A large complete graph in a space contains a link with large link invariant

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Abstract Let k be a non-negative integer. Then any embedding of the complete graph on $6\cdot 2^k$ vertices into a three-space contains a two-component link with the absolute value of its linking number greater than or equal to 2^k. Let j be a non-negative integer. Then any embedding of the complete graph on $48\cdot 2^j$ vertices into a three-space contains a knot with the absolute value of the second coefficient of its Conway polynomial greater than or equal to 2^{2j}.

2000 Mathematics Subject Classification. Primary 57M25, Secondary 57M15, 05C10.

Keywords and phrases. linking number, second coefficient of Conway polynomial, spatial graph, complete graph, intrinsic knotting, intrinsic linking.

§1. Introduction

Throughout this paper we work in the piecewise linear category. Let m be a natural number. By K_m we denote the complete graph on m vertices. Let G be a finite graph. A *cycle of* G is a subgraph of G that is homeomorphic to a circle. A cycle on n vertices is called an *n-cycle*. Let $L = J_1 \cup J_2$ be an oriented two-component link in the three-dimensional Euclidean space R^3. By $lk(L) = lk(J_1, J_2)$ we denote the linking number of L. Even when $L = J_1 \cup J_2$ is not oriented the absolute linking number $|lk(L)| = |lk(J_1, J_2)|$ is well-defined. Let J be a knot in R^3. By $a_2(J)$ we denote the second coefficient of the Conway polynomial of J.

In [1] Conway and Gordon showed the following theorem.

Theorem 1. [1] (1) *Let $f : K_6 \to R^3$ be an embedding. Then there are mutually disjoint 3-cycles γ_1 and γ_2 of K_6 such that $|lk(f(\gamma_1), f(\gamma_2))| \geq 1$.*

(2) *Let $f : K_7 \to R^3$ be an embedding. Then there is a 7-cycle γ of K_7 such that $|a_2(f(\gamma))| \geq 1$.*

In [2] Flapan extended this result as follows.

Theorem 2. [2] (1) *Let n be a natural number. Let $m = n(15n - 9)$. Let $f : K_m \to R^3$ be an embedding. Then there are mutually disjoint cycles γ_1 and γ_2 of K_m such that $|lk(f(\gamma_1), f(\gamma_2))| \geq n$.*

(2) *Let n be a natural number. Let p be a natural number with $p \geq 4\sqrt{n}$, and let $m = 2p(15p - 9)$. Let $f : K_m \to R^3$ be an embedding. Then there is a cycle γ of K_m such that $|a_2(f(\gamma))| \geq n$.*

By Theorem 2 (1) we have that for any natural number n there is a natural number m such that every embedding of K_m into R^3 contains a link whose absolute linking number is greater than or equal to n. In [2] m is given by $m = n(15n - 9)$. In this paper we show that $m = 12n$ is sufficient. By Theorem 2 (2) we have that for any natural number n there is a natural number m such that every embedding of K_m into R^3 contains a knot with the absolute value of the second coefficient of its Conway polynomial greater than or equal to n. In [2] m satisfies $m = 2p(15p - 9) \geq 8\sqrt{n}(60\sqrt{n} - 9) = 480n - 72\sqrt{n}$. In this paper we show that $m \geq 96\sqrt{n}$ is sufficient. Namely we show the following results.

Theorem 3. *Let k be a non-negative integer. Let $f : K_{6\cdot 2^k} \to R^3$ be an embedding. Then there are mutually disjoint $3\cdot 2^k$-cycles γ_1 and γ_2 of $K_{6\cdot 2^k}$ such that $|lk(f(\gamma_1), f(\gamma_2))| \geq 2^k$.*

Corollary 4. *Let n be a natural number. Let $f : K_{12n} \to R^3$ be an embedding. Then there are mutually disjoint cycles γ_1 and γ_2 of K_{12n} such that $|lk(f(\gamma_1), f(\gamma_2))| \geq n$.*

