Coexistence of coiled surfaces and spanning surfaces for knots and links

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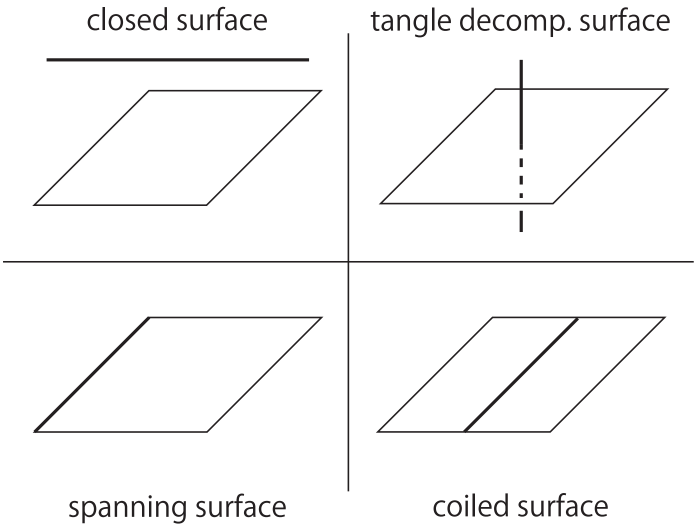
結び目理論にはまだまだ魅力的な予想が沢山ある。しかし、その殆どが**全ての結び目**に対する予想である為、非常に難しい。現時点では我々はまだ**全ての結び目**を扱える段階に入っていないのだ。

10 problems of knot theory

- Additivity problem (crossing number, unknotting number, e.t.c.)
- Nakanishi conjecture (unknotting on minimal diagrams)
- Lin's nugatory crossing conjecture
- Cabling conjecture
- Berge conjecture
- Fox's slice-ribbon conjecture
- Kashaev's volume conjecture
- Garoufalidis's Jones Slope conjecture
- Lopez conjecture
- **Neuwirth conjecture**

結び目理論の未解決問題１０ - NAVER まとめ
http://matome.naver.jp/odai/2133489552895746501

結び目 K と曲面 F の関係は、大きく 4 種類（$F \cap K = \emptyset$, $F \pitchfork K$, $\partial F = K$, $F \supset K$）に分類される。



Let K be a knot or link in S^3.

Essential surfaces

An orientable surface properly embedded in $E(K)$ is *essential* if it is incompressible, boundary-incompressible and not boundary-parallel. A non-orientable surface F properly embedded in $E(K)$ is *essential* if its frontier $cl(\partial N(F) - \partial E(K))$ is essential.

Note that any surface can be obtained by tubings and bandings along arcs in $\partial E(K)$ from essential surfaces and subsurfaces in $\partial E(K)$.

There are not so many geometrical properties satisfied by **all non-trivial knots**.

Geometrical properties satisfied by all non-trivial knots

Any non-trivial knot exterior contains:

- an essential non-separating surface (Frankl–Pontrjagin 1930, Seifert 1934).
- an essential separating surface (Culler–Shalen 1984).

For example, a torus knot exterior contains the fiber Seifert surface and the cabling annulus.

If an essential separating surface has two boundary components with integral ∂-slope, then it can be extended to a closed surface containing the knot. For example, the cabling annulus can be extended to a torus containing the torus knot.

The following conjecture asserts that **any non-trivial knot** exterior contains such an essential separating surface as above.

Conjecture 1 (Neuwirth Conjecture, 1964) *For* **any non-trivial knot** *K in the 3-sphere, there exists a closed surface S such that $S \supset K$, $S - K$ is connected, and $S \cap E(K)$ is essential.*

Knots satisfying the Neuwirth Conjecture

- Alternating knots (Aumann, 1956)
- Knots satisfying $g_I(K) < 2g(K)$, where $g_I(K)$ is an interpolating genus (Neuwirth, 1964)
- Generalized alternating knots (O, 2006)
- Knots with spanning surfaces obtained by Murasugi sums of essential non-orientable spanning surfaces (O, 2011)
- Montesinos knots (O–Rubinstein, 2012)
- All knots with 11 crossings or fewer except for $K11_n118$ and $K11_n126$ (O–Rubinstein, 2012)
- Knots with a degree one map to a knot satisfying the Neuwirth Conjecture (O–Rubinstein, 2012)

A procedure for constructing torus knots and links (Lyon, 1980)

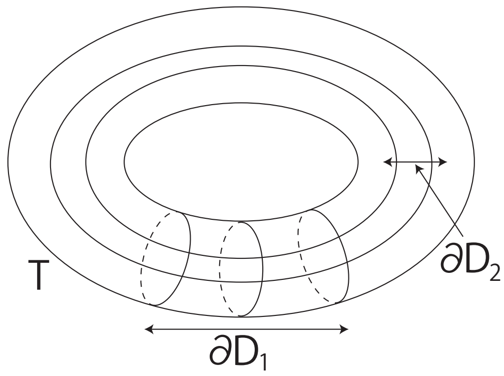
$S^3 = V_1 \cup_T V_2$, where T is a torus and V_i is a solid torus $(i = 1, 2)$

