ホモロジカルな表現と Alexander 多項式

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1 Introduction

1. Lawrence constructed the braid group representations on the configuration spaces. (Lawrence-Krammer representations.)
2. Bigelow and Mimachi gave new constructions of the Jones polynomial by means of the homological representation.
3. The representation space in the twisted homology group is equivalent to the linear skein associated with the Temperley-Lieb algebra.

We would like to construct the Alexander polynomial by means of the homological representation.

2 Twisted homology

To discuss twisted homology associated with a Selberg type integral, we employ definitions and properties from Mimachi's constructions.

Let $\mathrm{Conf}_{2m}(\mathbb{C})$ be the configuration space of $2m$ distinct points of the complex plane $\mathbb{C}$:

$$\mathrm{Conf}_{2m}(\mathbb{C}) = \{(z_1, \ldots, z_{2m}) \in \mathbb{C}^{2m} \mid z_i \neq z_j \text{ if } i \neq j\}.$$

For $z = (z_1, \ldots, z_{2m})$ in $\mathrm{Conf}_{2m}(\mathbb{C})$, let $\mathcal{L}_z$ be a local system on

$$\mathcal{T}_z = \mathbb{C}^m \backslash \bigcup_{1\le i<j\le m} \{t_i - t_j = 0\} \cup \bigcup_{\substack{1\le i\le m \\ 1\le j\le 2m}} \{t_i - z_j = 0\}$$

determined by

$$u(t) = \prod_{1\le i<j\le m} (t_i - t_j)^g \prod_{\substack{1\le i\le m \\ 1\le j\le 2m}} (t_i - z_j)^{\lambda_j} \qquad (g, \lambda_j \in \mathbb{R}),$$

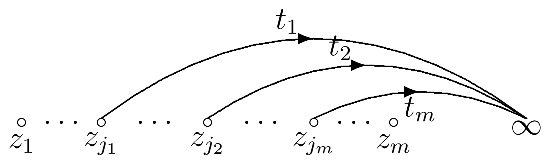
where $t = (t_1, \ldots, t_m) \in \mathcal{T}_z$ and g, λ_j are generic.

Let $H_m^{lf}(\mathcal{T}_z, \mathcal{L}_z)$ be the mth locally finite twisted homology group with coefficients in $\mathcal{L}_z$. Elements of $H_m^{lf}(\mathcal{T}_z, \mathcal{L}_z)$ are defined by ∂-closed locally finite twisted chains

$$C = \sum_\rho a_\rho \rho \otimes v_\rho \qquad (a_\rho \in \mathbb{C}),$$

where each ρ is an m-simplex and v_ρ is a section of $\mathcal{L}_z$ on ρ, called *loaded cycles* or *twisted cycles*. The boundary operator ∂ is a linear mapping over $\mathbb{C}$ defined by $\partial(\rho \otimes v_\rho) = \sum_{i=0}^m (-1)^i \rho^i \otimes v|_{\rho^i}$, where ρ is an m-simplex, ρ^i denotes the ith face of m-simplex ρ, and $v|_{\rho^i}$ is the restriction of v on ρ^i.

Define the element $\gamma_{j_1 j_2 \cdots j_m}(t;z)$ of $H_m^{lf}(\mathcal{T}_z, \mathcal{L}_z)$ by

$\gamma_{j_1 j_2 \cdots j_m}(t;z) =$  $\otimes\, u(t),$

where the arguments are fixed so that $\arg(t_i - t_j) = 0$ for $t_i > t_j$ and $\arg(t_i - z_j) = 0$ for $t_i > z_{2m}$.

Let $\widetilde{\gamma}_{j_1 j_2 \cdots j_m}(z)$ and $\widehat{\gamma}_{j_1 j_2 \cdots j_m}(z)$ be a symmetrization and an anti-symmetrization of $\gamma_{j_1 j_2 \cdots j_m}(t;z)$:

$$\widetilde{\gamma}_{j_1 j_2 \cdots j_m}(z) = \sum_{\sigma \in \mathfrak{S}_m} \gamma_{j_1 j_2 \cdots j_m}(t_{\sigma(1)}, \ldots, t_{\sigma(m)}; z),$$
$$\widehat{\gamma}_{j_1 j_2 \cdots j_m}(z) = \sum_{\sigma \in \mathfrak{S}_m} \mathrm{sgn}(\sigma) \gamma_{j_1 j_2 \cdots j_m}(t_{\sigma(1)}, \ldots, t_{\sigma(m)}; z),$$

where $\arg(t_{\sigma(i)} - t_{\sigma(j)}) = 0$ for $t_{\sigma(i)} > t_{\sigma(j)}$ and $\arg(t_{\sigma(i)} - z_j) = 0$ for $t_{\sigma(i)} > z_{2m}$. We set $\lambda_j = -g/2$ in $\widetilde{\gamma}_{j_1 j_2 \cdots j_m}(z)$ and $g = 0$ in $\widehat{\gamma}_{j_1 j_2 \cdots j_m}(z)$.

