3 実交点の twisted knot の分類表

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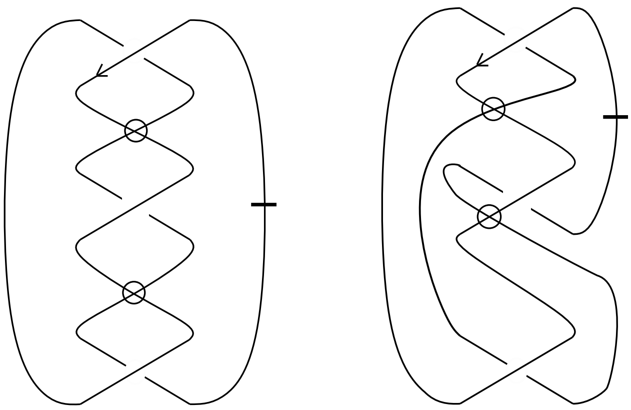
Twisted knots

Definition

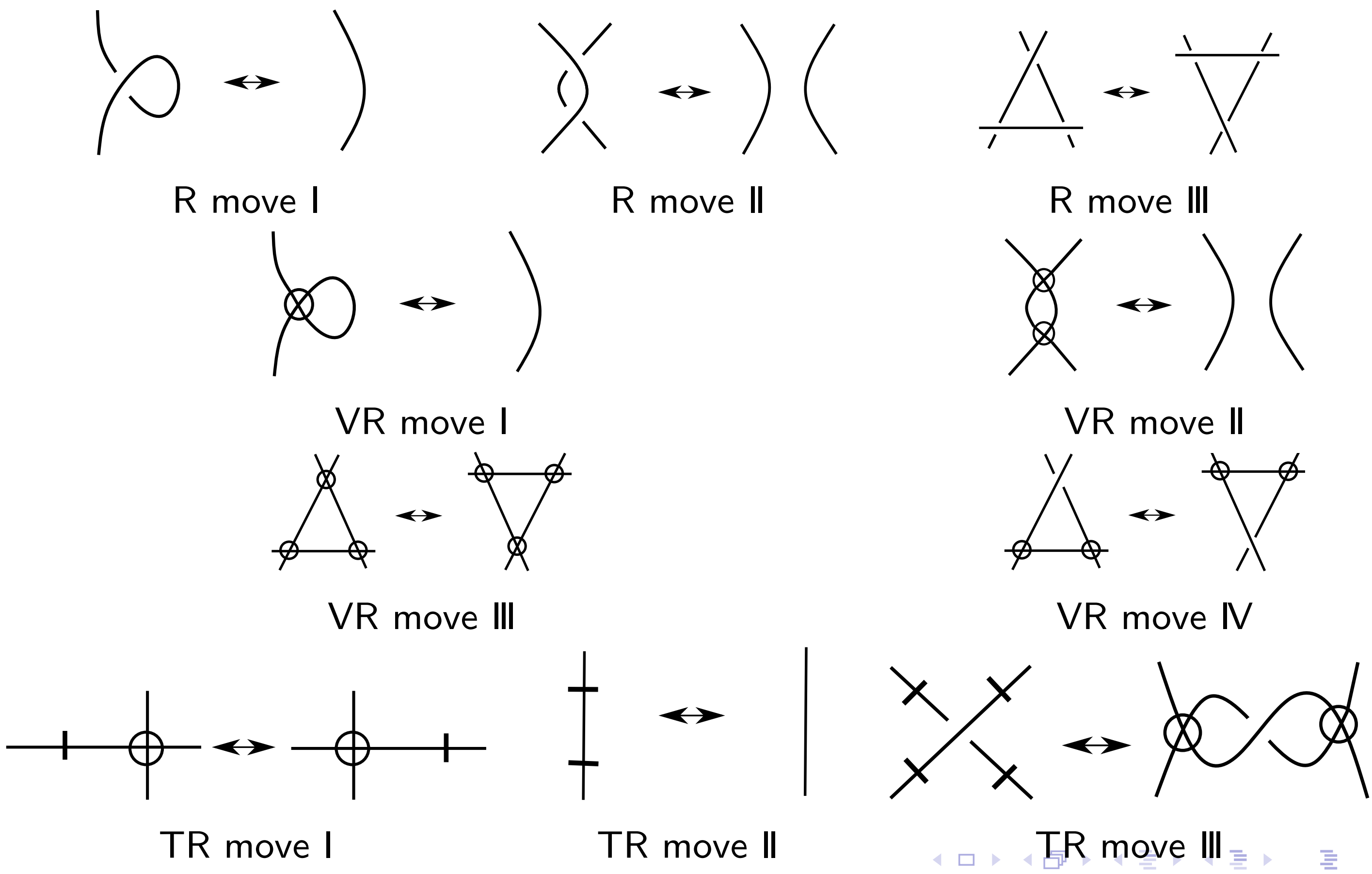
A twisted link diagram ⇔ A virtual link diagram (possibly) with bars.



