COSMETIC SURGERY ON LINKS

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1. Introduction

These notes are based on a talk given in Waseda University in December 25, 2012 as a part of the conference "musubime no suugaku 5". The work is available at [RY12b] (see also [RY12a] and [RR12]). The main subject of these papers is the *link volume*; however, our work on the link volume led us to considering surgery on links (with arbitrarily many components). In these notes we explain this work for the special case of links in S^3.

Let $L \subset M$ be a link in a closed, orientable, connected 3-manifold. By *surgery* on L we mean removing $N(L)$ from M and reattaching it, perhaps in a different way. Here and throughout this note, $N(L)$ denotes open normal neighborhood of L. A surgery on $L \subset M$ is called *cosmetic* if the resulting manifold is homeomorphic to M.

By *filling* the manifold $X = M \setminus N(L)$ we mean attaching solid tori to its boundary; possibly not all components are filled. Hence filling of X is more general than surgery on L (for example, X is a result of filling X, when no component is filled, but is not the result of surgery on L unless $L = \emptyset$). Below we shall see why this more flexible definition is needed.

Surgery on L (respectively, filling of X) is determined by a choice of slope on each component of L (respectively, of ∂X). We can identify the slopes on a given component with $\mathbb{Q} \cup \{1/0\}$. In addition, we use the symbol ∞ when a component of ∂X is not filled.

Before discussing cosmetic surgery on links, let us consider cosmetic surgery on knots in S^3. In 1988 Gordon and Leucke [GL89] famously proved that knots are determined by their complements. The precise statement is that there are no cosmetic surgeries on knots in S^3:

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Theorem 1 (Gordon–Luecke)**.** *Let $K \subset S^3$ be a non-trivial knot and α a slope. If $K(\alpha) \cong S^3$, the $\alpha = 1/0$.*

In terms of fillings, the theorem can be phrased as:

Theorem 2 (Gordon–Luecke)**.** *Let ∂X be a compact manifold so that ∂X is a single torus. If $X \ncong D^2 \times S^1$, then there is at most one slope α so that $X(\alpha) \cong S^3$.*

We remark that this is just one theorem in a very large body of knowledge regarding surgery. In all cases, the following philosophy must be followed (in [brackets] we explain how this philosophy applies to the two theorems above):

Philosophy: if we rule out certain links [the unknot], we obtain a constraint on the slopes [only one].

Below we describe results that apply to surgery on links, following this philosophy.

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2. The Farey graph

In this section we discuss the Farey graph. One source the reader may consult is the classic book of Hardy and Wright [HW08]. The Farey graph is a combinatorial object, but it is best seen in the hyperbolic plane. Fix a hyperbolic triangle T (see Figure 2 where T is shaded). Label its endpoints (which are, of course, three points at the ideal boundary of the hyperbolic plane) as $1/0$, $0/1$, and $1/1$. By reflecting T in one of its edge we obtain a new ideal triangle and label the new vertex $-1/1$, $1/2$, or $2/1$. continuing in this way, we obtain a tessellation of the hyperbolic plane by ideal triangles. A given edge e is on the boundary of two triangles, say T_1 and T_2; if the endpoints of e are labeled p/q and r/s, then the two other vertices of T_1 and T_2 are labeled $(p+r)/(q+s)$ and $(p-r)/(q-s)$ (the sign is determined by orientation, but we will not discuss this here). The following is a very entertaining exercise; it is not very hard, but does require and *idea*:

Exercise 3. *Every point of $\mathbb{Q} \cup \{1/0\}$ appears as the label of exactly one vertex of the Farey graph.*

The Ferey graph is the collection of vertices and edges of this tessellation. Although we described the Farey graph as embedded in the hyperbolic plane and part of its ideal boundary, we consider it as a combinatorial object. By construction it is connected;