We define vector spaces $\widetilde{V}_m, \widehat{V}_m \subset H_m^{lf}(\mathcal{T}_z, \mathcal{L}_z)$ by

$$\widetilde{V}_m = \{\widetilde{\gamma}_{j_1 j_2 \cdots j_m}(z) | 1 \le j_1 < \cdots < j_m \le 2m\},$$
$$\widehat{V}_m = \{\widehat{\gamma}_{j_1 j_2 \cdots j_m}(z) | 1 \le j_1 < \cdots < j_m \le 2m\}.$$

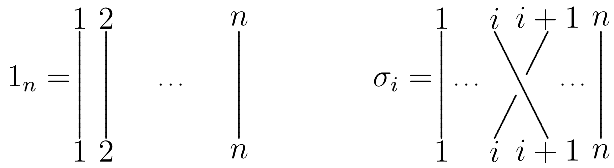
Remark. $\widetilde{V}_m \simeq TL_m$.

3 An action of the braid group

The braid group B_n on n strands is generated by $1_n, \sigma_1, \ldots, \sigma_{n-1}$, and the defining relations are

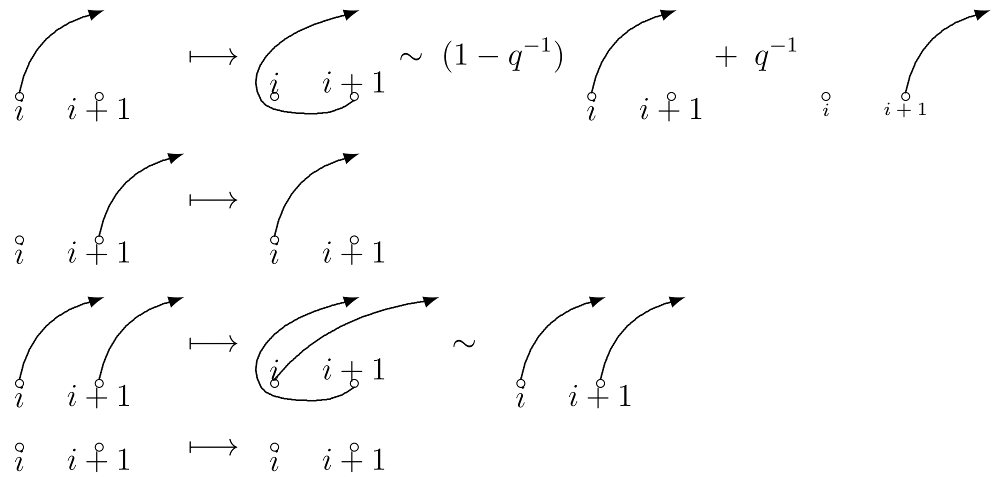
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad \text{if } 1 \le i \le n-2,$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad \text{if } |i-j| \ge 2.$$

A geometrical representation of B_n is given by n strands.

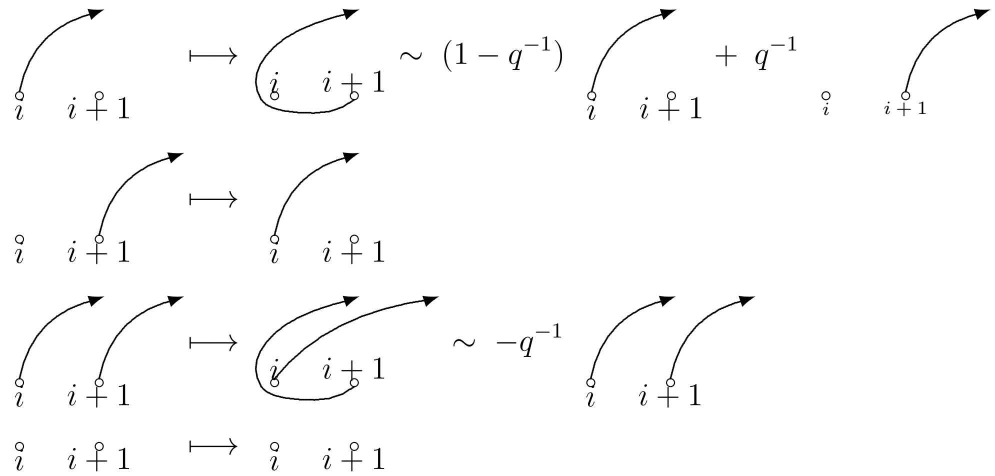


We fix a point $z \in \mathrm{Conf}_{2m}(\mathbb{C})$ as $0 < z_1 < z_2 < \cdots < z_{2m} < +\infty$, and we call each z_j the jth point. B_{2m} acts on $\widetilde{V}_m$ and $\widehat{V}_m$. The action of σ_i with respect to $\widetilde{\gamma}_{j_1 j_2 \cdots j_m}(z)$ and $\widehat{\gamma}_{j_1 j_2 \cdots j_m}(z)$ are given by

