Gap between the alternation number and the dealternating number

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A link is a disjoint union of circles embedded in $S^3$, a knot is a link with one component.
**Introduction**

**Definition**

A knot that possesses an alternating diagram is called an **alternating knot**, otherwise it is called a non-alternating knot.

![Alternating diagram](image1)

![Non-alternating diagram](image2)
Introduction

Definition

A knot that possesses an alternating diagram is called an alternating knot, otherwise it is called a non-alternating knot.

In 2015 Greene and Howie, independently, gave a characterization of alternating links.
Definition (Adams et al., 1992)

The dealternating number of a link diagram \( D \) is the minimum number of crossing changes necessary to transform \( D \) into an alternating diagram. The dealternating number of a link \( L \), denoted \( dalt(L) \), is the minimum dealternating number of any diagram of \( L \).

A link with dealternating number \( k \) is also called \( k \)-almost alternating. We say that a link is \textit{almost alternating} if it is 1-almost alternating.
Definition (Kawauchi, 2010)

The alternation number of a link diagram $D$ is the minimum number of crossing changes necessary to transform $D$ into some (possibly non-alternating) diagram of an alternating link.

The alternation number of a link $L$, denoted $\text{alt}(L)$, is the minimum alternation number of any diagram of $L$. 
Adams et al. showed that an almost alternating knot is either a torus knot or a hyperbolic knot.

\[ \text{alt}(L) = 1 \quad \text{dalt}(L) = 2 \]

\[ \text{alt}(L) \leq \text{dalt}(L) \]
Adams et al. showed that an almost alternating knot is either a torus knot or a hyperbolic knot.
To a link diagram $D$, Turaev associated a closed orientable surface embedded in $S^3$, called the \textit{Turaev surface}.

\begin{definition}[Turaev, 1987]
The \textit{Turaev genus}, $g_T(L)$, of a link $L$ is the minimal number of the genera of the Turaev surfaces of diagrams of $L$.
\end{definition}

\cite{Dasbach et al., 2008} $g_T(L) = 0$ if and only if $L$ is alternating.
Let \( L \in S^3 \) be an oriented link. The Khovanov homology of \( L \), denoted \( Kh(L) \), is a bigraded \( \mathbb{Z} \)-module with homological grading \( i \) and polynomial (or Jones) grading \( j \) so that \( Kh(L) = \bigoplus_{i,j} Kh^{i,j}(L) \).

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The coefficients of the monomials \( t^i q^j \) are shown.

\( j - 2i = s + 1 \) or \( j - 2i = s - 1 \), where \( s = 2 \) is the signature of \( 9_{42} \).
\[ \delta = j - 2i \] so that \( Kh(L) = \bigoplus_{\delta} Kh^{\delta}(L) \).

Let \( \delta_{\text{min}} \) be the minimum \( \delta \)-grading where \( Kh(L) \) is nontrivial and \( \delta_{\text{max}} \) be the maximum \( \delta \)-grading where \( Kh(L) \) is nontrivial.

\( Kh(L) \) is said to be \([\delta_{\text{min}}, \delta_{\text{max}}]\)-thick, and the Khovanov width of \( L \) is defined as

\[ w_{Kh}(L) = \frac{1}{2} (\delta_{\text{max}} - \delta_{\text{min}}) + 1. \]
\begin{align*}
    \text{alt}(K) & \leq \text{dalt}(K). \\
    g_T(K) & \leq \text{dalt}(K). \\
    w_{Kh}(K) - 2 & \leq g_T(K). \\
    \widehat{w}_{HF}(K) - 1 & \leq g_T(K).
\end{align*}

(2) [Abe and Kishimoto, 2010];
(3) [Champanerkar et al., 2007] and [Champanerkar and Kofman, 2009];
(4) [M. Lowrance, 2008].
\[
|\sigma(K) - s(K)| \leq \frac{\text{alt}(K)}{2}.
\] (5)

\[
|\sigma(K) - s(K)| \leq g_T(K).
\] (6)

Skein relation

\[
0 \leq \sigma(K_+) - \sigma(K_-) \leq 2.
\] (7)

\[
0 \leq s(K_+) - s(K_-) \leq 2.
\] (8)

where \(\sigma(K)\) and \(s(K)\) are the signature and Rasmussen \(s\)-invariant of a knot \(K\), respectively, and both invariants are equal to 2 for the positive trefoil knot.