Theorem 5. *Let k be a non-negative integer. Let $f : K_{48\cdot 2^k} \to R^3$ be an embedding. Then there is a cycle γ of $K_{48\cdot 2^k}$ such that $|a_2(f(\gamma))| \geq 2^{2k}$.*

Corollary 6. *Let n be a natural number. Let m be a natural number with $m \geq 96\sqrt{n}$. Let $f : K_m \to R^3$ be an embedding. Then there is a cycle γ of K_m such that $|a_2(f(\gamma))| \geq n$.*

Remark. Let n be a natural number. Then we have shown that there is a natural number m such that every every embedding of K_m contains a link whose absolute linking number is greater than or equal to n. The problem of deciding the smallest such m is still open. A similar problem for the second coefficient of the Conway polynomial of a knot is also still open. See [2].

Historical remark. The authors started this work without knowing Flapan's work [2]. The authors have shown Theorem 3 and Corollary 4 before knowing Flapan's work. Though the proof of Theorem 3 looks similar to that in [2], the proof is done independently. Then the authors have noticed Flapan's work. Then the authors have shown Theorem 5 and Corollary 6. The proof of Theorem 5 is essentially the same as that of Theorem 2 (2) in [2]. The proof of Theorem 2 (2) in [2] is based on Theorem 2 (1). The proof of Theorem 5 is based on Theorem 3. This is the only difference between them.

§2. Proofs

Lemma 7. *Let n and l be natural numbers with $2 \leq n < l$. Let G_l be a graph whose vertices are $v_1, v_2, \ldots, v_l$, $u_1, u_2, \ldots, u_l$ and whose edges are $v_1v_2, v_2v_3, \ldots, v_{l-1}v_l, v_lv_1$, $u_1u_2, u_2u_3, \ldots, u_{l-1}u_l, u_lu_1$, $v_1u_1, v_2u_2, \ldots, v_lu_l$. Let α be the l-cycle containing $v_1, v_2, \ldots, v_l$ and β the l-cycle containing $u_1, u_2, \ldots, u_l$. We give orientations to α and β by the cyclic orderings $v_1, v_2, \ldots, v_{l-1}, v_l, v_1$ and $u_1, u_l, u_{l-1}, \ldots, u_2, u_1$ respectively. Let $f : G_l \to R^3$ be an embedding. Let L be an oriented link in $R^3 - f(G_l)$. Suppose that the total linking number $lk(f(\alpha \cup \beta), L) \geq n$. Then there is an oriented $2l$-cycle γ of G_l such that $lk(f(\gamma), L) \geq n$.*

Proof. Let c_i be the 4-cycle of G_l containing $v_i, u_i, v_{i+1}, u_{i+1}$ for $i = 1, 2, \ldots, l-1$ and let c_l be the 4-cycle of G_l containing v_l, u_l, v_1, u_1. We give orientations to these 4-cycles so that $\alpha + \beta + c_1 + c_2 + \cdots + c_l = 0$ as elements of the first homology group of G. Then we have $lk(f(\alpha \cup \beta), L) + lk(f(c_1), L) + lk(f(c_2), L) + \cdots + lk(f(c_l), L) = 0$. We will show that there exists $i \in \{1, 2, \ldots, l\}$ such that $lk(f(c_i), L) \geq n - lk(f(\alpha \cup \beta), L)$. Suppose contrary that $lk(f(c_i), L) \leq n - lk(f(\alpha \cup \beta), L) - 1$ for each $i \in \{1, 2, \ldots, l\}$. Then we have

$$\begin{aligned}
0 &= lk(f(\alpha \cup \beta), L) + lk(f(c_1), L) + lk(f(c_2), L) + \cdots + lk(f(c_l), L) \\
&\leq lk(f(\alpha \cup \beta), L) + l(n - lk(f(\alpha \cup \beta), L) - 1) \\
&\leq lk(f(\alpha \cup \beta), L) + (n+1)(n - lk(f(\alpha \cup \beta), L) - 1) = n(n - lk(f(\alpha \cup \beta), L)) - 1 < 0.
\end{aligned}$$