$\widetilde{\gamma}_{j_1 j_2 \cdots j_m}(z)$:



$\widehat{\gamma}_{j_1 j_2 \cdots j_m}(z)$:



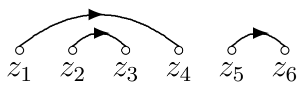
where $\lambda_j = -g/2$ in $\widehat{\gamma}_{j_1 j_2 \cdots j_m}(z)$, $q = \mathrm{e}^{\pi\sqrt{-1}g}$, and "$\sim$" means that they are homologically equivalent.

4 Standard twisted cycles and an ordering on t_i's

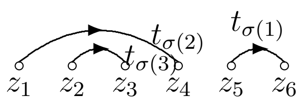
Example.

$$\widetilde{\gamma}_{5-6:1-4:2-3}(z)$$

$=$



$=\sum_{\sigma\in\mathfrak{S}_3}$



$$\otimes (z_6 - t_{\sigma(1)})^{-\frac{g}{2}} \prod_{1\leq j\leq 5} (t_{\sigma(1)} - z_j)^{-\frac{g}{2}}$$

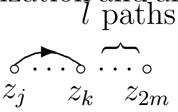
$$\prod_{4\leq j\leq 6} (z_j - t_{\sigma(2)})^{-\frac{g}{2}} \prod_{1\leq j\leq 3} (t_{\sigma(2)} - z_j)^{-\frac{g}{2}} \prod_{3\leq j\leq 6} (z_j - t_{\sigma(3)})^{-\frac{g}{2}}$$

$$\prod_{1\leq j\leq 2} (t_{\sigma(3)} - z_j)^{-\frac{g}{2}} \prod_{1\leq i<j\leq 3} (t_{\sigma(i)} - t_{\sigma(j)})^{g}$$

where $\arg(t_{\sigma(i)}-t_{\sigma(j)}) = 0$ for $t_{\sigma(i)} > t_{\sigma(j)}$, $\arg(t_{\sigma(1)}-z_j) = 0$ for $z_5 < t_{\sigma(1)<z_6}$, $\arg(t_{\sigma(1)}-z_j) = 0$ for $z_3 < t_{\sigma(2)<z_4}$, and $\arg(t_{\sigma(1)} - z_j) = 0$ for $z_2 < t_{\sigma(1)<z_3}$.

Let $\widetilde{\gamma}^l_{\cdots:j-k:\cdots}(z)$ and $\widehat{\gamma}^l_{\cdots:j-k:\cdots}(z)$ be a symmetrization and an anti-symmetrization of $\gamma^l_{\cdots:j-k:\cdots}(t;z)$:

$\gamma^l_{\cdots:j-k:\cdots}(t;z) = \ldots$



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Lemma 4.1. *$\widetilde{\gamma}^l_{\cdots:j-k:\cdots}(z)$ and $\widehat{\gamma}^l_{\cdots:j-k:\cdots}(z)$ are homologous to*

$$q^{m-\frac{k-1}{2}-l}\{\ \widetilde{\gamma}^l_{\cdots:j:\cdots}(z) - \widetilde{\gamma}^l_{\cdots:k:\cdots}(z)\ \},$$
$$q^{m-\frac{k-1}{2}}(-1)^l\{\ \widehat{\gamma}^l_{\cdots:j:\cdots}(z) - \widehat{\gamma}^l_{\cdots:k:\cdots}(z)\ \}.$$

5 The intersection form

The intersection forms

$$\widetilde{\langle,\rangle}, \widehat{\langle,\rangle} : H^{lf}_m(\mathcal{T}_z, \mathcal{L}_z) \times H^{lf}_m(\mathcal{T}_z, \mathcal{L}_z) \to \mathbb{C}$$

is the Hermitian form defined by

$$\widetilde{\langle C, C'\rangle} = \sum_{\rho,\tau} a_\rho \overline{a_\tau} \sum_{t\in\rho\cap\tau} I_t(\rho,\tau) v_\rho(t) v'_\tau(t)/|u|^2,$$
$$\widehat{\langle C, C'\rangle} = \sum_{\rho,\tau} a_\rho \overline{a_\tau} \sum_{t\in\rho\cap\tau} I_t(\rho,\tau),$$

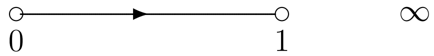
where

$$\mathrm{reg}C = \sum_{\rho} a_\rho \rho \otimes v_\rho(t), \quad C' = \sum_{\tau} a'_\tau \tau \otimes v'_\tau(t).$$

Here $\overline{}$ is complex conjugate, $I_x(\rho, \tau)$ is the topological intersection number at x.