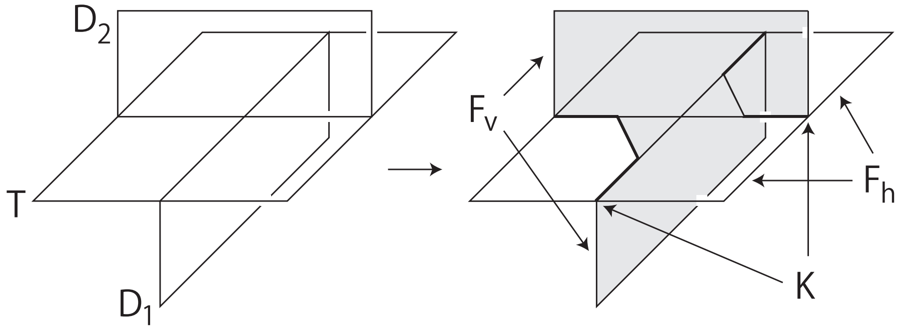
D_i : p_i mutually disjoint meridian disks of V_i $(i = 1, 2)$

By smoothing the intersections of ∂D_1 and ∂D_2 uniformly in T, we obtain a torus knot or link K of type (p_1, p_2).

For each point of $\partial D_1 \cap \partial D_2$, we add two triangle regions along this smoothing to $D_1 \cup D_2$, and then we obtain a Seifert surface F_v for K.

We also obtain cabling annuli $F_h = T \cap E(K)$.





Remarks

- $\chi(F_v) = |D_1| + |D_2| - |\partial D_1 \cap \partial D_2| = p_1 + p_2 - p_1 p_1$
- When K is a knot, then
  - $g(F_v) = (p_1 - 1)(p_2 - 1)/2 = g(K)$.
  - $\Delta(\partial F_v, \partial F_h) = |\partial D_1 \cap \partial D_2| = p_1 p_2$.
  - F_v is orientable and F_h is connected.

Definition of closed fake surfaces

We define three subsets of $\mathbb{R}^3$ as below.

$\Sigma_1 = \{(x, y, z) \in \mathbb{R}^3 | z = 0\}$

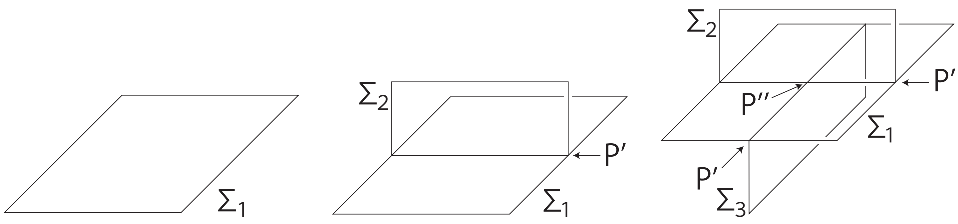
$\Sigma_2 = \{(x, y, z) \in \mathbb{R}^3 | y = 0, z \geq 0\}$

$\Sigma_3 = \{(x, y, z) \in \mathbb{R}^3 | x = 0, z \leq 0\}$

A finite 2-polyhedron P is called a *closed fake surface* (Ikeda, 1971) if each of its points has a neighborhood homeomorphic to one of the followings.

Type 1. Σ_1
Type 2. $\Sigma_1 \cup \Sigma_2$
Type 3. $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$

By P', we denote the set of points of Type 2 or 3. By P'', we denote the set of points of Type 3.



Type 1 Type 2 Type 3

A closed fake surface is *orientable* if each component of $P - P'$ is orientable.

Vertical and horizontal surfaces obtained by +-smoothing

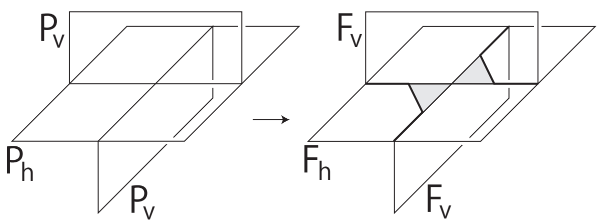
Vertical-horizontal decomposition

We say that a closed fake surface P embedded in S^3 has a *vertical-horizontal decomposition* $P = P_v \cup P_h$ if

- P_h is closed subsurfaces of P which corresponds to $(\Sigma_1, \mathbb{R}^3)$ at each neighborhood of points of Type 2 or 3, and
- P_v is subsurfaces of P which corresponds to $(\Sigma_2, \mathbb{R}^3)$ or $(\Sigma_3, \mathbb{R}^3)$ at each neighborhood of points of Type 2 or 3.