This is a contradiction. Thus we have that there exists $i \in \{1, 2, \ldots, l\}$ such that $lk(f(c_i), L) \geq n - lk(f(\alpha \cup \beta), L)$. Let $\gamma = \alpha + \beta + c_i$. Then γ is an oriented $2l$-cycle such that $lk(f(\gamma), L) = lk(f(\alpha \cup \beta), L) + lk(f(c_i), L) \geq n$. This completes the proof. □

Proof of Theorem 3. The proof is given by an induction on k. The case $k = 0$ is just Theorem 1 (1). Suppose that it is shown for k. Let $f : K_{6\cdot 2^{k+1}} \to R^3$ be an embedding. Let H_1, H_2 be mutually disjoint subgraphs of $K_{6\cdot 2^{k+1}}$ each of which is isomorphic to $K_{6\cdot 2^k}$. Then by the assumption there are mutually disjoint $3 \cdot 2^k$-cycles $\beta_1, \beta_2, \beta_3, \beta_4$ with $\beta_1 \cup \beta_2 \subset H_1$ and $\beta_3 \cup \beta_4 \subset H_2$ such that $|lk(f(\beta_1), f(\beta_2))| \geq 2^k$ and $|lk(f(\beta_3), f(\beta_4))| \geq 2^k$. Then it is easy to see that we can choose orientations of these cycles so that $lk(f(\beta_1 \cup$

$\beta_3), f(\beta_2 \cup \beta_4)) \geq 2^{k+1}$.

Then we take a subgraph H of $K_{6\cdot 2^{k+1}}$ that is isomorphic to $G_{3\cdot 2^k}$ in Lemma 7 so that β_1 and β_3 correspond to α and β in Lemma 7. We set $L = f(\beta_2 \cup \beta_4)$ and apply Lemma 7. Then we have an oriented $3 \cdot 2^{k+1}$-cycle $\gamma \subset H$ such that $lk(f(\gamma), L) \geq 2^{k+1}$.

Next we set $L' = f(\gamma)$ and take a subgraph H' of $K_{6\cdot 2^{k+1}}$ that is isomorphic to $G_{3\cdot 2^k}$ in Lemma 7 so that β_2 and β_4 correspond to α and β in Lemma 7. Then we apply Lemma 7 again and have an oriented $3 \cdot 2^{k+1}$-cycle $\gamma' \subset H'$ such that $lk(f(\gamma), f(\gamma')) \geq 2^{k+1}$. This completes the proof. □

Proof of Corollary 4. Let n be a natural number and $f : K_{12n} \to R^3$ an embedding. Then there is an integer k such that $2^{k-1} < n \leq 2^k$. Then we have $2^k < 2n$ and therefore $6 \cdot 2^k < 12n$. Therefore K_{12n} contains a subgraph that is isomorphic to $K_{6\cdot 2^k}$. Therefore by Theorem 3 we have that there are mutually disjoint cycles γ_1 and γ_2 of K_{12n} such that $|lk(f(\gamma_1), f(\gamma_2))| \geq 2^k \geq n$. This completes the proof. □

Let G_4 and G_8 be graphs as illustrated in Fig. 1.

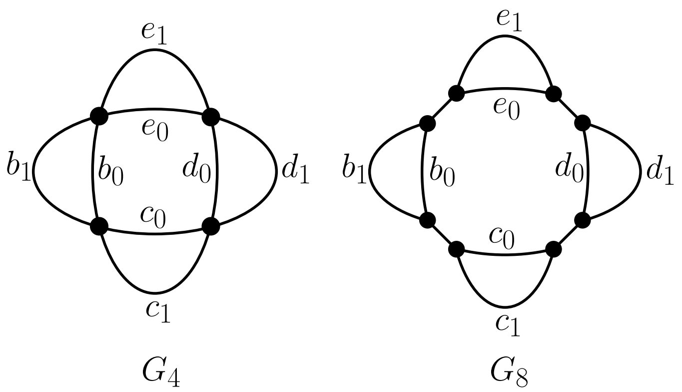

Fig. 1

Let $\Gamma(G_4)$ and $\Gamma(G_8)$ be the set of all 4-cycles of G_4 and the set of all 8-cycles of G_8 respectively. Let $i, j, k, l \in \{0, 1\}$. By $b_i c_j d_k e_l$ we denote the 4-cycle of G_4 or the 8-cycle of G_8 that contains the edges b_i, c_j, d_k and e_l. For each such cycle we define