A regularization $\mathrm{reg}\, C$ of $C = \overrightarrow{(0,1)} \otimes t^{-g/2}(1-t)^{-g/2}$ is given by

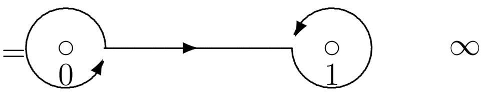
$C =$



$\mathrm{reg}\, C =$

$$\left\{ \frac{1}{\exp(-\pi\sqrt{-1}g)-1} S(\epsilon;0) + \overrightarrow{[\epsilon, 1-\epsilon]} - \frac{1}{\exp(-\pi\sqrt{-1}g)-1} S(1-\epsilon;1) \right\} \otimes u(t) \qquad C \in \widetilde{V}_m,$$

$=$



$$\{S(\epsilon;0) + S(1-\epsilon;1)\} \otimes u(t) \qquad C \in \widehat{V}_m,$$

$=$

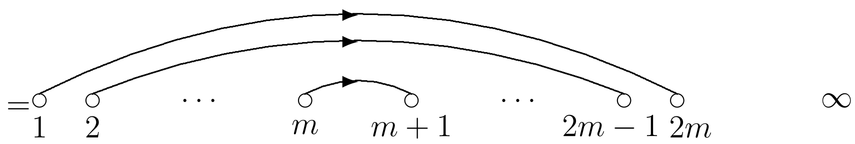
Examples.

$$\widetilde{\langle C, C \rangle} = + \frac{-1}{\exp(-\pi\sqrt{-1}g)-1} - \frac{+\exp(-\pi\sqrt{-1}g)}{\exp(-\pi\sqrt{-1}g)-1}$$
$$= -\frac{1}{q^{-1}-1} + \frac{-q^{-1}}{q^{-1}-1} = \frac{-1-q^{-1}}{q^{-1}-1}$$
$$\widehat{\langle C, C \rangle} = (-1) + (+1) = 0$$

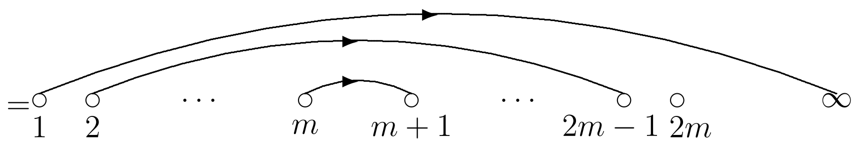
6 An invariant of knots and links

Let v_0 and v_0' be elements of $H_m^{lf}(\mathcal{T}_z, \mathcal{L}_z)$ defined by

$$v_0 = \widehat{\gamma}_{1-2m:2-(2m-1):\cdots:m-(m+1)}$$



$$v_0' = \widehat{\gamma}_{1:2-(2m-1):\cdots:m-(m+1)}$$



Let $\beta \in \langle \sigma_1, \ldots, \sigma_{m-1} \rangle$ for $m \geq 2$, $\hat{\beta}$ is the closure of β. $Al_{\hat{\beta}}(q)$ is defined by

$$Al_{\hat{\beta}}(q) = (-1)^{m-1} q^{-\frac{wl(\beta)}{2} + \frac{e(\beta)}{2} - \frac{m(m+1)}{4}} \frac{1}{m!} \langle \widehat{\beta \cdot v_0, v_0'} \rangle,$$

where wl is the word length of β and e is the exponent of β.

Theorem 6.1. *$Al_{\hat{\beta}}(q)$ is an invariant of knots and links. $Al_{\hat{\beta}}(q)$ satisfies*

$$Al_{\widehat{\beta_1 \sigma_i \beta_2}}(q) - Al_{\widehat{\beta_1 \sigma_i^{-1} \beta_2}}(q) = (q^{-1} - 1) Al_{\widehat{\beta_1 \beta_2}}(q)$$

Examples.

$$\begin{aligned}
Al_{\widehat{\sigma_1}}(q) &= 1 \\
Al_{\widehat{\sigma_1^{-1}}}(q) &= 1 \\
Al_{\widehat{\sigma_1^3}}(q) &= q^{-\frac{3}{2}}(1 - q^{-1} + q^{-2}), \\
Al_{\widehat{\sigma_1^{-3}}}(q) &= q^{-\frac{3}{2}}(1 - q^{-1} + q^{-2}), \\
Al_{\widehat{\sigma_1 \sigma_2^{-1} \sigma_1 \sigma_2^{-1}}}(q) &= q^{-3}(-1 + 3q^{-1} - q^{-2}).
\end{aligned}$$