ON THE HOMOMORPHISM INDUCED BY REGION CROSSING CHANGE

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1. Introduction

In 2010, Ayaka Shimizu [1] et al. defined a new operation on link diagram on the 2-sphere S^2. Let D be a link diagram on S^2. Then $|D|$ denotes the 4-valent graph obtained from D by replacing each crossings with a vertex. Then each component of $S^2 \setminus |D|$ is called a *region* of D. For a region R of D, $D(R)$ denotes the link diagram obtained from D by changing at all the crossings on the boundary of the R (Figure 1). In general, for each set $H = \{R_{i_1}, \ldots, R_{i_s}\}$ of regions of D, the link diagram obtained from D by *region crossing change* at H is defined to be $(\cdots(D(R_{i_1}))(R_{i_2})\cdots)(R_{i_s})$. It is known that this is well-defined, that is, the obtained link diagram is independent from the orders of the regions $R_{i_1}, \ldots, R_{i_s}$. For a convenience of the accessibility of H, we geometrically represent H on the diagram D by shading the regions which belong to H, and we call such representative a *coloring* of D.

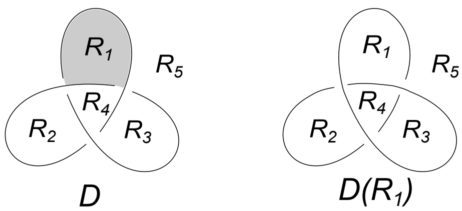


Figure 1

It is natural to ask what kind of link diagrams are obtained by the region crossing change on the given diagram. In [1] it is shown that for any knot diagram D we have: for any set of crossings of D there is a set of regions H such that exactly the crossings are changed by the region crossing change at H. On the other hand, it is remarked in [1] that 2-component trivial link diagram is not obtained from the link diagram of Figure 2 by any region crossing change.

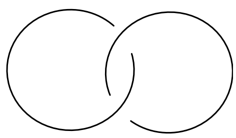


Figure 2

In this paper, for given link diagram, we study the sets of regions such that each of which gives trivial transformation of the link diagram, that is, the region crossing change at each set does not change the diagram at all.

Let D be a link diagram, and c a crossing of D. The crossing c is a *reducible* if there exists a circle E on S^2 which intersects D in only in c. We say that D is *irreducible* if it contains no reducible crossing.

Observation 1. *Let D be a link diagram, and $H = \{R_{i_1}, \cdots, R_{i_s}\}$ a set of regions of D. A crossing p is not changed by the region crossing change at H, if and only if the number of the regions that are adjacent to p is even. More precisely, in the case when p is an irreducible crossing, a neighborhood of p looks as one of Figure 3. Furthermore, in the case when p is a reducible crossing (as Figure 4) H looks as one of Figure 4.*

Figure 3

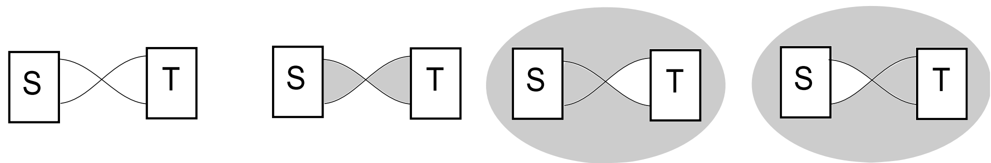


Figure 4

For the statement of the main result, we prepare some terminologies and results.

Definition 1. *We say that a coloring of D is a checker board coloring of D if the coloring satisfies the following conditions.*

(1) *For a small disk neighborhood of each crossing, $|D|$ divides it into four parts. Then one pair of part in opposition is shaded and the other is non-shaded,as in Figure 5(a).*
(2) *For any point p on $|D| \setminus \{crossings\}$, a small disk neighborhood of p is divided into two parts by $|D|$. Then one part is shaded and the other is non-shaded as in Figure 5(b).*

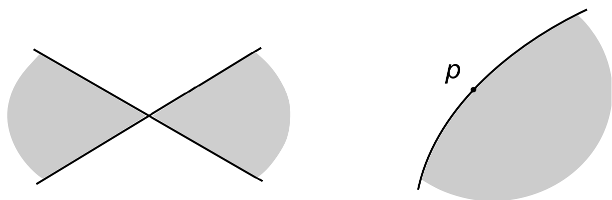


Figure 5

Remark 1. *It is well known that each link diagram admits exactly two checker board colorings.*

Let $D = k_1 \cup \cdots \cup k_n$ be a link diagram of an n-component link, where each k_i represents a component of the link. We fix a point ∞ in $S^2 \setminus |D|$. Then let $\mathcal{R}_i$ be the set of the regions of the knot diagram k_i and $\mathcal{R}$ the set of the regions of D. In general, for a set X, 2^X denotes the power set of X. Then ξ_i denotes the natural map $2^{\mathcal{R}_i} \to 2^{\mathcal{R}}$.