$$\varepsilon(b_i c_j d_k e_l) = \begin{cases} 1 & \text{if } i+j+k+l \text{ is even,} \\ -1 & \text{if } i+j+k+l \text{ is odd.} \end{cases}$$

Let $f : G_4 \to R^3$ be an embedding. Let $\lambda(f)$ be an integer defined by

$$\lambda(f) = |\sum_{\gamma \in \Gamma(G_4)} \varepsilon(\gamma) a_2(f(\gamma))|.$$

Let $f : G_8 \to R^3$ be an embedding. Let $\lambda(f)$ be an integer defined by

$$\lambda(f) = |\sum_{\gamma \in \Gamma(G_8)} \varepsilon(\gamma) a_2(f(\gamma))|.$$

Let B, C, D, and E denote the 2-cycles $b_0 \cup b_1$, $c_0 \cup c_1$, $d_0 \cup d_1$, and $e_0 \cup e_1$ respectively. It is shown in [3] that for any embedding $f : G_4 \to R^3$ we have the equation

$$\lambda(f) = |lk(f(B), f(D)) lk(f(C), f(E))|.$$

By contracting the four edges of G_8 we also have the same equation for any embedding $f : G_8 \to R^3$. Since $\Gamma(G_8)$ contains 16 cycles we have that

$$16\max\{|a_2(f(\gamma))||\gamma \in \Gamma(G_8)\} \geq \lambda(f) = |lk(f(B), f(D)) lk(f(C), f(E))|.$$

Therefore we have that

$$\max\{|a_2(f(\gamma))||\gamma \in \Gamma(G_8)\} \geq \frac{1}{16}|lk(f(B), f(D)) lk(f(C), f(E))|.$$

Proof of Theorem 5. Let H_1 and H_2 be mutually disjoint subgraphs of $K_{48 \cdot 2^k}$ each of which is isomorphic to $K_{24 \cdot 2^k} = K_{6 \cdot 2^{k+2}}$. Let $f : K_{48 \cdot 2^k} \to R^3$ be an embedding. Then by Theorem 3 there are mutually disjoint cycles γ_1 and γ_2 of H_1 and mutually disjoint cycles γ_3 and γ_4 of H_2 such that $|lk(f(\gamma_1), f(\gamma_2))| \geq 2^{k+2}$ and $|lk(f(\gamma_3), f(\gamma_4))| \geq 2^{k+2}$. Let H_3 be a subgraph of $K_{48 \cdot 2^k}$ that is isomorphic to G_8 that contains $\gamma_1, \gamma_3, \gamma_2$ and γ_4 in this cyclic order. Then we have that there is a cycle γ of H_3 such that $|a_2(f(\gamma))| \geq \frac{1}{16} 2^{k+2} \cdot 2^{k+2} = 2^{2k}$. This completes the proof. □

Proof of Corollary 6. Let n be a natural number. Let m be a natural number with $m \geq 96\sqrt{n}$ and let $f : K_m \to R^3$ be an embedding. Then there is a non-negative integer k such that $2^{2(k-1)} < n \leq 2^{2k}$. Then we have $2^{k-1} < \sqrt{n}$. Therefore we have $48 \cdot 2^k = 96 \cdot 2^{k-1} < 96\sqrt{n}$. Therefore K_m contains a subgraph isomorphic to $K_{48 \cdot 2^k}$. Then by Theorem 5 we have that there is a cycle γ of K_m such that $|a_2(f(\gamma))| \geq 2^{2k} \geq n$. This completes the proof. □

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