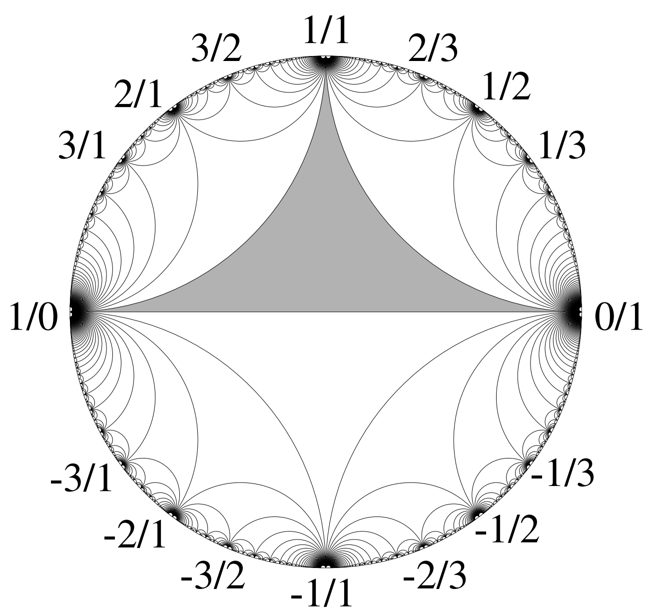

FIGURE 1. The Farey graph

hence we can get from any vertex (say labeled p/q) to any other vertex (say labeled r/s). The minimal number of edges needed to form a oath connecting them is called the *distance* between the two, denoted by

$$d(p/q, r/s).$$

Note that this is not at all the distance induced by the hyperbolic metric: the vertices are all on the ideal boundary and therefore have infinite distance in that sense. It is less obvious, but also true, that this distance is quite different from the distance in the circle: for example, the positive integers (that is, the points of the circle that are labeled $1/1$, $2/1$, $3/1$ and so on) converge to the point $1/0$ *as points of the circle*. Whoever, for every positive integer n,

$$d(n/1, 1/0) = 2.$$

Equipped with the function d, $\mathbb{Q} \cup \{1/0\}$ becomes a metric space. The following basic fact about the Farey is very important to our discussion, and is left as an exercise:

Exercise 4. *The Farey graph is unbounded.*

Our version of "small" sets of slopes are **bounded** sets. It follows from the exercise above that bounded set has an infinite complement.

3. SURGERY ON LINKS: SOME EXAMPLES

As we saw above, Gordon and Luecke show that a non-trivial knot in S^3 admits no non-trivial cosmetic surgery. For links, the situation is worse. Consider the Hopf link H (see Figure 3). Then $H(p/q, r/s) \cong S^3$ if and only if $ps - qr = \pm 1$. We see that if K is a component of H, then any slope of K can be completed to a cosmetic surgery (in fact, in infinitely many ways). Below we will assume that L is not the Hopf link. If L is a link and H is a sublink of L, we can obtain S^3 by filling $1/0$ on the components of $L \setminus H$ and p/q, r/s on H, as above.

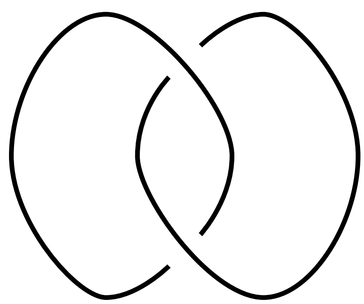

FIGURE 2. The Hopf Link

The Whitehead link W (see Figure 3) provides an interesting example. For all n, $W(1/0, 1/n) \cong S^3$. This is because the Whitehead link has an unknotted component. Below, we will *not* assume that L has no unknotted components. Instead of considering this infinite family, it would be better to consider their common multislope: $(1/0, \infty)$. (Recall that ∞ means *do not fill.*) We will return to this below, when we define minimally non hyperbolic fillings.

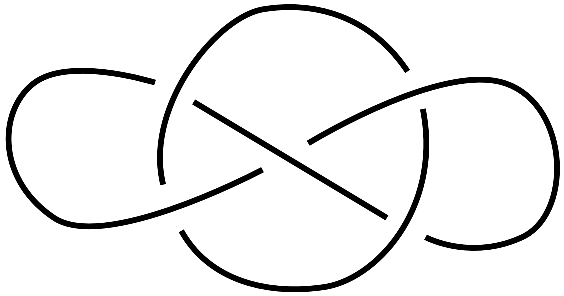

FIGURE 3. The Whitehead Link

4. STATEMENT OF THE MAIN RESULT

Let X be a compact manifold so that ∂X consists of the tori $T_1, \ldots, T_n$. A filling of X is determined by a *multislope*, that is, a vector of the form $(\alpha_1, \ldots, \alpha_n)$, where for each i, α_i is a slope on T_i. We allow that possibility $\alpha_i = \infty$ (in which case T_i is not filled).

