# On Genus Ranges of 4-Regular Rigid Vertex Graphs

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## 1 Introduction

This is a brief summary of [3]. The genus of a graph (also known as minimum genus) is a well known notion in topological graph theory, and has been studied for a variety of graphs (e.g., [11]). It represents the minimum genus of a surface in which a graph can be embedded. For the rest of the paper, all surfaces are assumed to be orientable. The genus range of a given graph is the set of all possible genera of surfaces in which the graph can be embedded cellularly (with open disk complements [9]). We study the genus ranges of four-regular graphs with rigid vertices (rigid graphs), and with a single transverse component – thus for the rest of the paper, every vertex of a graph is assumed to have valency 4, a cyclic order of edges is specified (up to reverse) at every vertex and preserved by embeddings, and the graph can be regarded as an image of a single immersed circle when traveled along the graph.

Such graphs can be specified by double occurrence words, otherwise known as unsigned Gauss codes, and are closely related to virtual knot diagrams [10]. Spatial embeddings of these graphs are called singular knots and used for finite type invariants [2] in knot theory. These graphs have also been used to model sequence rearrangements in DNA molecules [1]. Biological questions in these settings are translated in terms of rigid graphs and combinatorial questions about Hamiltonian polygonal paths – paths of rigid graphs that make "90° turn" at every rigid vertex, and visit every vertex exactly once. Thus the genus ranges are of interest from biological viewpoint as well.

Two main questions on genus ranges of such graphs are considered here:

**Problem 1.1.**
(a) Characterize the sets of integers that appear as genus ranges of graphs with $n$ four-valent vertices for each positive integer $n$.
(b) Characterize the graphs with a given set of genus range.

After briefly reviewing the notions and setting notations in Section 2, we present the statement and sketch proofs of the main theorem in Section 3.

# 2 Terminology and Preliminaries

In this section, concepts used in this paper are recalled, notations are established, and their basic properties are listed. See [3] for details.

The *genus range* $\mathrm{gr}(\Gamma)$ of a graph $\Gamma$ is the set of values of genera over all surfaces $F$ into which $\Gamma$ is embedded cellularly, i.e., embeddings with open disk complements. We denote the set of all genus ranges of graphs with $n$ vertices by $\mathcal{GR}_n$.

## 2.1 Double Occurrence Words and Ribbon Graphs

One way to obtain a cellular embedding of a graph $\Gamma$ in a compact orientable surface is by connecting bands (ribbons) along the graph. This construction is called a *ribbon construction*, and is outlined below.

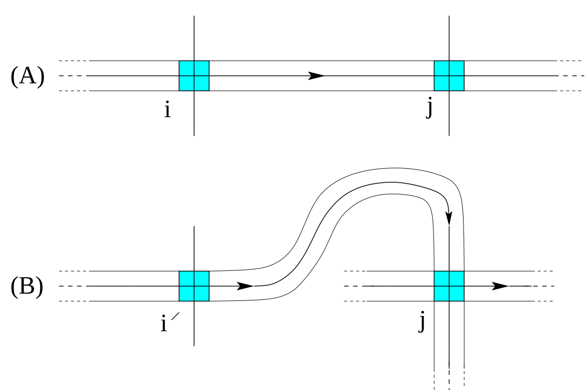


Figure 1: Ribbon construction

A path in a graph is called a *transverse path*, or simply a *transversal*, if it is the image of a circle in the graph, where the circle goes through every vertex "straight". A double occurrence word (DOW, unsigned Gauss code) is related to a rigid graph via transversals: Starting from a point on the graph, travel along the transversal, and write down the sequence of vertices in the order they are encountered along the transversal. This gives a DOW. Conversely, for a given DOW containing $n$ letters, a rigid graph is constructed with $n$ vertices (labeled with the letters) by tracing the labeled vertices in the order of their appearances in the DOW. It is known that equivalence classes of DOWs are in one-to-one correspondence with isomorphism classes of such graphs.

