Fold maps, topological information of their Reeb spaces and their source manifolds.
(折り目写像とその Reeb 空間の位相的情報と定義域多様体)

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Introduction and preliminaries

Main theme and notation

Main theme. The **Reeb space** of a generic smooth map whose codimension is minus and its **global algebraic topological property: homology groups and cohomology rings**
and application to **algebraic and differential topology of manifolds**.

Reeb space.

- The space defined as the **space of all connected components of inverse images of smooth maps**.
- Fundamental and important tools in the theory of Morse functions and its higher dimensional version to investigate the source manifolds, **inheriting fundamental invariants of manifolds** such as homology groups etc. in considerable cases.

Notation and terminologies.

$m > n \geq 1$: integers
M : a closed and connected manifold of dimension m
$f : M \to \mathbb{R}^n$: a (smooth) map
$S(f)$:the set of all singular points (the *singular set*)
$f(S(f))$ $(N - f(S(f)))$: the *singular* (resp. *regular*) *value set*
All the manifolds and maps between them are smooth and of class C^{∞} unless otherwise stated.

Fold maps

Definition 1

f: a *fold map* $\leftrightarrow$ At each singular point p f is of the form
$(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i(p)} {x_k}^2 - \sum_{k=m-i(p)+1}^{m} {x_k}^2)$
for an integer $0 \leq i(p) \leq \frac{m-n+1}{2}$.

$n = 1 \leftrightarrow$ Morse function.

Proposition 1

1. *The integer $i(p)$ is unique (we call $i(p)$ the index of p).*
   (f : special generic $\leftrightarrow$ f : *a fold map s.t.* $i(p) = 0$ *for each* p)
2. *The set of all singular points of an index is a smooth submanifold of dimension $n-1$ and $f|_{S(f)}$ is an immersion.*

$n = 1 \rightarrow$ The number of singular points of an index (defined by respecting the orientation of the target and defined uniquely) tells us about homology groups (the classical theory of Morse functions).

Remarks on fold maps and stable maps

- The restriction to the singular set of a fold map is transversal. $\Leftrightarrow$ The map is *stable*.
  f : *stable* $\leftrightarrow$ For a smooth map f' obtained by a slight perturbation, there always exists a pair of diffeomorphisms (Φ, ϕ) satisfying $f \circ \Phi = \phi \circ f'$.
  $\Leftrightarrow$ The types of the singular set and the singular value set are invariant under slight perturbations.

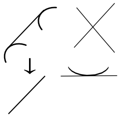

Figure 1: Singular points of fold maps and singular value sets of a stable fold map and a fold map being not stable (into the plane).

- Stable Morse functions (Morse functions such that at distinct singular points, the values are always distinct,) always exist densely on any closed manifold.

Reeb spaces

Reeb spaces

Definition 2

X, Y : topological spaces
$p_1, p_2 \in X$
$c : X \to Y$: a continuous map
$p_1 \sim_c p_2 \leftrightarrow p_1$ and p_2 are in the same connected component of $c^{-1}(p)$ for some $p \in Y$.
$\to$The relation is an equivalence relation.
$W_c := X/\sim_c$: the *Reeb space* of c.

$\to$ The Reeb space often inherits information of the source manifold (homology groups etc.).
$q_c : X \to W_c$: the quotient map
$\bar{c}$: a map uniquely define so that $c = \bar{c} \circ q_c$

Examples (Morse functions and their Reeb spaces)

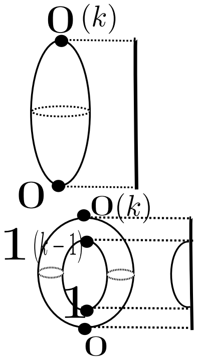

Figure 2: The Reeb spaces of Morse functions on a k-dim. homotopy sphere (except 4-dim. exotic spheres, which are undiscovered,) and a torus or $S^1 \times S^k$ (the numbers represent indices of singular points: ones in the brackets represent indices for Morse functions explained before).

The first homology groups of the source manifold and the Reeb space agree ($k > 1$).