Definition

Twisted links ⇔ Equivalence classes of twisted link diagrams under extended R moves (R moves, VR moves and twisted R moves).

Extended R moves



Purpose

To construct a table of prime twisted knots with 3 classical crossings.

⟨Known result⟩

- A table of virtual knots with crossing number up to 6 was constructed by Jeremy Green.

[Jeremy Green, https://www.math.toronto.edu/drorbn/Students/GreenJ/]

- A table of twisted knots with crossing number up to 2 was constructed by N.Kamada.

[N. Kamada, ***On twisted knots***, Contemporary Mathematics, Knot Theory and its Applications,670(2016), pp.328-341]

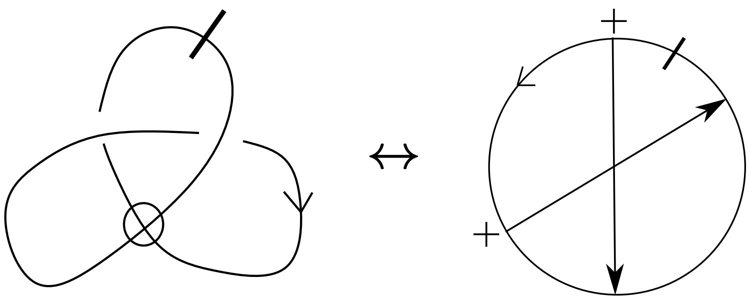
A Gauss chord diagram of twisted knot

Definition

A Gauss chord diagram of virtual knot $\Leftrightarrow$ The pre-image of the immersed loop of a knot diagram with oriented chords whose starting and end points are on a circle.The starting point (or end point) of a chord indicates an over arc (or an under arc). Each chord is labeled with $+$ or $-$, which is the sign of a crossing corresponding the chord.

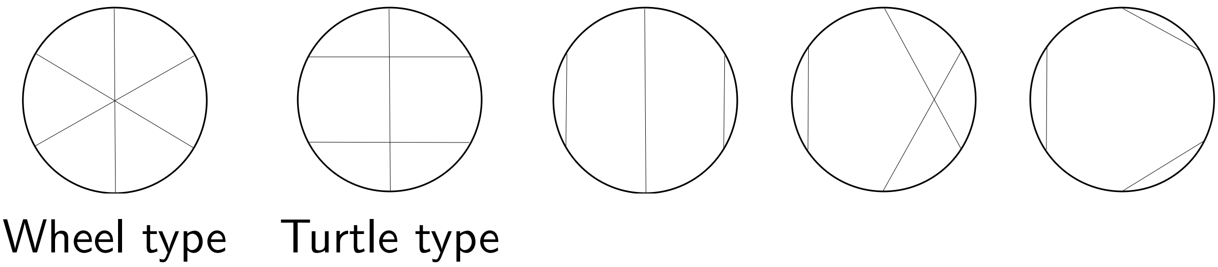
A twisted knot diagram is presented by a Gauss chord diagram with some bars on the arcs which correspond to edges of a twisted link diagram.

Example



A classification of twisted knots with 3 classical crossings

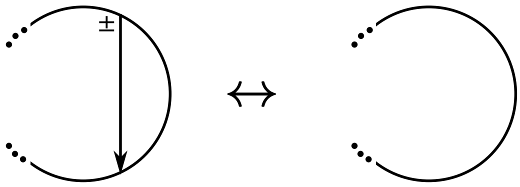
1. List up to projections of Gauss chord diagrams with 3 classical crossings, where a projection of a Gauss chord diagram is a circle with chords.