Embeddings of such graphs in surfaces are constructed as follows: To each vertex $i$, $i = 1, \ldots, n$, we associate a square with coordinate axes that coincide with the edges incident to the vertex as depicted in Figure 1. Each side of the square corresponds to an edge incident to the vertex. For an edge $e$ with the end points $i$ and $j$, we connect the sides of the squares at $i$ and $j$ corresponding to $e$ by a band. The bands are attached in such a way that the resulting surface is orientable. In Figure 1(A), the connection by a band is described when the vertex $j$ immediately follows $i$ in $w$, where $j$ is the first occurrence in $w$. In Figure 1(B), one possibility of connecting a band to the vertex $j$ at its second occurrence is shown. By continuing the band attachment along a transversal, one obtains a compact surface with boundary. The resulting surface is called a surface *obtained from ribbons*, *obtained by the ribbon construction*, or simply a *ribbon graph*. This notion has been studied in literature for general graphs (see, for example, [8]). By capping the boundaries by disks, one obtains a cellular embedding of $\Gamma$ in an orientable, closed (without boundary) surface. For a given cellular embedding of a graph $\Gamma$, its neighborhood is regarded as a surface obtained from ribbons as described above. Note that there are two choices in connecting a band to a vertex $j$ at its second occurrence, either from the top as in Figure 1(B), or from the bottom. Hence $2^n$ ribbon graphs (possibly not all distinct) can be constructed that correspond to all cellular embeddings of a given graph. Therefore, the genus range of a given graph can be computed by finding genera of all surfaces constructed from ribbons. In [5, 6], the two possibility of connecting bands are represented by signs ($\pm$), and signed Gauss codes were used to specify the two choices.

## 2.2 Computer Calculations

In this section we present calculations of genus ranges of rigid graphs. Calculations are based on a description of boundary curves of ribbon graphs in [7].

**Remark 2.1.** Computer calculations show that the sets of all possible genus ranges of $n$ letters for $n = 2, \ldots, 7$ are as presented in Table 1. For $n = 8$, only the set $\{0, 1, 2, 3, 4\}$ appears in addition to those for $n = 7$.

From computer calculations, observations on patterns and conjectures are formulated in [3]. To put these calculations into perspective, we denote by $[a, b]$ the set $\{a, \ldots, b\}$ for integers $0 \leq a \leq b$, and define the *consecutive power set*

Table 1: Computer calculations of genus ranges

| | $\mathcal{GR}_n$ : |
|---|---|
| $n=2$ | $\{0\},\{1\}$. |
| $n=3$ | $\{0\},\{1\},\{0,1\},\{1,2\}$. |
| $n=4$ | $\{0\},\{1\},\{0,1\},\{1,2\},\{0,1,2\}$. |
| $n=5$ | $\{0\},\{1\},\{2\},\{0,1\},\{1,2\},\{2,3\},\{0,1,2\},\{1,2,3\}$. |
| $n=6$ | $\{0\},\{1\},\{2\},\{3\},\{0,1\},\{1,2\},\{2,3\},\{0,1,2\},\{1,2,3\},\{0,1,2,3\}$. |
| $n=7$ | $\{0\},\{1\},\{2\},\{3\},\{0,1\},\{1,2\},\{2,3\},\{3,4\},$ |
| | $\{0,1,2\},\{1,2,3\},\{2,3,4\},\{0,1,2,3\},\{1,2,3,4\}$. |

of $\{0,1,\ldots,n\}$ for a positive integer $n$, denoted by $\mathcal{CP}(n)$, to be the set of all consecutive positive integers: $\mathcal{CP}(n)=\{\,[a,b]\,|\,0\le a\le b\le n\}$.

## 2.3 Computing the Genus Ranges

First we recall the well-known Euler characteristic formula, establishing the relation between the genus and the number of boundary components. The Euler characteristic $\chi(F)$ of a compact orientable surface $F$ of genus $g(F)$ and the number of boundary components $b(F)$ are related by $\chi(F)=2-2g(F)-b(F)$. As a complex, $\Gamma$ is homotopic to a 1-complex with $n$ vertices and $2n$ edges, and $F$ is homotopic to such a 1-complex. Hence $\chi(F)=n-2n=-n$. Thus we obtain the following well known formula, which we state as a lemma, as we will use it often.

**Lemma 2.2.** *Let $F$ be a surface for a graph $\Gamma$ obtained by the ribbon construction. Let $g(F)$ be the genus, $b(F)$ be the number of boundary components of $F$, and $n$ be the number of vertices of $\Gamma$. Then we have $g(F)=(1/2)(n-b(F)+2)$.*

Thus we can compute the genus range from the set of the numbers of boundary components of each ribbon graph, $\{\,b(F)\,|\,F \text{ is a ribbon graph of } G\,\}$. Note that $n$ and $b(F)$ have the same parity, as genera are integers.