When $P' = \emptyset$, we define $P_h = P$ and $P_v = \emptyset$.

Let P be a closed fake surface embedded in S^3 with a vertical-horizontal decomposition $P = P_v \cup P_h$.



+-smoothing of a closed fake surface P

For each neighborhood of a point of Type 3, we add two triangle regions $\{(x,y,z) \in \mathbb{R}^3 | xy \geq 0, |x+y| \leq 1\}$ to P_v.

Then we obtain by the +-smoothing on each point of P'' :

F_v : the *vertical surfaces* obtained from P_v

F_h : the *horizontal surfaces* obtained as $P_h \cap E(K)$

K : a knot or link obtained as ∂F_v

We note that $\chi(F_v) = \chi(P_v) - |P''|$ and $\chi(F_h) = \chi(P_h)$. The +-smoothing does not depend on orientations of $P - P'$. The −-*smoothing* of P can be similarly defined, and the results for the +-smoothing also hold for the −-smoothing.

Definition of essential closed fake surfaces

Compressing disk, monogon, bigon

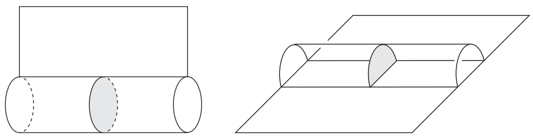
We say that a disk D with $D \cap P = \partial D$ is

- a *compressing disk* for P if ∂D is essential in P.
- a *monogon* if $\partial D \subset P_h - P''$ and $|\partial D \cap P'| = 1$.
- a *bigon* if the boundary of D is decomposed into two arcs $\alpha \subset P_v$ and $\beta \subset P_h$ and at least one of α, β is an essential arc in P.

Essential closed fake surfaces

A closed fake surface $P = P_v \cup P_h$ embedded in S^3 is said to be *essential* if

- $S^3 - P$ is irreducible,
- P has no compressing disk,
- P has no monogon,
- P has no bigon, and
- P_h has no 2-sphere component.



monogon bigon

The strong Neuwirth conjecture

Let K be a knot or link in S^3, and $F \supset K$ be closed surfaces embedded in S^3.

Coiled surface, Neuwirth surface

- We say that F is *coiled surfaces* for K if F is essential in the exterior $E(K)$.
- We say that a coiled surface F is a *Neuwirth surface* if $F - C$ is connected for each component C of K.

We remark that any knot or link has coiled surfaces since it bounds essential Seifert surfaces F and $\partial N(F)$ gives coiled surfaces.

Similarly, if a knot bounds an essential non-orientable spanning surface F, then $\partial N(F)$ gives a Neuwirth surface.

Therefore, the following Strong Neuwirth Conjecture is plausible.

Conjecture 2 (Strong Neuwirth Conjecture) *Any prime non-torus knot bounds an essential non-orientable spanning surface.*

The strong Neuwirth conjecture implies the Neuwirth conjecture.

Knots satisfying the strong Neuwirth Conjecture

- Alternating knots (Aumann, 1956)
- Generalized alternating knots (O, 2006)
- Knots with spanning surfaces obtained by Murasugi sums of essential non-orientable spanning surfaces (O, 2011)
- Montesinos knots (O–Rubinstein, 2012)
- All knots with 11 crossings or fewer except for $K11_n118$ and $K11_n126$ (O–Rubinstein, 2012)

Main theorem

Theorem 3 *Suppose that P is an essential orientable closed fake surface embedded in the 3-sphere S^3 with a vertical-horizontal decomposition $P = P_v \cup P_h$.*

Let F_v and F_h be the vertical surfaces and the horizontal surfaces respectively obtained from P by the $+$-smoothing, and K be the knot or link obtained from P' by the $+$-smoothing.

Then F_v and F_h are essential in $E(K)$. Moreover when K is a knot, we have that $\Delta(\partial F_v, \partial F_h) = |P''|$ and if F_v is orientable, then F_h is connected.

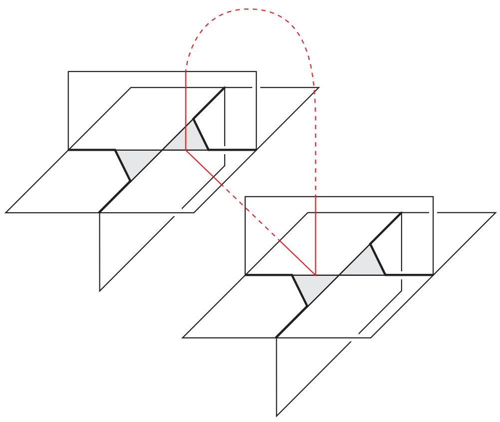
Uniformly twisted knots and links

We say that a knot or link K is *uniformly twisted* if it can be obtained from P' of an essential orientable closed fake surface P embedded in S^3 with a vertical-horizontal decomposition $P = P_v \cup P_h$ by the $+$-smoothing or $-$-smoothing.

