Bing doubling and the colored Jones polynomial

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1 Introducton

Bing doubling [1] is an operation which gives the satellite $B(K)$ of a knot K as illustrated below.

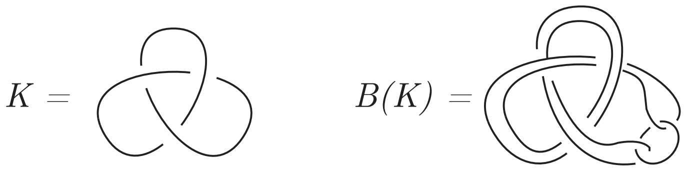

Figure 1: Trefoil K and its Bing double $B(K)$

We are interested in the relationship between topological properties of links and algebraic properties of quantum invariants. In this note, we give an algebraic property of colored Jones polynomials of Bing doubles (Theorem 2.1), and a divisibility property of the unified Witten-Reshetikhin-Turaev invariants [2] of integral homology spheres which are obtained from S^3 by surgery along Bing doubles (Theorem 2.3).

2 Results

For $m \geq 1$, let V_m denote the m-dimensional irreducible representation of the quantized enveloping algebra $U_h(sl_2)$ of sl_2. Let $\mathcal{R}$ denote the $\mathbb{Q}(q^{\frac{1}{2}})$-algebra

$$\mathcal{R} = \mathrm{Span}_{\mathbb{Q}(q^{\frac{1}{2}})}\{V_m \mid m \geq 1\}$$

with the multiplication induced by the tensor product. For $X_1, \ldots, X_n \in \mathcal{R}$, let $J_{L;X_1,\ldots,X_n}$ be the colored Jones polynomial of an n-component link L with the ith component L_i colored by X_i (cf. [2]).

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For $l \geq 0$, set

$$P_l = \prod_{i=0}^{l-1}(V_2 - q^{i+\frac{1}{2}} - q^{-i-\frac{1}{2}}) \in \mathcal{R},$$
$$P'_l = \frac{1}{\{l\}!}P_l \in \mathcal{R},$$
$$\tilde{P}'_l = q^{-\frac{1}{4}l(l-1)}P'_l \in \mathcal{R}.$$

For $k \geq 0$, set

$$\mathcal{P}_k = \mathrm{Span}_{\mathbb{Z}[q,q^{-1}]}\{\tilde{P}'_l \mid l \geq k\}.$$

Set

$$\hat{\mathcal{P}} = \varprojlim_{k\geq 0} \mathcal{P}_0/\mathcal{P}_k,$$
$$\omega^{\pm 1} = \sum_{l=0}^{\infty}(\pm 1)^l q^{\pm\frac{1}{4}l(l+3)}P'_l \in \hat{\mathcal{P}}.$$

Let M be the integral homology sphere obtained by surgery along an algebraically-split link $L = L_1 \cup \cdots \cup L_n$ in S^3 with framings $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$. Habiro [2] defined the unified Witten-Reshetikhin-Turaev invariant $J_M \in \widehat{\mathbb{Z}[q]}$ of M by

$$J_M = J_{L^0;\omega^{\epsilon_1},\ldots,\omega^{\epsilon_n}} \in \widehat{\mathbb{Z}[q]},$$

where L^0 is the link obtained from L by changing all framings to 0, and $\widehat{\mathbb{Z}[q]}$ is the Habiro ring defined by

$$\widehat{\mathbb{Z}[q]} = \varprojlim_{n\geq 0} \mathbb{Z}[q]/((1-q)(1-q^2)\cdots(1-q^n)).$$

We use the following q-integer notations:

$$\{i\} = q^{i/2} - q^{-i/2}, \quad \{i\}_n = \{i\}\{i-1\}\cdots\{i-n+1\},$$
$$\{n\}! = \{n\}_n, \quad \begin{bmatrix} i \\ n \end{bmatrix} = \{i\}_n/\{n\}!,$$

for $i \in \mathbb{Z}, n \geq 0$.

