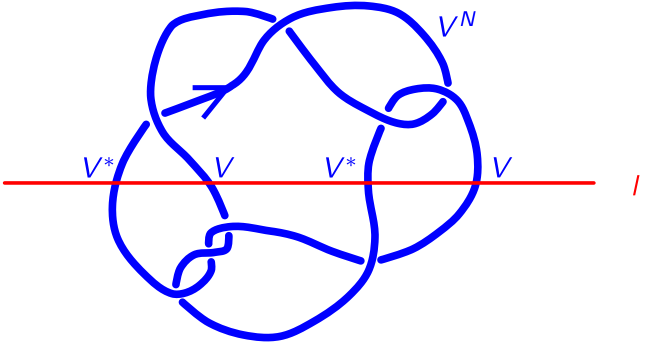
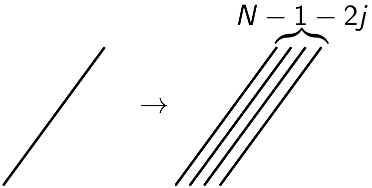


$$V_l = V^* \otimes V \otimes V^* \otimes V$$

Theorem 2.2

Let K be a framed knot. Then,

$$J_{K,N}(q) = \sum_{j=0}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^j \binom{N-1-j}{j} J_{K^{N-1-2j},2}(q).$$



Second Root of Unity

The results in this section hold for arbitrary knots.

Theorem 3.1 (M.)

Let K be a 0-framed knot. Then

$$|J_{K,N}(-1)| = \begin{cases} 1 & (N \text{ is an odd number}) \\ \det(K) & (N \text{ is an even number}) \end{cases}$$

Proof.

By the cabling formula, we have

$$
\begin{aligned}
|J_{K,N}(-1)| &= \left| \sum_{j=0}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^j \binom{N-1-j}{j} J_{K^{N-1-2j},2}(-1) \right| \\
&= \left| \sum_{j=0}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^j \binom{N-1-j}{j} \Delta_{K^{N-1-2j}}(-1) \right|
\end{aligned}
$$

By a well-known formula for the Alexander polynomial

$$
\Delta_{K^{N-1-2j}}(-1) = \Delta_K((-1)^{N-1-2j}) \cdot \Delta_{T_{N-1-2j,0}}(-1),
$$

Proof.

we obtain

$$\Delta_{K^{N-1-2j}}(-1) = \begin{cases} 1 & (N-1-2j=0) \\ \Delta_K(-1) & (N-1-2j=1) \\ 0 & (N-1-2j \neq 0,1) \end{cases}$$

Using this result,

$$\begin{aligned} |J_{K,N}(-1)| &= \left| \sum_{j=0}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^j \binom{N-1-j}{j} \Delta_{K^{N-1-2j}}(-1) \right| \\ &= \left| \sum_{\substack{j \in \mathbb{Z}_{\geq 0} \\ N-1-2j=0 \text{ or } 1}} (-1)^j \binom{N-1-j}{j} \Delta_{K^{N-1-2j}}(-1) \right| \end{aligned}$$

Proof.

$$\therefore |J_{K,N}(-1)| = \begin{cases} 1 & (N : \mathrm{odd}) \\ \Delta_K(-1) & (N : \mathrm{even}) \end{cases}$$

□

Theorem 3.2 (M.)

Let K be a knot. Then we have

$$J_{K,N}(e^{\frac{2}{3}\pi\sqrt{-1}}) = \begin{cases} 0 & (N = 6l) \\ 1 & (N = 6l+1) \\ 1 & (N = 6l+2) \\ 0 & (N = 6l+3) \\ -1 & (N = 6l+4) \\ -1 & (N = 6l+5) \end{cases}$$

Proof.

By the cabling formula, we have

$$J_{K,N}\left(e^{\frac{2}{3}\pi\sqrt{-1}}\right) = \sum_{j=0}^{\lfloor\frac{N-1}{2}\rfloor}(-1)^j\binom{N-1-j}{j}J_{K^{N-1-2j},2}\left(e^{\frac{2}{3}\pi\sqrt{-1}}\right)$$

$$= \sum_{j=0}^{\lfloor\frac{N-1}{2}\rfloor}(-1)^j\binom{N-1-j}{j}.$$

Proof.

So it suffices to show that

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^j\binom{n-j}{j} = \begin{cases} 1 & (n=6l) \\ 1 & (n=6l+1) \\ 0 & (n=6l+2) \\ -1 & (n=6l+3) \\ -1 & (n=6l+4) \\ 0 & (n=6l+5), \end{cases}$$

where $n = N - 1$.

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So it suffices to show that

$$a_n = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} = \begin{cases} 1 & (n = 6l) \\ 1 & (n = 6l+1) \\ 0 & (n = 6l+2) \\ -1 & (n = 6l+3) \\ -1 & (n = 6l+4) \\ 0 & (n = 6l+5), \end{cases}$$

where $n = N - 1$. Let us denote the left hand side with a_n.

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Idea 1

Constructing a generating function. Namely, a formal power series $f(x)$ in which the coefficient of x^n is a_n.

Proof.