Definition 2. *Let $D = k_1 \cup \cdots \cup k_n$, $\mathcal{R}_i$, $\mathcal{R}$, ξ_i be as above. A coloring of D is called a componentwise checker board coloring associated with k_i, if the coloring is the image of a checker board coloring of k_i by ξ_i. Then H_i denotes the componentwise checker board coloring associated with k_i, such that the region containing ∞ is not shaded.*

Remark 2. *Suppose that D is irreducible. Then, by Observation 1, we see that, for each i, the region crossing change at H_i gives the trivial transformation of the link diagram D.*

Let $\mathcal{C} = \{c_1, \cdots, c_m\}$ be the set of the crossings of D. By using Euler characteristic, we see that the number of the regions of D is $m + 2$. Then $R_1, \ldots, R_{m+2}$ denote the regions of D. Recall that $2^{\mathcal{R}}$ denotes the power set of $\mathcal{R}$ and $2^{\mathcal{C}}$ denotes the power set of $\mathcal{C}$. We introduce an addition on $2^{\mathcal{R}}$ and $2^{\mathcal{C}}$ by symmetric difference, that is, if A and B are elements of $2^{\mathcal{R}}$ (resp. $2^{\mathcal{C}}$), then the addition of A and B denoted by $A + B$ is given by: $A + B = (A \setminus B) \cup (B \setminus A)$. Furthermore scalar multiplication by $\mathbf{Z}_2$ is defined by: $0 \cdot A = \emptyset$, $1 \cdot A = A$ for each $A \in 2^{\mathcal{R}}$(resp.$2^{\mathcal{C}}$). It is easy to show that these structures make $2^{\mathcal{R}}$(resp. $2^{\mathcal{C}}$) a $\mathbf{Z}_2$-linear space, where $\{R_1\}, \ldots, \{R_{m+2}\}$(resp. $\{c_1\}, \ldots, \{c_m\}$) is a basis of $2^{\mathcal{R}}$(resp. $2^{\mathcal{C}}$). Next we define a map φ from $2^{\mathcal{R}}$ to $2^{\mathcal{C}}$. For a subset H of $\mathcal{R}$, we define $\varphi(H)$ to be the set consisting of the crossings which are changed by the region crossing change at H. It is easy to see that the map φ is a $\mathbf{Z}_2$-linear map. We note that $\ker \varphi$ consists of the sets of regions such that the region crossing change at each set gives the trivial transformation.Then the main result of this paper is:

Theorem 1. *Let $D = k_1 \cup \cdots \cup k_n$, $\mathcal{R}$, $\mathcal{C}$, $\varphi : 2^{\mathcal{R}} \to 2^{\mathcal{C}}$, $H_1, \ldots, H_n$ be as above. Suppose that D is irreducible. Then, $\{H_1, \ldots, H_n, \mathcal{R}\}$ is a basis of* $\ker \varphi$.

2. Incidence matrix

Throughout this section, let $D = k_1 \cup \cdots \cup k_n$, $\mathcal{R} = \{R_1, \ldots, R_{m+2}\}$, $\mathcal{C} = \{c_1, \ldots, c_m\}$, $\varphi : 2^{\mathcal{R}} \to 2^{\mathcal{C}}$, $H_1, \cdots, H_n$ be as Section 1. Note that m is the number of the crossings of D. The *incidence matrix* $A(D)$ is the $(m+2) \times m$-matrix whose (i, j)-entry is given by:

$$m_{ij} = \begin{cases} 1 & \text{if } c_j \in \partial R_i \\ 0 & \text{if } c_j \notin \partial R_i \end{cases}$$

(We note that $A(D)$ is the same as the incidence matrix defined by Cheng and Gao in [2] up to permutations of rows and columns.) Then in [2] (see, also [3]) the following is shown.

Theorem 2. *Let D, $A(D)$, m be as above. Then the number of the components of D is given by $m - \mathrm{rank}_{\mathbf{Z_2}} A(D) + 1$.*

A representative of φ

Let $f_{\mathcal{R}}; 2^{\mathcal{R}} \to \mathbf{Z_2}^{m+2}$ be the isomorphism given by;

$$\begin{aligned} f_{\mathcal{R}}(\{R_1\}) &= (1,0,\cdots,0) \in \mathbf{Z_2}^{m+2} \\ f_{\mathcal{R}}(\{R_2\}) &= (0,1,0\cdots,0) \in \mathbf{Z_2}^{m+2} \\ &\vdots \\ f_{\mathcal{R}}(\{R_{m+2}\}) &= (0,\cdots,0,1) \in \mathbf{Z_2}^{m+2} \end{aligned}$$