We use the notation $\widehat{\alpha}_i$ to indicate that α_i is replaced by ∞. Our main result is:

Theorem 5. *Let X be as above. Let A be the set of multislopes of X fulfilling the following two conditions:*

(1) $\forall (\alpha_1, \ldots, \alpha_n) \in A, \quad X(\alpha_1, \ldots, \alpha_n) \cong S^3.$
(2) $\forall (\alpha_1, \ldots, \alpha_n) \in A$ *and* $(\forall i,\ 2 \leq i \leq n)$,

$$X(\widehat{\alpha}_1, \alpha_2, \ldots, \widehat{\alpha}_i, \ldots, \alpha_n) \ncong T^2 \times [0,1].$$

Then the set

$$\{\alpha_1 \mid (\exists \alpha_2, \ldots, \alpha_n)\ (\alpha_1, \ldots, \alpha_n) \in A\}$$

is bounded.

We refer the reader to [RY12b] for the proof; in the remainder of this note we will describe the main tools used in the proof, which is inductive.

Before that, let us elaborate a little on the statement of the theorem. In this theorem we consider all possible fillings that yield S^3 (condition (1)), subject to condition (2). Condition (2) can be understood in two ways: one way (as stated) say that is we fill all the components of ∂X *except* T_1 and T_i, the resulting manifold is not $T^2 \times [0,1]$. Equivalently, we consider all fillings of X that result in S^3 so that the cores of the solid tori attached to T_1 and T_i do not form a Hopf link. It is clear from the discussion in Section 3 that is Condition (2) is not satisfied, any slope on $_1$ can be completed to a multislope yielding

S^3. In other words, condition (2) was simply designed to ensure that the cores of the solid tori attached to X do not contain a Hopf sublink where the core of the solid torus attached to T_1 is one of its components.

5. PARTIAL FILLINGS AND MINIMALLY NON HYPERBOLIC FILLINGS

Recall the example of the Whitehead link W discussed in Section 3. In that example we saw that the multislope $\{(1/0,/1/n)\}_{n\in\mathbb{Z}}$ all yield S^3. Instead of considering the infinite set of multislopes $\{(1/0,/1/n)\}_{n\in\mathbb{Z}}$, it is more efficient to consider the single multislope $(1/0,\infty)$ which common to all the multislopes above in the sense that for all n

$$(1/0,\infty) = (1/0,\widehat{1/n}).$$

This motivates the following definition:

Definition 6. Let $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\alpha' = (\alpha'_1,\dots,\alpha'_n)$ be multislopes of $\partial X = T_1,\dots,T_n$. If for each i, $\alpha'_i \in \{\alpha_i,\infty\}$, we say that α' is a *partial filling* of α. We write $\alpha' \subset_{p.f.} \alpha$.

It is well known that the manifold $X = S^3 \setminus N(W)$ is hyperbolic. Note that both $X(1/0,1/n)$ and $X(1/0,\widehat{1/n}) = X(1/0,\infty)$ are not hyperbolic. In some sense, the latter is "simpler", as we fill fewer boundary components. We define a minimally non hyperbolic filling as "simplest" non hyperbolic filling in the sense, that is:

Definition 7. Suppose that X is hyperbolic. A multislope α of X is called *minimally non hyperbolic* if the following two conditions hold:

(1) $X(\alpha)$ is not hyperbolic, and—
(2) For every partial filling $\alpha' \subset_{p.f.} \alpha$ with $\alpha' \neq \alpha$, $X(\alpha')$ is hyperbolic.