Theorem 2.1. *Let K be a knot with 0-framing. For $i, j \geq 0$, we have*

$$J_{B(K);P'_i,P'_j} = \sum_{l\geq 0} a^{(l)}_{i,j} J_{K;P'_l},$$

where

$$a^{(l)}_{i,j} = \delta_{i,j}(-1)^i \frac{\{2i+1\}!\{l\}!}{\{2l+1\}!}\lambda_{l,i},$$
$$\lambda_{l,i} = \sum_{k=0}^{l}(-1)^k \begin{bmatrix} 2l+1 \\ k \end{bmatrix} \begin{bmatrix} 2l+i-2k+1 \\ 2i+1 \end{bmatrix}.$$

For an n-component link $L = L_1 \cup \cdots \cup L_n$ with 0-framing, let $B_u(L)$, $u = 1, \ldots, n$, be the link obtained from L by replacing L_u with its Bing double. We can generalize Theorem 2.2 as follows.

Theorem 2.2. *Let $L = L_1 \cup \cdots \cup L_n$ be an n-component link with 0-framing. For $W_1, \ldots, W_n \in \mathcal{R}$, $u = 1, \ldots, n$, and $i, j \geq 0$, we have*

$$J_{B_u(L);W_1,\ldots,W_{u-1},P'_i,P'_j,W_{u+1},\ldots W_n} = \sum_{l \geq 0} a_{i,j}^{(l)} J_{L;W_1,\ldots,W_{u-1},P'_l,W_{u+1},\ldots W_n}.$$

For $m \geq 1$, let $\Phi_m(q) = \prod_{d|m}(q^d - 1)^{\mu(\frac{m}{d})} \in \mathbb{Z}[q]$ denote the mth cyclotomic polynomial, where $\prod_{d|m}$ denotes the product over all the positive divisors d of m, and μ is the Möbius function.

Theorem 2.3. *Let K be a knot with 0-framing, and M the integral homology sphere obtained from S^3 by surgery along $B(K)$ with framing 1 or -1 for each component. We have*

$$J_M - 1 \in \Phi_1^2 \Phi_2^2 \Phi_3 \Phi_4 \Phi_6 \widehat{\mathbb{Z}[q]}.$$

3 Proof

3.1 Proof of Theorem 2.2

In this section, we prove Theorem 2.2. It is enough to prove the following two claims.

Claim 3.1. *There are $x_{i,j}^{(l)} \in \mathbb{Q}(q^{1/2})$, $i, j, l \geq 0$, satisfying*

$$J_{B_u(L);W_1,\ldots,W_{u-1},P'_i,P'_j,W_{u+1},\ldots W_n} = \sum_{l \geq 0} x_{i,j}^{(l)} J_{L;W_1,\ldots,W_{u-1},P'_l,W_{u+1},\ldots W_n},$$

for any n-component link $L = L_1 \cup \cdots \cup L_n$ with 0-framing, $W_1, \ldots, W_n \in \mathcal{R}$, and $u = 1, \ldots, n$.

Claim 3.2. *We have $x_{i,j}^{(l)} = a_{i,j}^{(l)}$ for $i, j, l \geq 0$.*

Proof of Claim 3.1. We prove the claim for $n = 1$. The proof is similar for $n \geq 2$. Let K be a knot with 0-framing. Since $\{P'_l\}_{l \geq 0}$ is a basis of $\mathcal{R}$, the colored Jones polynomial $J_{B(K);P'_i,P'_j}$ is a linear sum of colored Jones polynomials $J_{B(K);V_t,V_u}$ in $\mathcal{R}$. It is known that a colored Jones polynomial $J_{B(K);V_t,V_u}$ is a linear sum of colored Jones polynomial $J_{K;V_s}$ in $\mathbb{Z}[q^{1/2}, q^{-1/2}]$, thus a linear sum of colored Jones polynomials $J_{K;P'_l}$ in $\mathcal{R}$. Consequently, $J_{B(K);P'_i,P'_j}$ is a linear sum $J_{K;P'_l}$ in $\mathcal{R}$. Moreover, in each step, the coefficients of the linear sum do not depend on the knot K. Hence we have the assertion. □

We prove Claim 3.2 by using Lemmas 3.3, 3.4, and Proposition 3.5 as follows.