Examples (fold maps and their Reeb spaces)

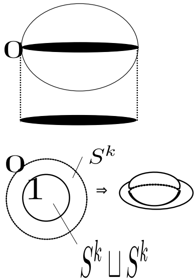

Figure 3: The Reeb spaces of fold maps; a canonical projection of a unit sphere of dim. k, which is one of the simplest special generic maps, and a stable fold map from $S^2 \times S^k$ into the plane (each number represents index of each singular point and manifolds represent inverse images of corresponding points

$k \geq 1$ and

- In the former case, the j-th homology groups of the source manifold and the Reeb space agree for $j < k$.
- In the latter case the first and the second homology groups of the source manifold and the Reeb space agree for $k > 2$.

Special generic maps and round fold maps

Fundamental properties of special generic maps

Fact 1 (Saeki (1993))

$\exists f : M \to \mathbb{R}^n$: *special generic*

$\Leftrightarrow$

- $\exists$ *a compact smooth manifold* W_f *s.t.* $\partial W_f \neq \emptyset$ *which we can immerse into* $\mathbb{R}^n$.
- M *is obtained by gluing the following two manifolds by a bundle isomorphism between the* S^{m-n}*-bundles over the boundary* ∂W_f.
  - *A smooth* S^{m-n}*-bundle over* W_f.
  - *A linear* D^{m-n+1}*-bundle over* ∂W_f.

W_f is regarded as the **Reeb space** of f.

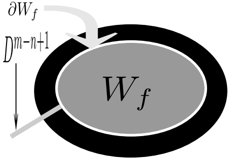

Figure 4: The image of a special generic map.

Special generic maps into the plane

Fact 2 (Saeki (1993))

M : a homotopy sphere of dimension ≥ 2 (not exotic 4-dim. one)
$\Rightarrow \exists f : M \to \mathbb{R}^2$:special generic map s.t. W_f is homeomorphic to D^2.
On the other hand, manifolds admitting such special generic maps are such homotopy spheres.

Example 1 (Saeki (1993) etc.)

Figure 5 represents the image of a special generic map into the plane on a manifold represented as a connected sum of total spaces of (three) suitable smooth bundles with fibers being homotopy spheres over S^1 (product bundles are OK).
$\to$ Conversely, a manifold admitting such a map is such a manifold.

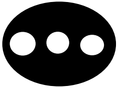

Figure 5: A special generic map into the plane.

Special generic maps and differentiable structures

Fact 3 (Saeki (1993))

Each exotic homotopy sphere of dimension $m > 3$ does not admit a special generic map into $\mathbb{R}^{m-3}$, $\mathbb{R}^{m-2}$ and $\mathbb{R}^{m-1}$.

Fact 4 (Wrazidlo (2017))

7-dim. oriented homotopy spheres of 14 types (of 28 types) do not admit special generic maps into $\mathbb{R}^3$.

Fact 5 (Saeki and Sakuma (1990s–2000s))

$\exists$ *pairs of* **homeomorphic** 4*-dim. closed manifolds satisfying the following*:
for each pair, both manifolds admit fold maps into $\mathbb{R}^3$,
one admits a special generic map into $\mathbb{R}^3$ and the other not.

$\to$ **Special generic maps into spaces whose dimensions are larger than 2 often restrict the diffeomorphism types**, which makes special generic maps attractive.

Round fold maps

Definition 3

$m > n \geq 2$ $\quad f$: a fold map

f: a <u>round</u> fold map $\leftrightarrow$ $f|_{S(f)}$: embedding $\quad f(S(f))$: concentric

Example 2

Maps in FIGURE 3: (the target space must be of dim. larger than 1).

Round fold maps and monodromies

Definition 4

A round fold map has

- *a trivial monodromy*. $\leftrightarrow$ Consider the inverse image of the complement P of an open disc in the connected component diffeomorphic to an open disc of the regular value set. If we consider the composition of the restriction map of f to the inverse image and the canonical projection to the boundary ∂P, then it gives a trivial smooth bundle.
- *locally trivial monodromies*. $\leftrightarrow$ Consider each connected component of the singular value set and small closed tubular neighborhood, If we consider the composition of the restriction map of f to the inverse image of the neighborhood and the canonical projection to the component, then it gives a trivial smooth bundle.