In Theorem 3, if F_v is non-orientable, then K satisfies the strong Neuwirth conjecture, and hence the Neuwirth conjecture. Otherwise, F_h is connected and hence it is a Neuwirth surface for K. Hence, we have the following corollary.

Corollary 4 *A uniformly twisted knot satisfies the Neuwirth conjecture.*

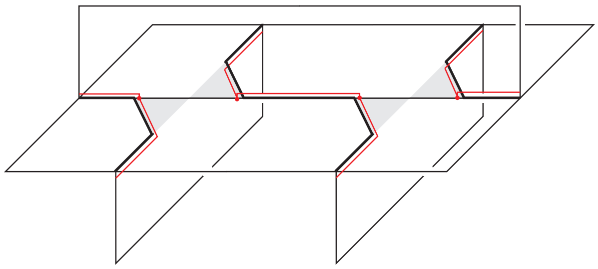
Essence of Proof



The boundary of an outermost disk δ in a compressing disk for F_h or F_v with respect to $D \cap (F_v \cup F_h)$ (Case 1)

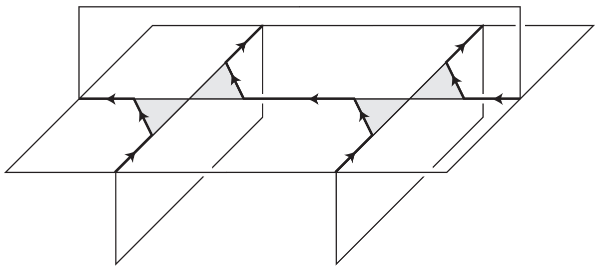
Hereafter, we assume that K is a knot.

It can be observed that $\min |\partial F_v \cap \partial F_h| = 2|P''|$. Thus the distance $\Delta(\partial F_v, \partial F_h)$ is equal to $|P''|$.



+-smoothing of a closed fake surface P

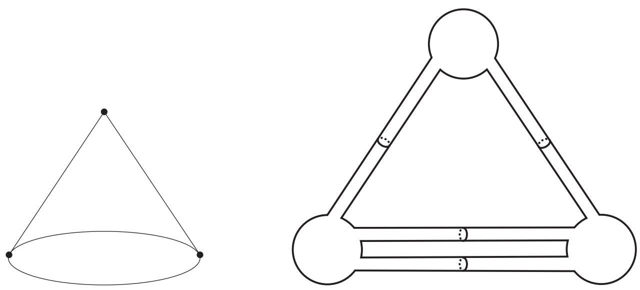
Suppose that F_v is orientable. Then K can be oriented by the orientation of F_v. This shows that F_h is connected.



An orientation of K induced by F_v

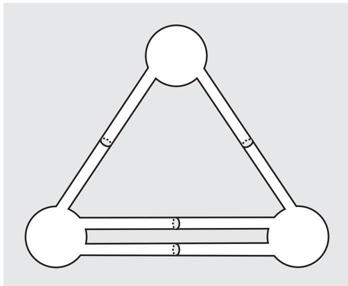
Example

Let S^2 be a 2-sphere embedded in S^3 and G be a 2-connected graph embedded in S^2 with at least one edge. Then the closed surface $P_h = \partial N(G)$ decomposes S^3 into two handlebodies V_1 and V_2, where V_1 contains G.



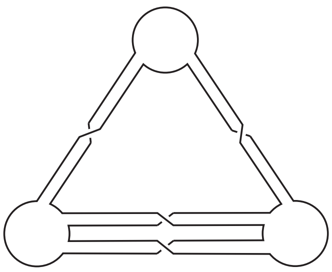
a 2-connected graph G a closed surface P_h

Let P_v be a union of meridian disks of V_1 which are dual to edges of G, and meridian disks of V_2 as the regions of $S^2 - intV_1$. Then we obtain an essential orientable closed fake surface P with the vertical-horizontal decomposition $P = P_v \cup P_h$.



a closed fake surface P

Let F_v and F_h be the vertical surfaces and the horizontal surfaces respectively obtained from P by the $+$-smoothing or $-$-smoothing, and K be the knot or link obtained from P' by the $+$-smoothing or $-$-smoothing.



a uniformly twisted knot K

Problem and Conjecture

Problem 5 *Does there exist a non-splittable and prime knot or link which is not uniformly twisted?*

Conjecture 6 *For any non-trivial, non-split link L, there exists closed surfaces F containing L such that each component of L is non-separating in F and F is essential in $E(L)$.*