Let A and H be the Borromean rings and the Hopf link, respectively, depicted in Figure 2.

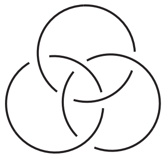
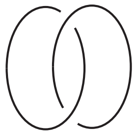
$A =$  $H =$ 

Figure 2: Borromean rings

Lemma 3.3 (Habiro [2, Corollary 14.2])**.** *For $i, j, k \geq 0$, we have*

$$J_{A;P'_i,P'_j,P'_k} = \delta_{i,j,k}(-1)^i\{2i+1\}_{i+1}/\{1\}.$$

For $l \geq 0$, set

$$S_l = \prod_{i=1}^{l}(V_2 - (q^i + 1 + q^{-i})) \in \mathcal{R}.$$

For $x, y \in \mathcal{R}$, set

$$\langle x, y\rangle = J_{H;x,y}.$$

Lemma 3.4 (Habiro [2, Proposition 6.6])**.** *For $i, j \geq 0$, we have*

$$\langle S_l, P_m\rangle = \delta_{l,m}\{2m+1\}_{2m}.$$

We prove the following proposition later.

Proposition 3.5. *For $m, n \geq 0$, we have*

$$S_l = \sum_{m\geq 0}\lambda_{l,m}P_m = \sum_{m\geq 0}\lambda_{l,m}\{m\}!P'_m,$$

Proof of Theoem 2.2. By Lemma 3.3 and Proposition 3.5, we have

$$\begin{aligned}
J_{A;S_l,P'_i,P'_j} &= \sum_m \lambda_{l,m}J_{A;P_m,P'_i,P'_j} \\
&= \sum_m \lambda_{l,m}\{m\}!J_{A;P'_m,P'_i,P'_j} \\
&= \delta_{i,j}\lambda_{l,i}\{i\}!(-1)^i\{2i+1\}_{i+1}/\{1\} \\
&= \delta_{i,j}\lambda_{l,i}(-1)^i\{2i+1\}!/\{1\}
\end{aligned}$$

On the other hand, note that $A = B_2(H)$. Hence we have

$$\begin{aligned}
J_{A;S_l,P'_i,P'_j} &= J_{B_2(H);S_l,P'_i,P'_j} \\
&= \sum_{k\geq 0} x_{i,j}^{(k)}\langle S_l, P'_k\rangle \\
&= \sum_{k\geq 0} x_{i,j}^{(k)}\langle S_l, P_k\rangle/\{k\}! \\
&= x_{i,j}^{(l)}\{2l+1\}_{2l}/\{l\}!
\end{aligned}$$

Here the second identity follows from Claim 3.1 and the last identity follows from Lemma 3.4.

Consequently, we have

$$x_{i,j}^{(l)} = \delta_{i,j}(-1)^i \frac{\{2i+1\}!\{l\}!}{\{2l+1\}!}\lambda_{l,i} = a_{i,j}^{(l)}.$$

Hence we have the assertion. □

We prove Prroposition 3.5 by using the following lemmas.