Let $f_{\mathcal{C}}; 2^{\mathcal{C}} \to \mathbf{Z_2}^m$ be the isomorphism given by;

$$\begin{aligned} f_{\mathcal{C}}(\{c_1\}) &= (1,0,\cdots,0) \in \mathbf{Z_2}^{m} \\ f_{\mathcal{C}}(\{c_2\}) &= (0,1,0\cdots,0) \in \mathbf{Z_2}^{m} \\ &\vdots \\ f_{\mathcal{C}}(\{c_m\}) &= (0,\cdots,0,1) \in \mathbf{Z_2}^{m} \end{aligned}$$

Then $\psi : \mathbf{Z_2}^{m+2} \to \mathbf{Z_2}^m$ denotes the homomorphism $f_{\mathcal{C}} \circ \varphi \circ (f_{\mathcal{R}})^{-1}$. It is clear $A(D)$ represents the linear map ψ which is equivalent to φ. By Homomorphism Theorem in linear algebra this fact together with Theorem 2 implies

$$\begin{aligned} \dim_{\mathbf{Z_2}} \ker \varphi &= (m+2) - \mathrm{rank}_{\mathbf{Z_2}} A(D) \\ &= (m+2) - (m-n+1) \\ &= n+1, \end{aligned} \tag{1}$$

where n is the number of the component of D.

3. Proof of theorem

Lemma 1. *Let $D = k_1 \cup \cdots \cup k_n$, $\mathcal{R}$, $\mathcal{C}$, $\varphi : 2^{\mathcal{R}} \to 2^{\mathcal{C}}$, $H_1, \ldots, H_n$ be as in Section 1. $\{H_1, \ldots, H_n, \mathcal{R}\}$ is linearly independent.*

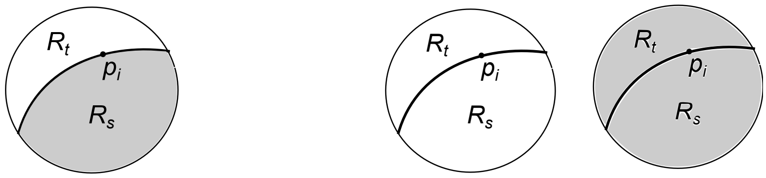
Proof. Suppose that the following equation is satisfied.

$$\varepsilon_1 H_1 + \varepsilon_2 H_2 + \cdots + \varepsilon_n H_n + \varepsilon_{n+1} \mathcal{R} = 0 \qquad (\varepsilon_i \in \mathbf{Z_2}) \tag{2}$$

Recall that a point ∞ in $S^2 \setminus |D|$ is fixed, and the region containing ∞ in not shaded by H_i $(i = 1, \ldots, n)$ (Definition 2). On the other hand, the region containing ∞ is shaded by $\mathcal{R}$. These together with (2) implies $\varepsilon_{n+1} = 0$. Then, for each i $(= 1, \ldots, n)$, we take a point p_i in $|D| \setminus \{crossings\}$ such that $p_i \in k_i$. A small disk neighborhood of p_i is divided into two components by $|D|$. One component is contained in a region shaded by H_i, say R_s, and the other is not contained in any region shaded by H_i. Then R_t denotes the region containing the other component. On the other hand, we see that both of the regions R_s and R_t are shaded or non-shaded by $H_j (j \neq i, j \in \{1, \ldots, n\})$ by Definition 2 of componentwise checker board coloring. These facts together with (2) imply $\varepsilon_i = 0$. As the conclusion, we have shown that

$$\varepsilon_1 = \cdots = \varepsilon_n = 0$$

. Hence $\{H_1, \ldots, H_n, \mathcal{R}\}$ is linearly independent. □



The coloring of H_i near p_i The coloring of H_j near p_i

Figure 6

Now we give a proof Theorem 1. Suppose that D is irreducible. Then by Remark 2, $H_i \in \ker\varphi$ $(i = 1, \dots, n)$, and $\mathcal{R} \in \ker\varphi$. By Lemma 1, $\{H_1, \dots, H_n, \mathcal{R}\}$ is linearly independent. This fact together with (1) implies $\{H_1, \dots, H_n, \mathcal{R}\}$ is a basis of $\ker\varphi$. This completes the proof of Theorem 1.

References

[1] A.Shimizu. *Region crossing change is an unknotting operation*, to appear in Journal of the Mathematical Society of Japan.(arXiv:1011.6304)
[2] Z.Cheng, H.Gao. *On region crossing change and incidence matrix*, arXiv:1101.1129.
[3] Z.Cheng. *When is region crossing change an unknotting operation?*, arXiv:1201.1735.

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