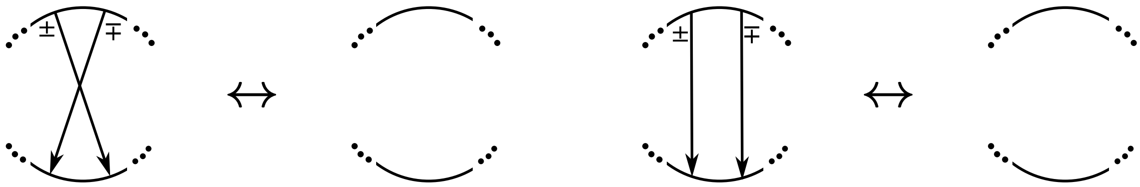
A classification of twisted knots with 3 classical crossings

2. Give directions, signs and bars to chords.
3. Remove equivalent Gauss chord diagrams under extended R moves.

R move I

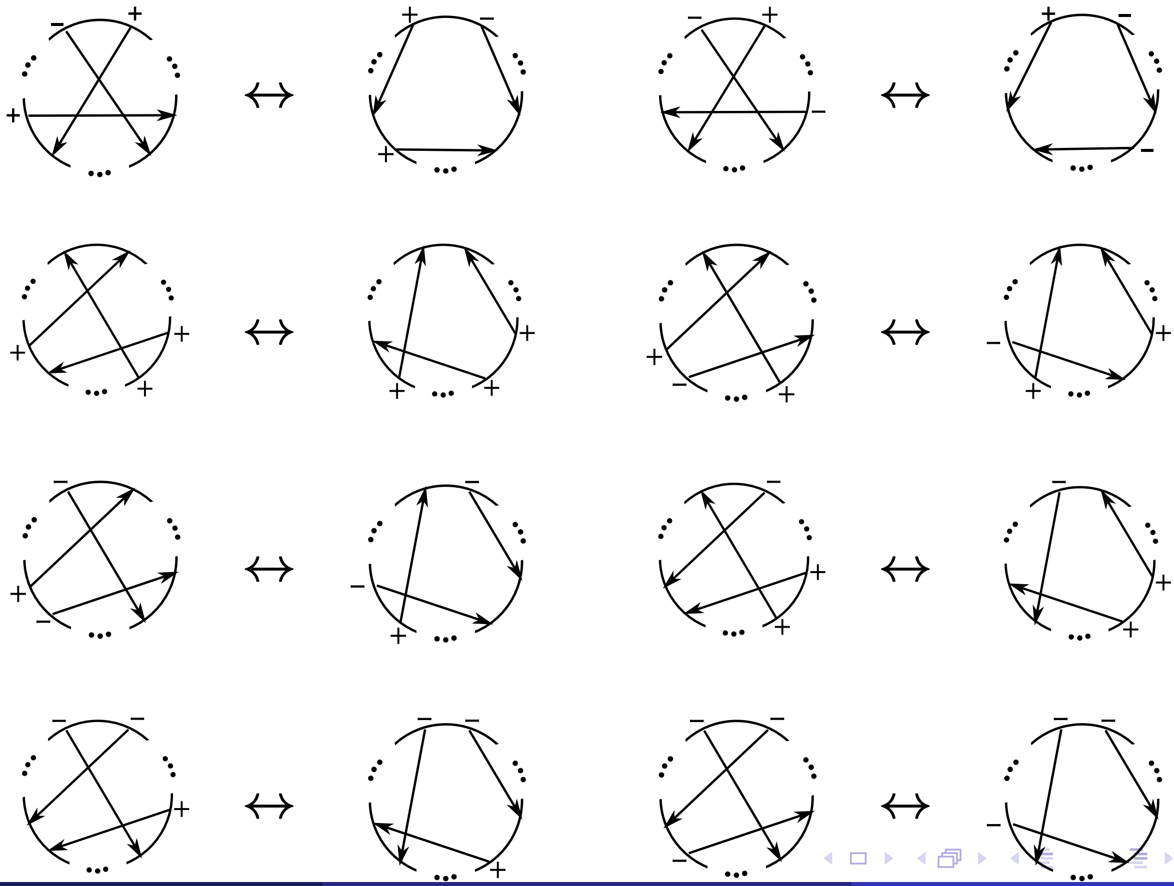


R move Ⅱ



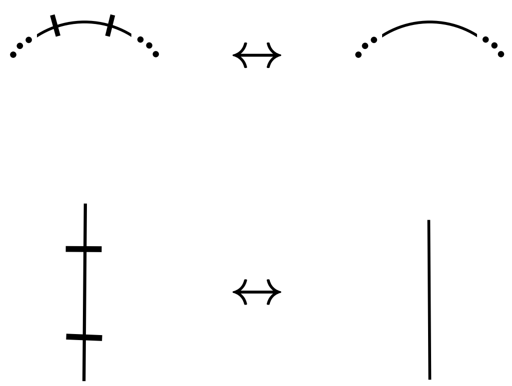
A classification of twisted knots with 3 classical crossings