Lemma 3.6. *For $m \geq 0$, we have*

$$S_l = \sum_{k=0}^{l}(-1)^i \begin{bmatrix} 2l+1 \\ i \end{bmatrix} V_{2l-2k}.$$

Proof. We use an induction on l. For $m = 0, 1$, we have

$$\begin{aligned} S_0 &= 1, \\ S_1 &= (V_2 - (q + 1 + q^{-1})) = V_2 - [3]. \end{aligned}$$

For $m \geq 2$, we have

$$\begin{aligned} S_m =& S_{m-1}(V_2 - (q^m + 1 + q^{-m})) \\ =& \sum_{k=0}^{m-1}(-1)^k \begin{bmatrix} 2m-1 \\ k \end{bmatrix} V_{2m-2k-2}(V_2 - (q^m + 1 + q^{-m})) \\ =& \sum_{k=0}^{m-2}(-1)^k \begin{bmatrix} 2m-1 \\ k \end{bmatrix} (V_{2m-2k} + V_{2m-2k-2} + V_{2m-2k-4} - (q^m + 1 + q^{-m})V_{2m-2k-2}) \\ &+ (-1)^{m-1} \begin{bmatrix} 2m-1 \\ m-1 \end{bmatrix} (V_2 - (q^m + 1 + q^{-m})) \\ =& \sum_{k=0}^{m-2}(-1)^k \begin{bmatrix} 2m-1 \\ k \end{bmatrix} (V_{2m-2k} - (q^m + q^{-m})V_{2m-2k-2} + V_{2m-2k-4}) \\ &+ (-1)^{m-1} \begin{bmatrix} 2m-1 \\ m-1 \end{bmatrix} (V_2 - (q^m + 1 + q^{-m})) \\ =& V_{2m} - (q^m + q^{-m})V_{2m-2} - [2m-1]V_{2m-2} \\ &+ \sum_{k=2}^{m-1} \left((-1)^{k-2} \begin{bmatrix} 2m-1 \\ k-2 \end{bmatrix} - (-1)^{k-1} \begin{bmatrix} 2m-1 \\ k-1 \end{bmatrix} (q^m + q^{-m}) + (-1)^k \begin{bmatrix} 2m-1 \\ k \end{bmatrix} \right) V_{2m-2k} \\ &+ (-1)^{m-2} \begin{bmatrix} 2m-1 \\ m-2 \end{bmatrix} - (-1)^{m-1} \begin{bmatrix} 2m-1 \\ m-1 \end{bmatrix} (q^m + 1 + q^{-m}) \\ =& \sum_{k=1}^{m}(-1)^k \begin{bmatrix} 2m+1 \\ k \end{bmatrix} V_{2m-2k}. \end{aligned}$$

□

The following lemma is observed in the proof of [2, Proposition 6.6].

Lemma 3.7 (Habiro [2])**.** *For* $m, n \geq 0$, *we have*

$$\langle V_m, S_n \rangle = \{m+n+1\}_{2n+1}/\{1\}.$$

Proof of Proposition 3.5. For $m, n \geq 0$, we have

$$\begin{aligned}\langle S_m, S_n \rangle &= \sum_l \langle \lambda_{m,l} P_l, S_n \rangle \\ &= \lambda_{m,n} \{2n+1\}_{2n}.\end{aligned}$$

Hence we have

$$\begin{aligned}\lambda_{m,n} &= \langle S_m, S_n \rangle / \{2n+1\}_{2n}. \\ &= \sum_{k=1}^{m} (-1)^k \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \langle V_{2m-2k}, S_n \rangle / \{2n+1\}_{2n} \\ &= \sum_{k=1}^{m} (-1)^k \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \{2m+n-2k+1\}_{2n+1} / \{2n+1\}! \\ &= \sum_{k=1}^{m} (-1)^k \begin{bmatrix} 2m+1 \\ k \end{bmatrix} \begin{bmatrix} 2m+n-2k+1 \\ 2n+1 \end{bmatrix},\end{aligned}$$

where the second identity follows from Lemma 3.6 and the third identity follows from 3.7. Hence we have the assertion. □

References

[1] R. H. Bing, A homeomorphism between the 3-sphere and the sum of two solid horned spheres, Ann. of Math. (2) **56** (1952) 354–362.

[2] K. Habiro, A unified Witten-Reshetikhin-Turaev invariants for integral homology spheres. Invent. Math. **171** (2008), no. 1, 1–81.