Clearly any non hyperbolic filling admits a minimally non hyperbolic partial filling. Thus there are enough minimally non hyperbolic partial fillings for our needs. On the other hand, there aren't too many:

Proposition 8. *Any hyperbolic manifold X admits only finitely many minimally non hyperbolic fillings.*

For a proof of Proposition 8, see [RY12b] (in version 1 of the arxiv this is Proposition 4.2).

6. $T(X)$

The proof of the main theorem (and several others) proceeds by induction on the number of vertices of a directed rooted tree associated with X, denoted as $T(X)$. For a detailed description of $T(X)$ see [RY12b] (in version 1 of the arxiv, $T(X)$ is constructed in Section 5). Each vertex of $T(X)$ is labeled, and the label is a manifold. We emphasize that a vertex contains more information than the manifold it represents; it also contains information about how the manifold was obtained (generally, the filling slopes). Thus distinct vertices may have homeomorphic label.

The construction of the tree is recursive, that is, if the root is X and one of the direct descendants of X is, say, X_1, then we place $T(X_1)$ at that point). If necessary we shift all the labels of $T(X_1)$. For that reason, we will only describe the essential construction here: for a manifold X, we describe the direct descendants of X.

Before getting into the detail we describe the levels. The levels are counted modulo 3 and obey the following rule:

(1) Geometric manifolds are arranged along levels $3n$.
(2) Composite manifold are arranged along levels $3n+1$.
(3) JSJ manifolds are arranged along levels $3n+2$ (that is, irreducible manifolds with non empty torus decomposition).
(4) The root itself is at level 0, 1, or 2 (the tree is constructed recursively).

The manifold X, which is the root of $T(X)$, is at level 0, 1, or 2. The edges out of X are the following:

(1) When X is hyperbolic, edges out of X correspond to minimally non hyperbolic fillings and end at levels 1, 2, or 3.
(2) Sol and Seifert manifolds are leaves.
(3) When X is composite, edges out of X correspond to the component of its prime decomposition and end at levels 2 and 3.
(4) Edges out of a JSJ manifold at level 2 correspond to the component of its JSJ decomposition and are at level 3.

Induction is made possible by

Proposition 9. *$T(X)$ is finite.*

For a proof, see [RY12b] (in version 1 of the arxiv this is Proposition 5.1).

7. CONCLUDING REMARKS

$T(X)$ can be used to control the non hyperbolic fillings of a hyperbolic manifold. In order to consider only finitely many fillings, we only consider minimally non hyperbolic partial fillings; the cost is that in the next step we are forced to consider fillings of non hyperbolic manifold manifolds. The following steps (fillings Seifert manifolds, reducible manifolds, and JSJ manifolds) are quite different in nature and the JSJ case can lead to many cases that one needs to consider. Nevertheless, it leads to theorems such as Theorem 5. We refer the reader to [RY12b] to other applications (in particular, Sections 6–10 of version 1 on the arxiv). Some are quite different than Theorem 5; for example, in Section 6 we consider cosmetic surgeries on a link $L \subset T^2 \times [0,1]$ and show that the images of a bounded set of slopes of $T^2 \times \{1\}$ form a bounded set of slopes of $T^2 \times \{0\}$, and in Sections 8 and 9 we study certain fillings that yield hyperbolic manifolds.

REFERENCES

[GL89] C. McA. Gordon and J. Luecke. Knots are determined by their complements. *J. Amer. Math. Soc.*, 2(2):371–415, 1989.

[HW08] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Oxford University Press, Oxford, sixth edition, 2008. Revised by D. R. Heath-Brown and J. H. Silverman, With a foreword by Andrew Wiles.

[RR12] J. Remigio-Juárez and Y. Rieck. The Link Volumes of some prism manifolds. *ArXiv e-prints*, May 2012. arxiv.org/1205.2783.

[RY12a] Y. Rieck and Y. Yamashita. The Link Volume of 3-Manifolds. *ArXiv e-prints*, May 2012. arxiv.org/1205.1274.

[RY12b] Yo'av Rieck and Yasushi Yamashita. The link volume of hyperbolic 3-manifolds. Available at http://arxiv.org/abs/1211.1839, 2012.

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