R move Ⅲ

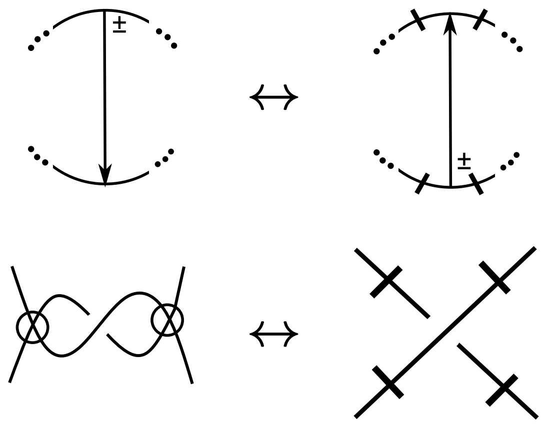


A classification of twisted knots with 3 classical crossings

TR move II



TR move III



26 Wheel types.
61 Turtle types.

A classification of twisted knots with 3 classical crossings

4 Construct diagrams $D^{\#}$, $-D$, and $-D^{\#}$ for a twisted link diagram D in the table, where $D^{\#}$ is a mirror image of D, and $-D$ is a diagram obtained by reversing the orientation of D.
Calculate multivariable polynomial invariants (Arrow polynomial or Miyazawa polynomial) $X_D \in \mathbb{Z}[A, A^{-1}, M, d_1, d_2, \ldots]$ of them.

Remark $D \sim D^{\#*}$ and $D^{\#} \sim D^*$ as a twisted link. (D^* is the reflection of D)
(N.Kamada, S.Kamada, Virtual links which are equivalent as twisted links, preprint, https://arxiv.org/abs/1804.06198)
There are virtual links such that $D^{\#} \not\sim D^*$.

20 Wheel types and 21 Turtle types are determined by multivariable polynomial invariants.

A classification of twisted knots with 3 classical crossings

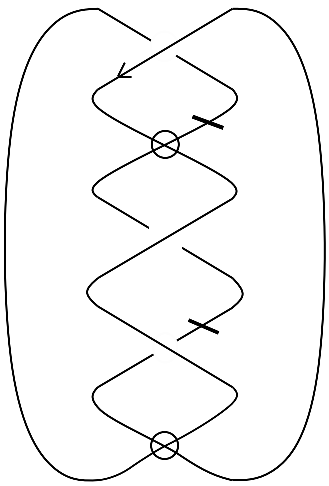
5. Calculate WJKSS invariants (Sawollek invariant for double covering diagram) of 6 Wheel types and 40 Turtle types.

$\boxed{\text{21 Turtle types are determined by WJKSS invariants.}}$

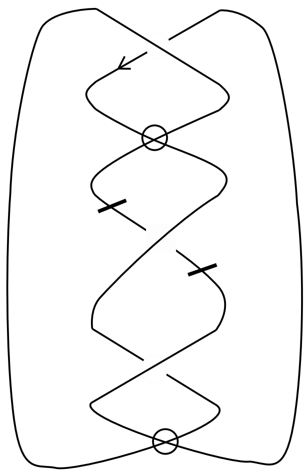
6. Apply extended R moves to each pair of twisted knot diagrams whose invariants (multivariable polynomial invariant and WJKSS invariant) are equal.

$\boxed{\text{2 Wheel types and 3 Turtle types are determined by applying extended R moves.}}$

Example : Deforming diagrams (3_{21} in table)



D

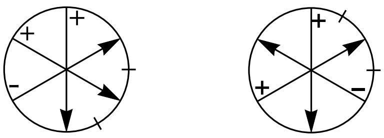


D'

	multivariable polynomial invariant	WJKSS invariant
D	$\frac{(A^4+1)(A^8M^2+A^4(2M^2+1)-(2A^8+3A^4+1)M^2+M^2)}{A^{10}}$	$\frac{(y-1)(y+1)(x-y)(x+y)}{y^2}$
D'	$\frac{(A^4+1)(A^8M^2+A^4(2M^2+1)-(2A^8+3A^4+1)M^2+M^2)}{A^{10}}$	$\frac{(y-1)(y+1)(x-y)(x+y)}{y^2}$

Example : Deforming diagrams (3_{21} in table)