Round fold maps and monodromies (figures)

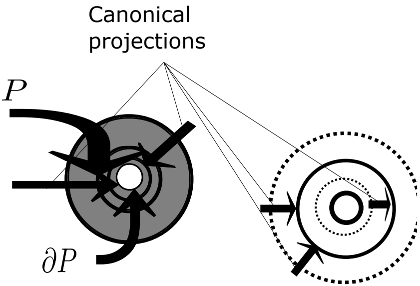

Figure 6: A round fold map having a trivial monodromy and locally trivial monodromies.

Having a trivial monodromy. $\rightarrow$ **Globally product.**
Having locally trivial monodromies. $\rightarrow$ **Locally product.**
$\rightarrow$ There are several examples of maps having local trivial monodromies and not having trivial monodromies (2014 (K)).

Manifolds admitting round fold maps (1)

An *almost-sphere*.
$\leftrightarrow$ A homotopy sphere obtained by gluing two standard closed discs on the boundaries.
$\Leftrightarrow$ Homotopy spheres which are not 4-dim. exotic spheres, undiscovered.

Fact 6 (2013 (K))

$m > n \geq 2$ M : m-dim. manifold
M is represented as a total space of a smooth bundle with fibers being an almost-sphere Σ over a standard sphere of dim. n.
$\Leftrightarrow \exists f : M \to \mathbb{R}^n$: a round fold map having a trivial monodromy s.t.

- *The number of connected components of the singular value set has 2 connected components.*
- *The inverse image of a point in the connected component being an open disc of the regular value set is $\Sigma \sqcup \Sigma$.*

$\rightarrow$ A generalized map of the latter one in Figure 3

Manifolds admitting round fold maps (2)

Fact 7 (2013 (K))

$m > n \geq 2$ M : m-dim. manifold
M is represented as a connected sum of total spaces of smooth bundles with fibers being S^{m-n} over a standard sphere of dim. n.
$\Rightarrow \exists f : M \to \mathbb{R}^n$: a round fold map having locally trivial monodromies s.t.

- *The number of connected components of the singular value set has $l+1$ components.*
- *The inverse image of a point in the component being an open disc of the regular value set is a disjoint union of $l+1$ copies of S^{m-n}.*

$\rightarrow$ A generalization of the map just before and the Reeb space is a bouquet of l n-dim. spheres.

Example 3 (2013 (K))

$(m, n) = (7, 4)$ M : an oriented homotopy sphere. $\rightarrow$ We can apply.

$\rightarrow$ Differentiable structures are affected by topological types of the maps (**not necessarily special generic**).

Surgerying maps and changes of homology groups of the Reeb spaces

Obtaining new maps and manifolds by surgeries and application to global studies of Reeb spaces and manifolds

Explicitly constructing manifolds and smooth maps systematically is fundamental, important and **difficult** in the singularity theory and differential topological theory of manifolds.

3-manifolds and 4-manifolds
→ Construction by surgeries (Dehn surgeries etc.) is fundamental and important,
→ Stable fold maps and more general generic maps are constructed systematically under several situations.
Higher dimensional manifolds and maps
→ Algebraic topologically it is not so difficult owing to the situations that **the dimenisions are high** and that we can freely move handles or local manifolds.
→ Explicit **construction and representations** of such manifolds are **difficult to know systematically**.
→ Explicit stable fold maps and more general generic maps are also **difficult to obtain** in general except the presented examples etc.

Construction is far more difficult than existence in general.

Bubbling surgeries by Kobayashi

Definition 5 (Kobayashi (2011–2))

A **bubbling surgery** is an operation of exchanging a stable (fold) map into another one by removing an open ball with its inverse image in the regular value set and attach a new map with a standard sphere coinciding with the new connected component of the new singular value set consisting of values at singular points equivalent to ones of fold maps.

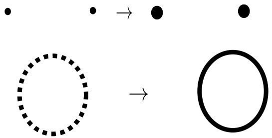

Figure 7: Bubbling surgeries (n=1 and $n = 2$ cases).