Next we compute the number of boundary components of ribbon graphs for a given graph. In Figure 2(A), the boundary curves of a ribbon graph of a graph, near a vertex, are indicated. The arrows of these boundary curves indicate orientations of the boundary components induced from a chosen orientation of the ribbon graph. If in Figure 1(B), the direction of entering the vertex has been changed from top to bottom (or vise versa), the ribbon graph changes. This change in the ribbon graph is illustrated in Figure 2(B). Note that the new ribbon graph is orientable, as indicated by the arrows on the new boundary

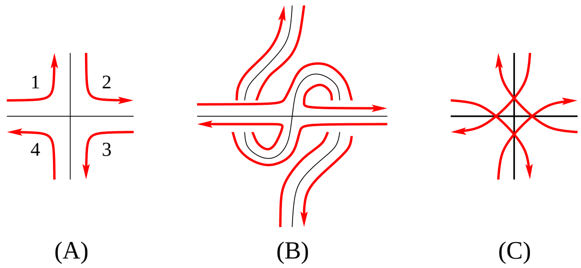


Figure 2: Changing the connection at a vertex

components. We use a schematic image in Figure 2(C) to indicate the changes of connections of the boundary components illustrated in Figure 2(B). We call this operation a *connection change*. Thus starting from one ribbon graph for a given graph $\Gamma$, one obtains its genus range by computing the number of boundary components for the surfaces obtained by switching connections at every vertex ($2^n$ possibilities for a graph with $|\Gamma| = n$ for a positive integer $n$).

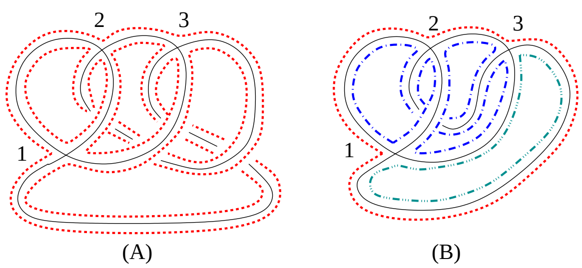


Figure 3: Boundary curves of ribbon neighbohoods of the graph with DOW 121323.

**Example 2.3.** In Figure 3, boundary curves of two ribbon graphs are shown for the same graph representing the word 121323. In Figure 3(A), one sees that the boundary curve is connected, and by Lemma 2.2, its genus is 2. In Figure 3(B), where the connection at vertex 3 is changed, one sees that the boundary curves consist of 3 components, and hence its genus is 1. The genus range of this graph turns out to be $\{1, 2\}$.

## 2.4 Basic Lemmas

Observations from computer calculations lead to finding properties of genus ranges. We were able to prove some of such properties, that were useful in determining genus ranges. Such basic lemmas are listed below without proofs. Again see [3] for details.

**Lemma 2.4.** *The genus range of any graph consists of consecutive integers.*

**Corollary 2.5.** *For any* $n \in \mathbb{N}$, *we have* $\mathcal{GR}_{2n-1}, \mathcal{GR}_{2n} \subset \mathcal{CP}(n)$.

**Lemma 2.6.** *For any DOW* $w$, *the corresponding genus range of* $\Gamma_w$ *is equal to that of* $\Gamma_{w'}$ *where* $w' = waa$ *and* $a$ *is a letter that does not appear in* $w$.

**Corollary 2.7.** *If a set* $A$ *appears as the genus range in* $\mathcal{GR}_n$ *for a positive integer* $n$, *then* $A$ *appears as a genus range in* $\mathcal{GR}_m$ *for any integer* $m > n$.

The following constructions are used for inductive proofs.

**Definition 2.8.** Let $\Gamma_1$ and $\Gamma_2$ be graphs. A graph $\Gamma$ is said to be obtained from $\Gamma_1$ and $\Gamma_2$ by a *cross sum* if it is formed by connecting the two graphs to the figure-eight graph as depicted in Figure 4.

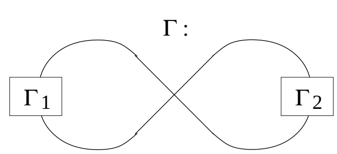


Figure 4: Cross sum

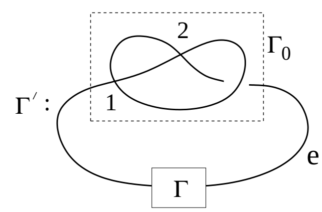


Figure 5: Connecting a pretzel

**Lemma 2.9.** *Let* $\Gamma_1$ *and* $\Gamma_2$ *be graphs. If* $\Gamma$ *is obtained from* $\Gamma_1$ *and* $\Gamma_2$ *by a cross sum, then* $\text{gr}(\Gamma) = \{\, g_1 + g_2 \mid g_1 \in \text{gr}(\Gamma_1),\ g_2 \in \text{gr}(\Gamma_2) \,\}$.