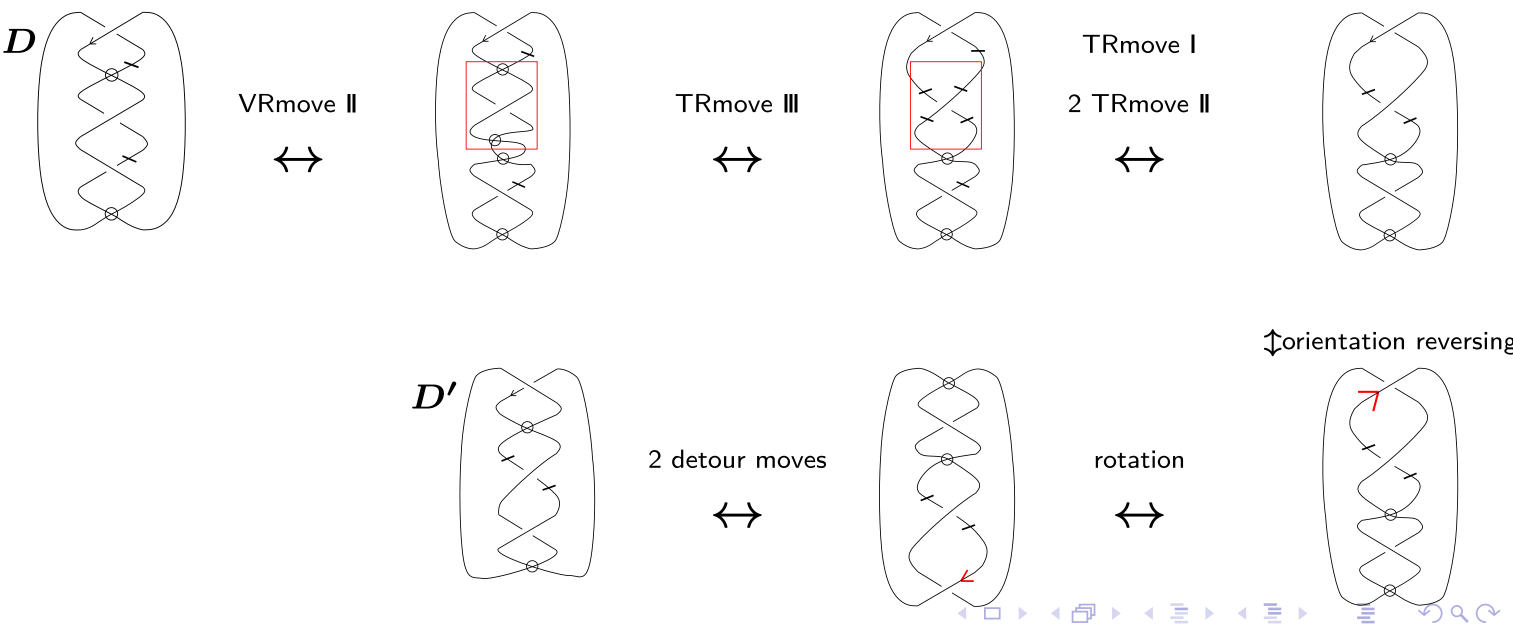
orientation reversing pair



D D'

Example of wheel type

D' is equivalent to $-D$

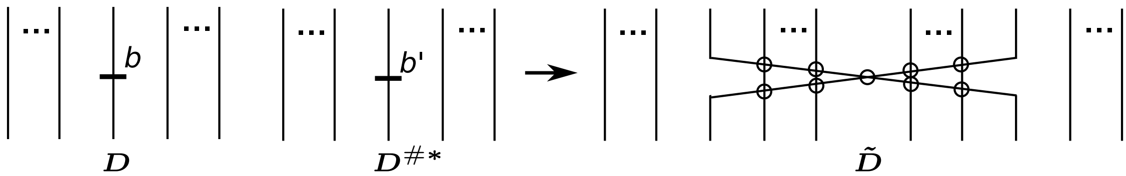


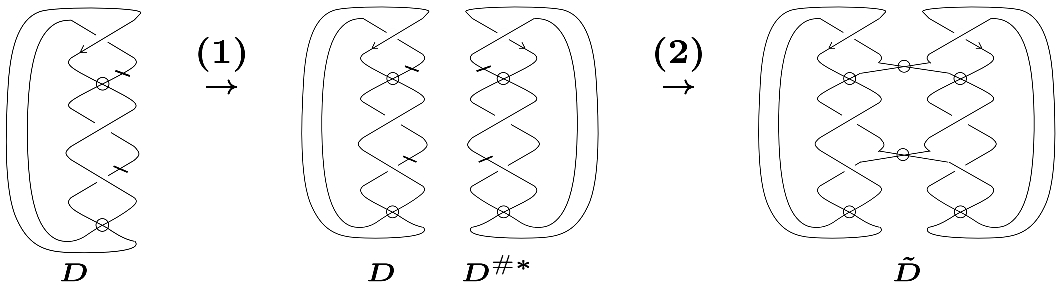
A classification of twisted knots with 3 classical crossings

7 Calculate multivariable polynomial invariants for double covering diagrams of the remaining diagrams (15 Turtle types).

