Ideal tetrahedral decomposition of hyperbolic chain link complement

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Joint work with
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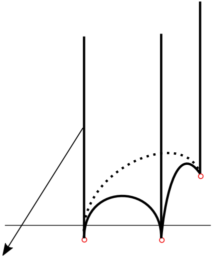
Hyperbolic 3-space

Hyperbolic 3-space $\mathbb{H}^3$

$$(\{(x+yi,t)\in\mathbb{C}\times\mathbb{R}\mid t>0\}, \frac{dx^2+dy^2+dt^2}{t^2})$$

The geodesics in $\mathbb{H}^3$ are vertical lines which meet the plane $\mathbb{C}$ at a right angle, and semi-circles with center on $\mathbb{C}$. An ideal tetrahedron is a tetrahedron in $\mathbb{H}^3$ with all four vertices on $\partial\mathbb{H}^3$.

Geodesic and ideal tetrahedron



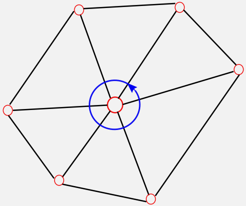
Hyperbolic 3-manifold

Definition

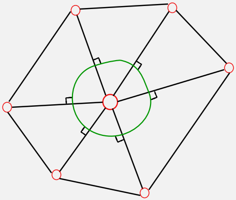
A 3-manifold, M is hyperbolic if M is locally modelled on $\mathbb{H}^3$.

Fact [Epstein-Penner (1988)]

A hyperbolic 3-manifold M can be obtained by glueing hyperbolic ideal polyhedra $P_1, \ldots, P_n$ such that



angle sum is 2 π.



completeness.

Open problem

Question

Let M be a noncompact hyperbolic 3-manifold of finite volume. Can we decompose M into ideal tetrahedra?

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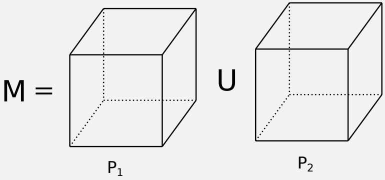
Example [Difficulty]



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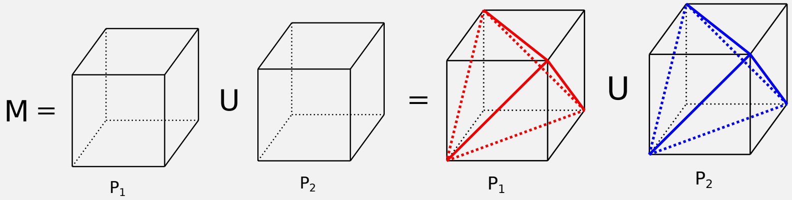


Theorem

Theorem [Yoshida, Osaka J. Math., 1996]

Suppose that a noncompact hyperbolic 3-manifold M is obtained by glueing two convex ideal polyhedra P_1 and P_2 in such a way that every face of P_1 is pasted with a face of P_2. Then, M can be decomposed into ideal tetrahedra.

Hyperbolic Link

Definiton

L is a hyperbolic link if $S^3 \setminus L$ is a hyperbolic manifold.

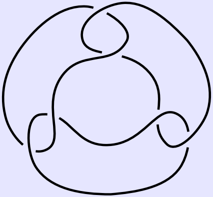
Fact (Thurston)

L is a hyperbolic link in S^3 if and only if L is neither a satellite link nor a Seifelt link.

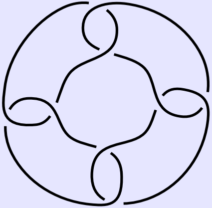
n-chain link

Definition

$n = 3$



$n = 4$



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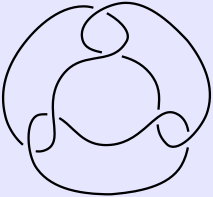

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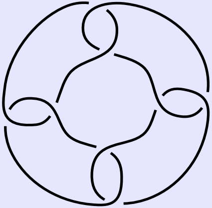


We focus on alternating n-chain links. It is known that all alternating chain links are hyperbolic for $n \geq 3$.