Example 4

A round fold map is obtained by a finite iteration of bubbling surgeries starting from a canonical projection of an unit sphere. In the case where $n = 1$ holds, we can naturally extend the definition of a round fold map.

Bubbling operations

$f : M \to N$: a stable fold map.
$S \subset \mathbb{R}^n - f(S(f))$; a bouquet of a finite number of closed, connected and orientable submanifolds.
$N_i(S) \subset N(S) \subset N_o(S)$; regular neighborhood of S.
Q; a component of $f^{-1}(N_o(S))$ s.t. $f|_Q$ is a bundle over $N_o(S)$.
M'; an m-dim. closed manifold s.t. $M - \mathrm{Int}\, Q$ is a compact smooth submanifold.
$f' : M' \to N$ a stable fold map.
$f|_{M - \mathrm{Int}\, Q} = f'|_{M - \mathrm{Int}\, Q}$
(The singular value set of f') $= f(S(f)) \sqcup \partial N(S)$.
$f'|_{(M' - (M - Q)) \bigcap f'^{-1}(N_i(S))}$; smooth bundle over $N_i(S)$.

Definition 6 (K (2015–))

A *bubbling operation*. $\leftrightarrow$ A procedure of constructing f' from f.
S is called the *generating polyhedron* of the operation.
A bubbling operation is *normal*. $\leftrightarrow$ The generating polyhedron is a manifold (*generating manifold*).
A bubbling operation is an *M-bubbling operation*. $\leftrightarrow$ The last smooth bundle consists of 2 components.

Explicit bubbling operations and changes of Reeb spaces

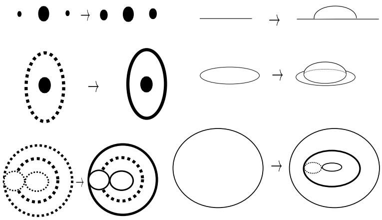

Figure 8: Changes of singular value sets (three figures in the left) and Reeb spaces (three figures in the right) by bubbling operations

$\rightarrow$ The three figures in the left show cases $(n, k) = (1, 0), (2, 0), (3, 1)$ where k is the dim. of a generating manifold (polyhedron).
$\rightarrow$ The three figures in the right show cases $(n, k) = (1, 0), (2, 0), (2, 1)$ of M-bubblings.

Example 5

A round fold map in Fact 7 is obtained by a finite iteration of M-bubbling operations starting from a canonical projection of an unit sphere.

More explicit theme

Studies of

- Construction of explicit maps, Reeb spaces and source manifolds.
- (Potential) application to geometry of manifolds.

by bubbling operations based on constructive related studies by Kobayashi .

Problem 1

Study the resulting topologies of Reeb spaces (and consequently manifolds) obtained by finite iterations of such operations starting from fundamental maps.

$\rightarrow$ We can change (co)homology groups (rings) flexibly in suitable cases.

Simple results for M-bubbling operations

Fact 8 (K (2015–8))

M : an m-dim. closed manifold.
$\{G_k\}_{k=1}^{n}$: a family of free finitely generated Abelian groups s.t. $\sum_{k=1}^{n-1} \mathrm{rank}\, G_k \leq \mathrm{rank}\, G_n$ and $\mathrm{rank}\, G_n \neq 0$.
*$\Rightarrow \exists f' : M' \to \mathbb{R}^n$: a fold map on an m-dim. manifold M' obtained by a finite iteration of **normal** bubbling operations to a fold map $f : M \to \mathbb{R}^n$ s.t. $H_k(W_{f'};\mathbb{Z}) \cong H_k(W_f;\mathbb{Z}) \oplus G_k$.*

$\to$ Take standard spheres as generating manifolds suitably.

Fact 9 (K (2015–8))

In the previous fact, if we drop "normal" and "$\sum_{k=1}^{n-1} \mathrm{rank}\, G_k \leq \mathrm{rank}\, G_n$", then the same fact holds.

Remark 1 (K (2015–8))

For **suitable** explicit sequences of the groups containing ones with **torsions**, we can show several similar facts.