Double covering diagram

(1) For a twisted link diagram D, put $D^{\#*}$(reflection and mirror image) on right of D.
(2) Deform these diagrams as in the following figure where a bar b of D correspond to a bar b' of $D^{\#*}$.





Theorem [N.Kamada, S.Kamada]

$D \sim D' \Rightarrow \widetilde{D} \sim \widetilde{D'}$
$X_{\widetilde{D}}$ is an invariant of D.

13 Turtle types are determined by this method.
2 Turtle types are still to be determined.

A classification of twisted knots with 3 classical crossings

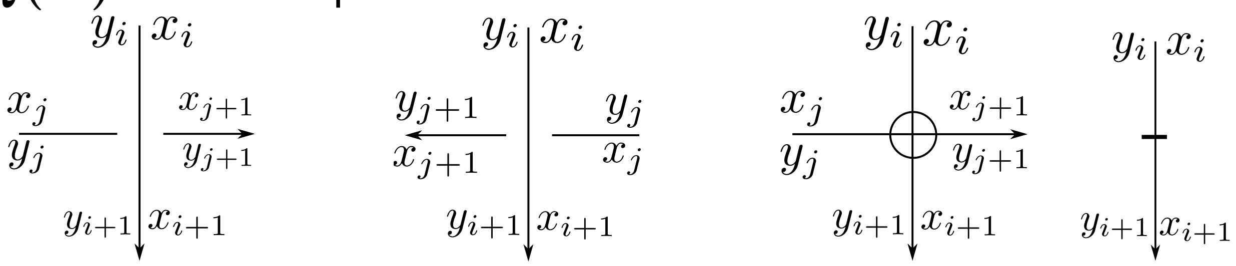
8 Calculate $\widetilde{\mathrm{col}}_3(D)$ for the remaining diagrams (2 Turtle types).

The quandle is a set X with a binary operator $*$, which satisfies the following.

- For any $x \in X$,$x * x = x$
- For any $y \in X$, the map $S_y : X \longrightarrow X$ defined by $x \mapsto x * y$ is a bijection.
- For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

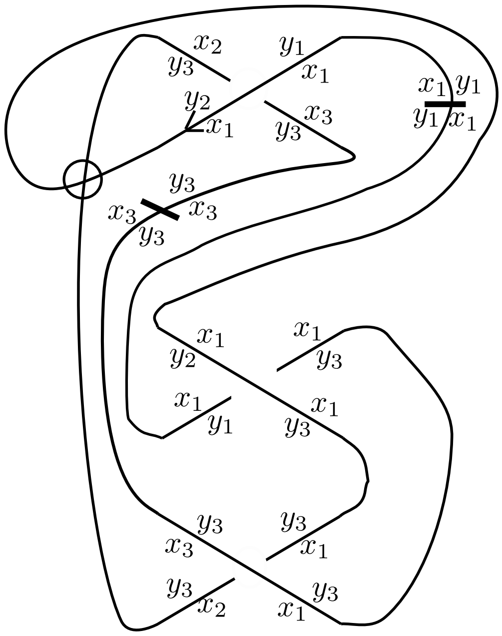
A classification of twisted knots with 3 classical crossings

$Q(D)$: twisted quandle of D.

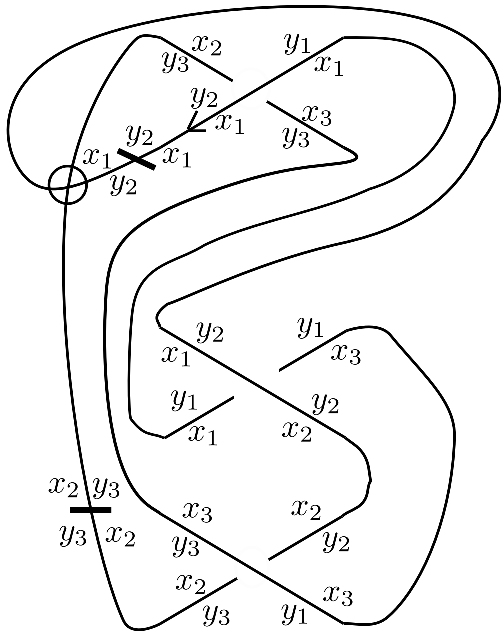


$$\begin{array}{llll}
x_{i+1} = x_i & x_{i+1} = x_i & x_{i+1} = x_i & \\
x_{j+1} = x_j * x_i & x_j = x_{j+1} * x_i & x_{j+1} = x_j & x_{i+1} = y_i \\
y_{i+1} = y_i * y_j & y_i = y_{i+1} * y_j & y_{i+1} = y_i & y_{i+1} = x_i \\
y_{j+1} = y_j & y_{j+1} = y_j & y_{j+1} = y_j &
\end{array}$$