We say that the boundary component $\delta$ of a ribbon graph $\Gamma$ *traces* the edge $e$ of $\Gamma$ if the boundary of the ribbon that contains $e$ is a portion of $\delta$. There are at most two boundary components that can trace an edge.

**Lemma 2.10.** *Let* $\Gamma$ *be a graph,* $\Gamma_0$ *be the graph corresponding to the word* 1212, *and* $\Gamma'$ *be the graph obtained by connecting an edge* $e$ *of* $\Gamma$ *with* $\Gamma_0$ *as depicted in Figure 5. Suppose* $\text{gr}(\Gamma) = [m, n]$ *for non-negative integers* $m$ *and* $n$, *(*$m \le n$*).*
(i) *Suppose for all ribbon graphs of* $\Gamma$, *the two boundary curves tracing edge* $e$ *belong to two distinct boundary components, then* $\text{gr}(\Gamma') = [m+1, n+1]$.
(ii) *Suppose for some ribbon graphs of* $\Gamma$, *the two boundary curves tracing edge* $e$ *belong to the same boundary component, then* $\text{gr}(\Gamma') = [m, n+1]$.

## 3 Main Theorem

In this section we present the main theorem:

**Theorem 3.1.** *There exists a positive integer* $\psi_k$ *for each positive integer* $k$ *such that:*
(i) *for any* $n \in \mathbb{N}$, $\mathcal{GR}_{2n-1} \supset [\, \mathcal{CP}(n) \setminus \{[0,n],[h,h] \,|\, \psi_{2n-1} < h \leq n\}\,]$, *and*
(ii) *for any* $n \in \mathbb{N}$, $\mathcal{GR}_{2n} \supset [\, \mathcal{CP}(n) \setminus \{[h,h] \,|\, \psi_{2n} < h \leq n\}\,]$.
*Furthermore,*
(iii) *for any* $n \in \mathbb{N}$, $[0,n],[n,n] \notin \mathcal{GR}_{2n-1}$.

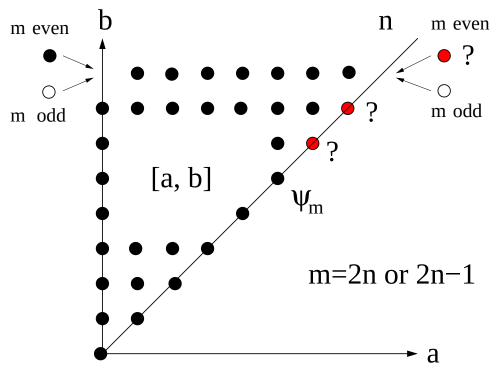


Figure 6: Main Theorem

The integers $\psi_k$ are explicitly described in [3]. See Figure 6 for the situation stated in Theorem 3.1. In this figure, a point $[a,b]$ with a black dot represents the existence of a graph with the genus range $[a,b]$. All points inside the visible, upper-left triangle consist of black dots. Points outside of the triangle do not satisfy the Euler characteristic formula (Lemma 2.2), so that there is no graph with these genus ranges. White dots represent that there do not exist graphs with such genus ranges, even though they do not violate Lemma 2.2. Red dots labeled with "?" represent that we do not know whether there are graphs with such ranges.
*Sketch Proof.* We describe an idea of proof for (i) and (ii). The proof is based on induction on $m$, the number of vertices ($m = 2n$ or $2n-1$ in Figure 6). Corollary 2.9 and Lemma 2.10 are used to proceed with induction for most of the genus ranges $[a,b]$ off the diagonal entries $[h,h]$. Since the "corner points" $[0,n]$ and $[n,n]$ are sometimes excluded by (iii), the genus ranges of the forms $[0,n+1]$ and $[n,n+1]$ become necessary to construct for induction. These ranges are realized by the following propositions.

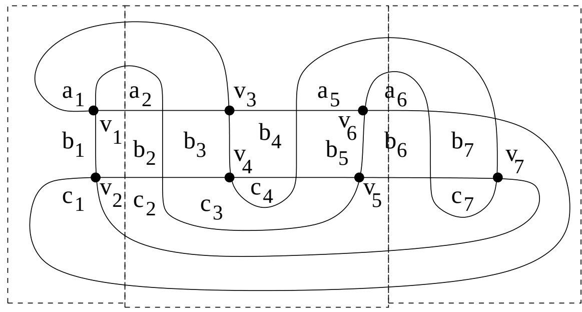


Figure 7: A graph with genus range $[0, 7]$

**Proposition 3.2.** *For any integer $n > 1$, there exists a graph $\Gamma$ of $2n$ vertices with* $\mathrm{gr}(\Gamma) = \{0, 1, \ldots, n\}$.