Theorem

Theorem.

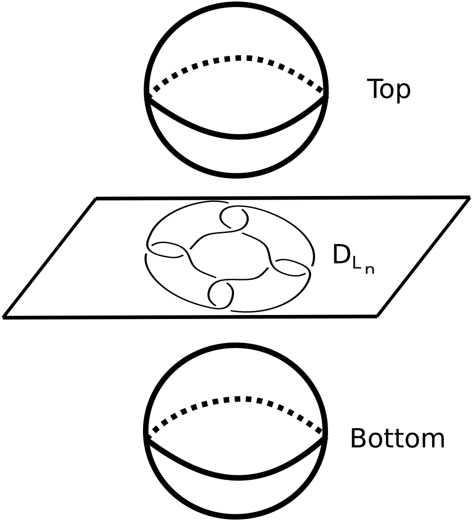
Let L_n be an alternating n-chain link and D_{L_n} the reduced alternating diagram of L_n. Let $P_1 \cup P_2$ denote the polyhedral decomposition of $S^3 \setminus L_n$ obtained from D_{L_n}. We consider the set

$$S_{P_1 \cup P_2} = \{(v_1, v_2) \mid v_1 \in V(P_1), v_2 \in V(P_2)\}$$

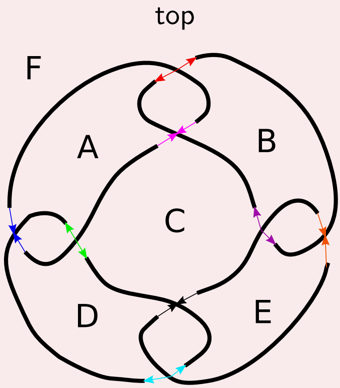
so that an ideal tetrahedral decomposition of $S^3 \setminus L_n$ can be obtained from the cone decompositions of P_1 and P_2 from v_1 and v_2, respectively. Then, we have the following.

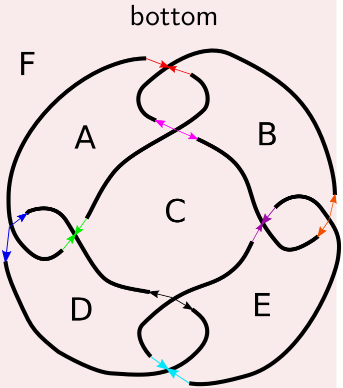
$$\#S_{P_1 \cup P_2} = \begin{cases} 18 & \text{if } n = 3 \\ 32 & \text{if } n = 4 \\ 2n^2 - 2n & \text{if } n \geq 5 \end{cases}$$

Polyhedral decomposition



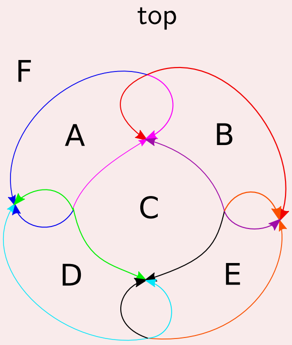
Outline of Proof [$n=4$]