$X_n = \{1, 2, \ldots, n\}$ a dihedral quandle $x * y = 2y - x$
$\widetilde{\mathrm{col}}_n(D)$: the number of homomorphisms from $Q(D)$ to X_n (coloring)



$\boldsymbol{D}$



$\boldsymbol{D'}$

$\widetilde{\mathrm{col}}_{\mathbf{3}}(\boldsymbol{D}) = \mathbf{3}$ $\widetilde{\mathrm{col}}_{\mathbf{3}}(\boldsymbol{D'}) = \mathbf{9}$

$\boxed{\text{2 Turtle types are determined by this method.}}$

Conclusion

Theorem 1

82 prime twisted knots with 3 classical crossings are determined.

Theorem 2

(i) 3_1, 3_4, 3_5, 3_7, 3_8, 3_9, 3_{10}, 3_{11}, 3_{12}, 3_{13}, 3_{14}, 3_{15}, 3_{16}, 3_{17}, 3_{18}, 3_{19}, 3_{20}, 3_{31}, 3_{40}, 3_{42}, 3_{44}, 3_{46}, 3_{47}, 3_{50}, 3_{58}, 3_{60}, 3_{62}, 3_{65}, 3_{66}, 3_{67}, 3_{68}, 3_{69}, 3_{70}, 3_{72}, 3_{74}, 3_{79}, and 3_{80} are invertible.

(ii) 3_{24}, 3_{25}, 3_{26}, 3_{28}, 3_{29}, 3_{30}, 3_{33}, 3_{34}, 3_{35}, 3_{36}, 3_{37}, 3_{38}, 3_{43}, 3_{45}, 3_{48}, 3_{49}, 3_{51}, 3_{52}, 3_{53}, 3_{54}, 3_{56}, 3_{57}, 3_{59}, 3_{61}, 3_{63}, 3_{64}, 3_{73}, 3_{75}, 3_{77}, and 3_{78} are non invertible.

Proof : (i) D is deformed to $-D$ by extended R moves.
(ii) The value of WJKSS invariant of D is different from that of $-D$.

Conclusion

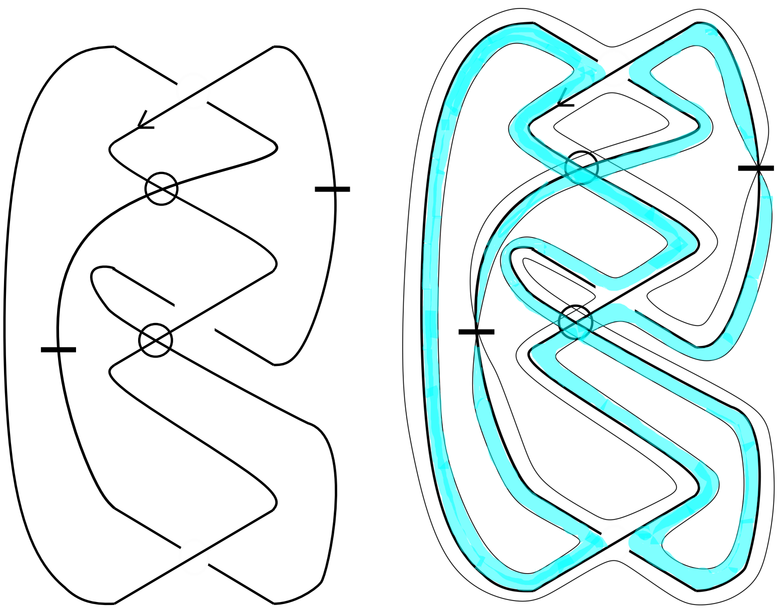
Proposition 3

All diagrams in our table are chiral .

Proof : It is proved by showing that the multivariable polynomial invariant X_D of D is not equal to that of mirror image of D .

Proposition 4

$\mathbf{3}_{44}$, $\mathbf{3}_{58}$, $\mathbf{3}_{67}$, $\mathbf{3}_{70}$, $\mathbf{3}_{81}$, $\mathbf{3}_{82}$ are checkerboard colorable, and other diagrams in our table are not checkerboard colorable.