A recent result –cohomologies of Reeb spaces obtained by finite iterations of M-bubbling operations–

A result on cohomology rings of Reeb spaces

Theorem 1 (K (2018))

$n \geq 3$ $m >> n$ f: a stable fold map obtained from a finite iteration of bubbling operations starting from a special generic map.

- *W_f is regarded as a regular neighborhood of a bouquet of spheres.*
- *The bouquet includes just $n_k \geq 0$ k-dim. spheres for $1 \leq k \leq n-1$.*
- *For integers $1 \leq j \leq n-1$ and $1 \leq j' \leq n_j$, an integer $A_{j,j'}$ is defined and if the relation $2(n-j) > n$ holds, then it is 0, 1 or -1.*

$\{G_j\}_{j=1}^{n}$: free finitely generated Abelian groups s.t. $G_1 = \{0\}$, $\mathrm{rank} G_n > 0$ and $G_j = \oplus_{j'=1}^{\mathrm{rank} G_n} G_{j,j'}$.
$\Rightarrow \exists f'$: obtained by a finite iteration of M-bubbling operations to f s.t.

1. *$H_j(W_{f'};\mathbb{Z}) \cong H_j(W_f;\mathbb{Z}) \oplus G_j$ $H^j(W_{f'};\mathbb{Z}) \cong H^j(W_f;\mathbb{Z}) \oplus G_j$.*
2. *H_i: a graded ring regarded as $\oplus_{j=1}^{n}(G_{j,i} \oplus H^j(W_f;\mathbb{Z}))$ as a group s.t. $H^*(W_{f'};\mathbb{Z})$ and $\frac{H^*(W_f;\mathbb{Z}) \oplus \oplus_{i=1}^{\mathrm{rank} G_i} H_i}{W_f}$ are isomorphic as rings where W_f identify all $H^*(W_{f'};\mathbb{Z})$'s appearing in the summand.*
3. *For H_i, consider the j'-th j-dim. sphere of the bouquet for W_f representing a generator of H_i, then for a generator of $G_{n-j,i}$, the product is $A_{j,j'}$ times a generator of a submodule of $H^n(W_{f'};\mathbb{Z})$.*

A remark on the previous theorem

Remark 2

For considerable cases, we obtain the following.

- $\exists$ (Families of cohomology rings with coefficient rings $\mathbb{Z}$ of Reeb spaces) mutually non-isomorphic s.t.
  The underlying homology groups are isomorphic.
- $\exists$ (Families of cohomology rings with coefficient rings $\mathbb{Z}$ of Reeb spaces) mutually non-isomorphic s.t.
  - The cohomology rings obtained by tensoring $\mathbb{Q}$ are isomorphic.
  - Each family includes infinite rings.

Additional remarks on the previous theorem

- If the Reeb space has considerable cohomological information of manifolds, then we consequently obtain families of source manifolds whose cohomology rings are mutually distinct and whose homology groups (and cohomology rings with coefficient rings $\mathbb{Q}$) are isomorphic.
  $\rightarrow$ If **inverse images of regular values are disjoint unions of spheres** and **several conditions on differential topological properties of maps** are satisfied, we can know such invariants of source manifolds completely from Reeb spaces (Saeki, Suzuoka and K 2000s-). $\rightarrow$ **Special generic maps**, **presented maps** etc. OK.
  $\rightarrow$ **Topology of Reeb spaces** and **cohomology rings of manifolds** closely related.
  $\rightarrow$ Compare this fact to a relation between **differentiable structures of homotopy spheres** and **differential topological properties of certain fold maps** presented before: Fact 5, Example 3 etc..
- $\exists$ pairs of source manifolds **cohomologically isomorphic** s.t. **characteristic classes** are distinct.

Future works

Future works

Problem 2

For an arbitrary graded ring satisfying an appropriate suitable condition, can we give a Reeb space (a manifold) whose cohomology ring is isomorphic to the ring.

$\rightarrow$ As the speaker thinks, it seems to be true that we can construct new rings one after another by introducing new types of surgery operations one after another.

Problem 3

How about the homology groups, the cohomology rings etc. of source manifolds, not of Reeb spaces, generally.

Problem 4

More precise information on Reeb spaces and source manifolds

Thank you.