On the minimal coloring number of the minimal diagram of torus links

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Let $L$ be a link, and $D$ a diagram of $L$.

**Z-coloring**

A map $\gamma : \{\text{arcs of } D\} \rightarrow \mathbb{Z}$ is called a **Z-coloring** on $D$ if it satisfies the condition $2\gamma(a) = \gamma(b) + \gamma(c)$ at each crossing of $D$ with the over arc $a$ and the under arcs $b$ and $c$. A Z-coloring which assigns the same color to all the arcs of the diagram is called a **trivial Z-coloring**.

$L$ is **Z-colorable** if $\exists$ a diagram of $L$ with a non-trivial Z-coloring.
Let $L$ be a $\mathbb{Z}$-colorable link.

**Minimal coloring number**

[1] For a diagram $D$ of $L$,

$$\text{mincol}_{\mathbb{Z}}(D) := \min\{\#\text{Im}(\gamma) \mid \gamma : \text{non-tri. } \mathbb{Z}\text{-coloring on } D\}$$

[2] $$\text{mincol}_{\mathbb{Z}}(L) := \min\{\text{mincol}_{\mathbb{Z}}(D) \mid D : \text{a diagram of } L\}$$
**Simple \( \mathbb{Z} \)-coloring**

\( \gamma : \) a \( \mathbb{Z} \)-coloring on a diagram \( D \) of a non-trivial \( \mathbb{Z} \)-colorable link \( L \)

If \( \exists d \in \mathbb{N} \) s.t. at each crossings in \( D \), the differences between the colors of the over arcs and the under arcs are \( d \) or 0, then we call \( \gamma \) a simple \( \mathbb{Z} \)-coloring.
Simple $\mathbb{Z}$-coloring

$\gamma$: a $\mathbb{Z}$-coloring on a diagram $D$ of a non-trivial $\mathbb{Z}$-colorable link $L$. If $\exists d \in \mathbb{N}$ s.t. at each crossings in $D$, the differences between the colors of the over arcs and the under arcs are $d$ or $0$, then we call $\gamma$ a simple $\mathbb{Z}$-coloring.

Theorem 1 [Ichihara-M., JKTR, 2017]

Let $L$ be a non-splittable $\mathbb{Z}$-colorable link. If there exists a simple $\mathbb{Z}$-coloring on a diagram of $L$, then $\text{mincol}_{\mathbb{Z}}(L) = 4$. 
Theorem 2 [M., to apper JKTR, Zhang-Jin-Deng]

Any $\mathbb{Z}$-colorable link has a diagram admitting a simple $\mathbb{Z}$-coloring.
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Any $\mathbb{Z}$-colorable link has a diagram admitting a simple $\mathbb{Z}$-coloring.

Colorally

$L$ : a $\mathbb{Z}$-colorable link

$$\text{mincol}_\mathbb{Z}(L) = \begin{cases} 2 & (L \text{ : splittable}) \\ 4 & (L \text{ : non-splittable}) \end{cases}$$
One of key moves of the proof

\[
\begin{align*}
&x \pm d_m \\
&x \mp d_m
\end{align*}
\]
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**Problem**

\[ \text{mincol}_\mathbb{Z}(D_m) = ? \] for a minimal diagram \( D_m \) of a \( \mathbb{Z} \)-colorable link.

[1] For an even integer $n \geq 2$, the pretzel link $P(n, -n, \cdots, n, -n)$ with at least 4 strands has a minimal diagram $D_m$ s.t. $\text{mincol}_{\mathbb{Z}}(D_m) = n + 2$.

[2] For an integer $n \geq 2$, the pretzel link $P(-n, n+1, n(n+1))$ has a minimal diagram $D_m$ s.t. $\text{mincol}_{\mathbb{Z}}(D_m) = n^2 + n + 3$.

[3] For even integer $n > 2$ and non-zero integer $p$, the torus link $T(pn, n)$ has a minimal diagram $D_m$ s.t. $\text{mincol}_{\mathbb{Z}}(D_m) = 4$. 
Theorem 4 [Ichihara-Ishikawa-M., In progress]

Let $p, q$ and $r$ be non-zero integers such that $|p| \geq q \geq 1$ and $r \geq 2$. If $pr$ or $qr$ are even, the torus link $T(pr, qr)$ has a minimal diagram $D_m$ s.t.

$$\text{mincol}_{\mathbb{Z}}(D_m) = \begin{cases} 4 & (r : \text{even}) \\ "5" & (r : \text{odd}) \end{cases}$$

Remark

A torus link $T(pr, qr)$ is $\mathbb{Z}$-colorable if and only if $pr$ or $qr$ are even.
Proof of Theorem 4 (In the case $r$:even)]

Let $D$ be the following minimal diagram of $T(pr, qr)$. 
In the following, we will find a \( \mathbb{Z} \)-coloring \( \gamma \) on \( D \) by assigning colors on the arcs of \( D \). We devide such arcs into \( q \) subfamilies \( x_1, \cdots, x_q \).
We first find a local $\mathbb{Z}$-coloring $\gamma$. In the case $r$ is even, we start with setting $\gamma(x_i) = (\gamma(x_{i,1}), \gamma(x_{i,2}), \cdots, \gamma(x_{i,r})) = (1, 0, \cdots, 0, 1)$ for any $i$. 
We can extend $\gamma$ on the arcs in the regions (1) and $(q + 1)$. 
We can extend $\gamma$ on the arcs in the regions $(2), (3), \ldots, (q)$. 
Now, $\gamma$ can be extended on all the arcs in the region depicted as follows.
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Since $D$ is composed of $p$ copies of the local diagram, it concludes that $D$ admits a $\mathbb{Z}$-coloring with only four colors 0, 1, 2 and 3.
Thank you for your attention.