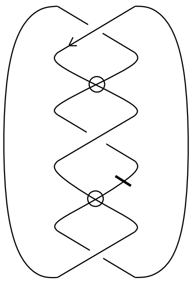


$\mathbf{3_{44}}$

$\boldsymbol{D}$: a twisted link diagram
$\boldsymbol{A(D)}$: an abstract link diagram associated with $\boldsymbol{D}$
$\boldsymbol{D}$: checkerboard colorable $\Leftrightarrow$ $\boldsymbol{A(D)}$ admits checkerboard coloring

Conclusion

Proof of Theorem 4 : There are checkerboard colorable diagrams of $\mathbf{3_{44}}$, $\mathbf{3_{58}}$, $\mathbf{3_{67}}$, $\mathbf{3_{70}}$, $\mathbf{3_{81}}$, $\mathbf{3_{82}}$ in the table.
For the rest of them, in their multivariable invariants there are terms which include M.



$\mathbf{3_2}$

$$X_D = -\frac{(A^4+1)(A^{12}+A^4-1)M}{A^{18}}$$

Theorem [N.Kamada]

If D is a twisted link diagram of a checkerboard colorable twisted link, then X_D is in $\mathbb{Z}[A, A^{-1}, d_1, d_2, \ldots]$.

Conclusion

Proposition 5

(i) The minimum numbers of bars of the following twisted knots are $\mathbf{1}$.
$\mathbf{3_1}$, $\mathbf{3_2}$, $\mathbf{3_3}$, $\mathbf{3_4}$, $\mathbf{3_5}$, $\mathbf{3_6}$, $\mathbf{3_{24}}$, $\mathbf{3_{25}}$, $\mathbf{3_{26}}$, $\mathbf{3_{27}}$, $\mathbf{3_{28}}$, $\mathbf{3_{29}}$, $\mathbf{3_{30}}$, $\mathbf{3_{31}}$, $\mathbf{3_{32}}$, $\mathbf{3_{33}}$, $\mathbf{3_{34}}$, $\mathbf{3_{35}}$, $\mathbf{3_{36}}$, $\mathbf{3_{37}}$, $\mathbf{3_{38}}$, $\mathbf{3_{39}}$, $\mathbf{3_{40}}$, $\mathbf{3_{41}}$, $\mathbf{3_{42}}$
(ii) The minimum numbers of bars of the following twisted knots are $\mathbf{2}$.
$\mathbf{3_7}$, $\mathbf{3_8}$, $\mathbf{3_9}$, $\mathbf{3_{10}}$, $\mathbf{3_{11}}$, $\mathbf{3_{12}}$, $\mathbf{3_{13}}$, $\mathbf{3_{14}}$, $\mathbf{3_{15}}$, $\mathbf{3_{16}}$, $\mathbf{3_{43}}$, $\mathbf{3_{45}}$, $\mathbf{3_{46}}$, $\mathbf{3_{47}}$, $\mathbf{3_{48}}$, $\mathbf{3_{49}}$, $\mathbf{3_{50}}$, $\mathbf{3_{51}}$, $\mathbf{3_{52}}$, $\mathbf{3_{53}}$, $\mathbf{3_{54}}$, $\mathbf{3_{55}}$, $\mathbf{3_{56}}$, $\mathbf{3_{57}}$, $\mathbf{3_{59}}$, $\mathbf{3_{60}}$, $\mathbf{3_{61}}$, $\mathbf{3_{62}}$, $\mathbf{3_{63}}$, $\mathbf{3_{64}}$, $\mathbf{3_{65}}$, $\mathbf{3_{66}}$, $\mathbf{3_{68}}$, $\mathbf{3_{69}}$, $\mathbf{3_{71}}$, $\mathbf{3_{72}}$

Proof : The number of bars increase (or decrease) always even number by any twisted Reidemeister moves.
(i) There are diagrams with 1 bar.
(ii) If the value of multivariable polynomial invariant of the twisted link diagram $\boldsymbol{D}$ has terms which include $\boldsymbol{M}$, $\boldsymbol{D}$ has some bars. There are diagrams with 2 bars and their multivariable polynomial invariants have such terms.

Problems

1. To construct the table of composite twisted knot diagrams with 3 classical crossings.
2. To determine invertibility of following diagrams.

$$\boxed{\begin{array}{c}\mathbf{3_2},\ \mathbf{3_6},\ \mathbf{3_{21}},\ \mathbf{3_{22}},\ \mathbf{3_{23}},\ \mathbf{3_{27}},\ \mathbf{3_{32}},\\ \mathbf{3_{39}},\ \mathbf{3_{41}},\ \mathbf{3_{55}},\ \mathbf{3_{71}},\ \mathbf{3_{76}},\ \mathbf{3_{81}},\ \mathbf{3_{82}}.\end{array}}$$

Thank you!