Seeing that

$$a_n = \binom{n}{0} - \binom{n-1}{1} + \cdots + (-1)^{\lfloor \frac{n}{2} \rfloor} \binom{n - \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor}$$

and

$$\begin{aligned}
(1-x)^0 &= \binom{0}{0} \\
(1-x)^1 &= \binom{1}{0} - \binom{1}{1}x \\
(1-x)^2 &= \binom{2}{0} - \binom{2}{1}x + \binom{2}{2}x^2 \\
(1-x)^3 &= \binom{3}{0} - \binom{3}{1}x + \binom{3}{2}x^2 - \binom{3}{3}x^3 \\
&\vdots
\end{aligned}$$

Proof.

we can slide these terms by multiplying x^n with $(1-x)^n$:

$$\begin{aligned}
x^0(1-x)^0 &= \binom{0}{0} \\
x^1(1-x)^1 &= \binom{1}{0}x - \binom{1}{1}x^2 \\
x^2(1-x)^2 &= +\binom{2}{0}x^2 - \binom{2}{1}x^3 + \binom{2}{2}x^4 \\
&\vdots
\end{aligned}$$

That is,

$$f(x) := \sum_{n=0}^{\infty} x^n(1-x)^n$$

is nothing but a generating function of a_n.

Proof.

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} x^n (1-x)^n \\
&= \frac{1}{1-x(1-x)} \\
&= \frac{1+x}{1+x^3} \\
&= (1+x)\sum_{k=0}^{\infty}(-1)^k x^{3k} \\
&= \sum_{k=0}^{\infty}(-1)^k (x^{3k}+x^{3k+1}) \\
&= \sum_{l=0}^{\infty}(x^{6l}+x^{6l+1}+0x^{6l+2}-x^{6l+3}-x^{6l+4}+0x^{6l+5})
\end{aligned}
$$

□

Definitions

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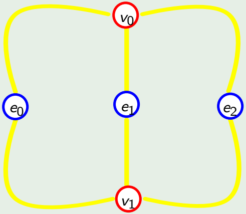
Definition 4.2 (Root Polytope)

Let G be a bipartite graph with color classes E and V. For $e \in E$ and $v \in V$, let $\mathbf{e}$ and $\mathbf{v}$ denote the standard generators of $\mathbb{R}^E \oplus \mathbb{R}^V$. Then the root polytope of G is defined to be

$$Q_G = \mathrm{Conv}\{\mathbf{e} + \mathbf{v} \mid ev \text{ is an edge of } G\}.$$

Example 4.3



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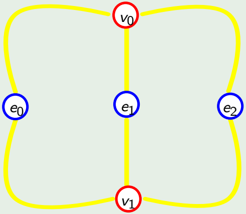

$\mathbf{e}_0 + \mathbf{v}_0$

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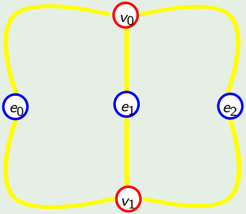

$\mathbf{e}_2 + \mathbf{v}_1$ $\mathbf{e}_1 + \mathbf{v}_1$

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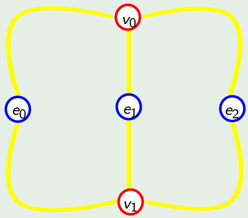
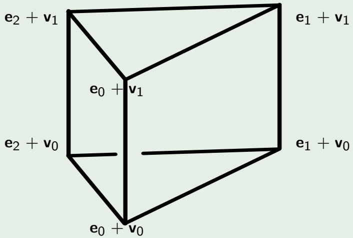

v_0
e_0
e_1
e_2
v_1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Triangulation

Definition 4.4

A collection of maximal simplices in Q_G is called a triangulation if their union is Q_G and the intersection of any two simplices is their common face.

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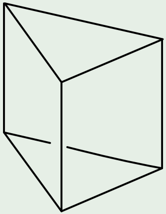

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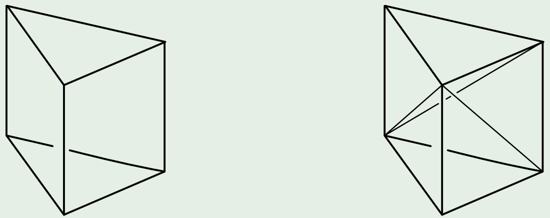

Definition 4.6 (hyper graph)

A hyper graph is a pair $\mathscr{H} = (V, E)$ of disjoint sets, where elements of E is a non-empty sebsets of any cardinality of V.

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$$\begin{array}{lcl} V & : & \text{vertex set} \\ E & : & \text{hyperedge set} \end{array}$$

- A hypergraph is a generalization of a graph.
- We obtain a bipartite graph $\mathrm{Bip}\,\mathscr{H}$ from a given hypergraph $\mathscr{H}$.
- A bipartite graph and hypergraphs are in one-to-two correspondence. We call such a bipartite graph associated bipartite graph.

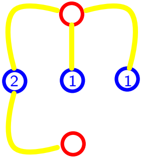
$\mathscr{H} = (V, E)$: a hypergraph so that Bip $\mathscr{H}$ is connected.

Definition 4.7 (hypertrees)

A hypertree in $\mathscr{H}$ is defined to be a function $f : E \to \mathbb{N} = \{0, 1, \dots\}$ so that a spanning tree of Bip $\mathscr{H}$ can be found with valence $f(e) + 1$ at each $e \in E$.

Example 4.8

Let $\mathscr{H} = (V, E)$ be a hypergraph whose vertex set is $V = \{v_0, v_1\}$ and hyperedge set $E = \{e_0, e_1, e_2\}$.
From a hypertree $\big(f(e_0), f(e_1), f(e_2)\big) = (1, 0, 0)$, we obtain the following spanning tree of Bip $\mathscr{H}$.



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$||$ (Postnikov)

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Theorem 4.9

When we deal with the bipartite graph G_D, the number of hypertrees coincide with the determinant of the alternating knot whose diagram is D. Namely,

the number of hypertrees of $\mathscr{H}$ ($\mathscr{H}$ is induced by G_D)
= the number of Kauffman states for D
= determinant of K.

Theorem 1 (M.)

The volume of the root polytope is equal to the determinant of the link if the given link is alternating.