This is proved by exhibiting a specific family of graphs that realize these genus ranges. The graph in this family for $n = 14$ with genus range $[0, 7]$ is shown in Figure 7. Since it is planar, its genus rage includes 0. Its boundary curves correspond to the complementary regions in the plane. By changing the connection at vertex $v_1$, the curves of diagonal regions get connected, and the number of boundary curves reduces by 2. Hence the genus increases by 1. By changing the connection at vertex $v_2$, the genus increases by 1 again. Repeating this process at vertices $v_1$ through $v_7$ in this order, the desired genus range $[0, 7]$ is realized. The proposition is proved by induction using a family of similar graphs.

The following proposition is also proved by providing a family of graphs. A *tangled cord* [4] with $n$ vertices and $2n$ edges, denoted $T_n$, is a special type of graphs illustrated in Figure 8. The graph $T_n$ corresponds to the DOW

$$1213243 \cdots (n-1)(n-2)n(n-1)n.$$

Specifically, $T_1, T_2$ and $T_3$ correspond to 11, 1212, and 121323 respectively, and the DOW of $T_{n+1}$ is obtained from the DOW of $T_n$ by replacing the last letter $(n)$ by the subword $(n+1)n(n+1)$. Figure 8 shows the structure of the tangled cord, where $T_n$ and $T_{n+1}$ are depicted in (a) and (b), respectively, indicating how the family is constructed inductively.

**Proposition 3.3.** *Let $T_n$ be the tangled cord with $n$ vertices. Then*

$$\mathrm{gr}(T_n) = \begin{cases} \left\{ \frac{n-2}{2}, \frac{n}{2} \right\} & \text{if } n \text{ is even,} \\ \left\{ \frac{n-1}{2}, \frac{n+1}{2} \right\} & \text{if } n \text{ is odd.} \end{cases}$$

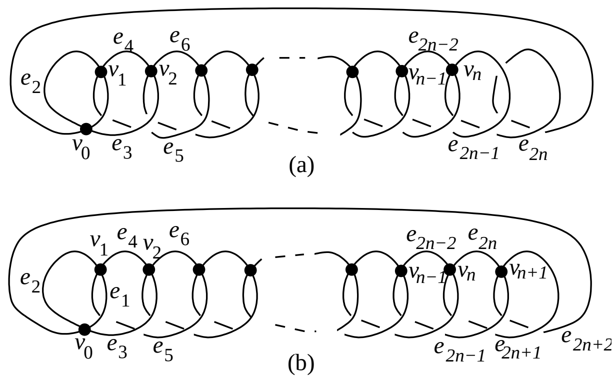


Figure 8: Tangled cord $T_n$

This proposition is proved by checking all connection possibilities inductively. The final step of the proof of Theorem 3.1 is to realize the diagonal ranges $[h, h]$ as much as possible by using Lemma 2.9.

**Remark 3.4.** From Theorem 3.1, we conjecture that for any $m \in \mathbb{N}$, there is an integer $\Psi_m \geq \psi_m$ such that

$$\begin{aligned} \mathcal{GR}_{2n} &= \mathcal{CP}(n) \setminus \{\, [h,h] \mid \Psi_{2n} < h \leq n \,\}, \\ \mathcal{GR}_{2n-1} &= \mathcal{CP}(n) \setminus \{\, [0,n], [h,h] \mid \Psi_{2n-1} < h \leq n \,\}. \end{aligned}$$

The highest possible genus along the diagonal that we can construct at this time are the numbers $\psi_m$ in [3], and we do not know whether ranges larger than $\psi_m$ can be realized. The conjecture says that the ranges are realized up to certain numbers $\Psi_m$ along the diagonal, which may be larger than $\psi_m$. The conjecture also says that all ranges larger than $\Psi_m$ are not realized, i.e., there is no "gap" in genus ranges along the diagonal.

For small values of $m \leq 20$, the conjecture holds with $\Psi_m = \psi_m$ for $m = 1, \ldots, 7, 9, 13$. At this time we are not able to determine if $[5,5]$ is in $\mathcal{GR}_m$ for $m = 10, 11$, $[6,6]$ for $m = 12$, $[7,7]$ for $m = 14, 15$, $[8,8]$ for $m = 16, 17, 18$, and $[9,9]$ for $m = 18, 19$. For $n = 100$, for example, we do not know if $[h,h] \in \mathcal{GR}_{100}$ for $h = 43, \ldots, 50$. We know that $[4,4]$ is not in $\mathcal{GR}_8$ only by computer calculations searching through all graphs with 8 vertices.

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