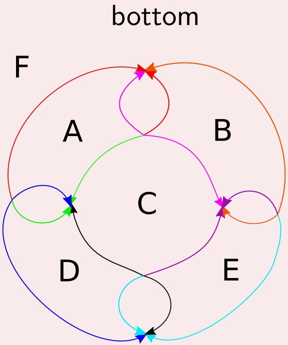




Polyhedral decomposition

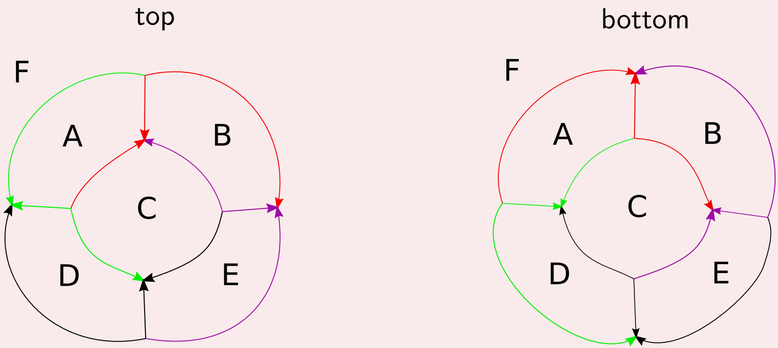
Shrink L_n to ideal vertices





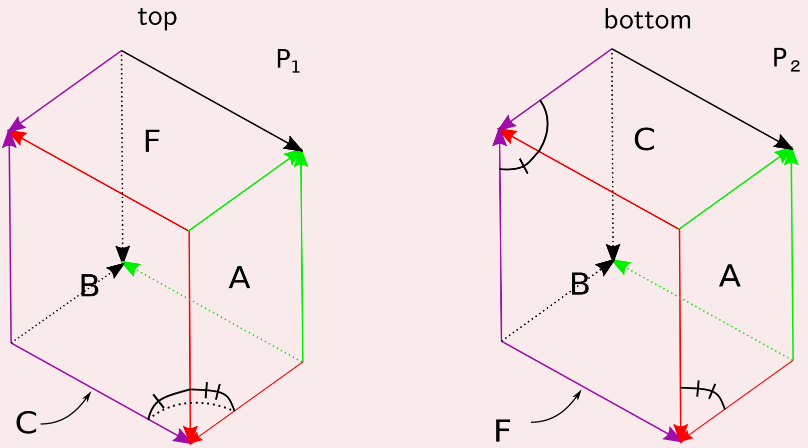
Polyhedral decomposition

Collapse bigon faces



Polyhedral decomposition

4-prisms.

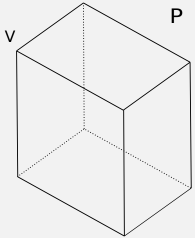


Cone decomposition

Cone decomposition of P from a vertex v

Devide the fases of the polyhedron P not containing v into triangles (in any way you like), and decompose P into the cone of these triangles from the vertex v.

Example



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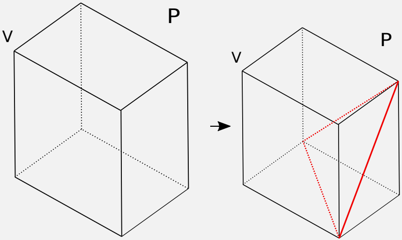

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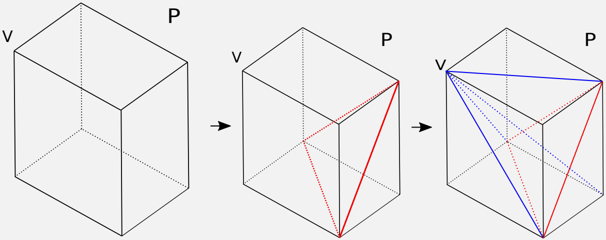
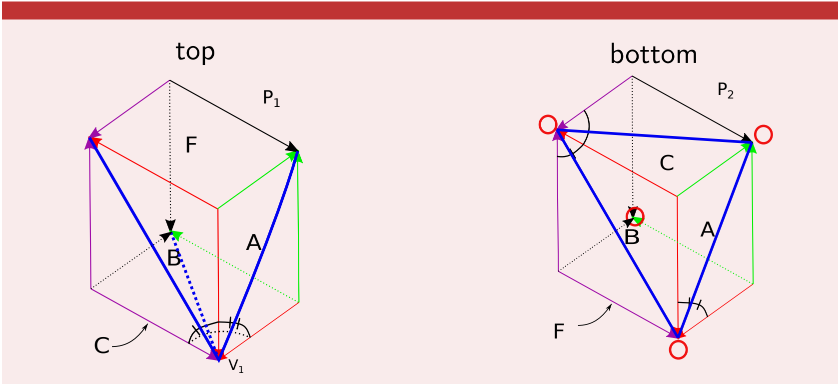


Tetrahedral decomposition



In Figure, we have $\#S_{P_1 \cup P_2} = 4$ for the vertex v_1 of P_1. Since P_1 has eight vertices, $\#S_{P_1 \cup P_2} = 32$.

Thank you for your attention!