(5) [Abe, 2009];
(6)[Dasbach and Lowrance, 2011];
(7) [Cochran and Lickorish, 1986];
(8) [Rasmussen, 2010].
\( \text{alt}(K) \) and \( \text{dalt}(K) \)

[Abe and Kishimoto, 2010] Examples where the alternation number equals the dealternating number.

[Lowrance, 2015] For all \( n \in \mathbb{N} \) there exists a knot \( K \), which is the iteration of Whitehead doubles of eight figure-eight knot, such that \( \text{alt}(K) = 1 \) and \( n \leq \text{dalt}(K) \).

[Guevara-Hernández, 2017] For all \( n \in \mathbb{N} \) there exist a knot family \( DS_n \) such that if \( K \in DS_n \) then \( \text{alt}(K) = 1 \) and \( \text{dalt}(K) = n \).
Families of knots

\[ N\left((\sigma_2\sigma_3)^{3(m+1)}\sigma_2\sigma_3^{-1}(\sigma_1\sigma_2)^{3n} \cdot c\right) \]

where \( l, m, n \in \mathbb{N} \).
Families of knots

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N((\sigma_2\sigma_3)^3(m+1)\sigma_2\sigma_3^{-1}(\sigma_1\sigma_2)^3n \cdot c)
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$$N(((\sigma_2\sigma_3)^3(m+1)\sigma_2\sigma_3^{-1}(\sigma_1\sigma_2)^3n \cdot c)$$

where $l, m, n \in \mathbb{N}$. 
\[ N(\left(\sigma_2 \sigma_3\right)^{3(m+1)} \sigma_2^{-1} \sigma_3 \sigma_2^{-1} \left(\sigma_1 \sigma_2\right)^{3n} \cdot c) \]

where \( l, m, n \in \mathbb{N} \).
\( w_{Kh}(K) - 2 \leq dalt(K) \)

**Theorem (Khovanov, 2010)**

There are long exact sequences

\[ \cdots Kh^{i-e-1,j-3e-2}(D_h) \to Kh^{i,j}(D_+) \to Kh^{i,j-1}(D_v) \to Kh^{i-3,j-3e-2}(D_h) \to \cdots \]

and

\[ \cdots Kh^{i,j+1}(D_v) \to Kh^{i,j}(D_-) \to Kh^{i-e+1,j-3e+2}(D_h) \to Kh^{i+1,j+1}(D_v) \to \cdots \]

When only the \( \delta = j - 2i \) grading is considered, the long exact sequence become

\[ \cdots Kh^{\delta-e}(D_h) \xrightarrow{f_+^{\delta-e}} Kh^\delta(D_+) \xrightarrow{g_+^\delta} Kh^{\delta-1}(D_v) \xrightarrow{f_+^{\delta-1}} Kh^{\delta-e-2}(D_h) \to \cdots \]

and

\[ \cdots Kh^{\delta+1}(D_v) \xrightarrow{f_-^{\delta+1}} Kh^\delta(D_-) \xrightarrow{g_-^\delta} Kh^{\delta-e}(D_h) \xrightarrow{h_-^{\delta-e}} Kh^{\delta-1}(D_v) \to \cdots \]

\( e = neg(D_h) - neg(D_+) \)

The crossings \( D_+, D_-, D_v, D_h \), respectively.
Let $D_+, D_-, D_v$ and $D_h$ be as above. Suppose $Kh(D_v)$ is $[v_{\text{min}}, v_{\text{max}}]$-thick and $Kh(D_h)$ is $[h_{\text{min}}, h_{\text{max}}]$-thick. Then $Kh(D_+)$ is $[\delta^+_{\text{min}}, \delta^+_{\text{max}}]$-thick, and $Kh(D_-)$ is $[\delta^-_{\text{min}}, \delta^-_{\text{max}}]$-thick, where

$$
\delta^+_{\text{min}} = \begin{cases} 
\min\{v_{\text{min}} + 1, h_{\text{min}} + e\} & \text{if } v_{\text{min}} \neq h_{\text{min}} + e + 1 \\
v_{\text{min}} + 1 & \text{if } v_{\text{min}} = h_{\text{min}} + e + 1 \text{ and } h^v_{\min} \text{ is surjective} \\
v_{\text{min}} - 1 & \text{if } v_{\text{min}} = h_{\text{min}} + e + 1 \text{ and } h^v_{\min} \text{ is not surjective},
\end{cases}
$$

$$
\delta^+_{\text{max}} = \begin{cases} 
\max\{v_{\text{max}} + 1, h_{\text{max}} + e\} & \text{if } v_{\text{max}} \neq h_{\text{max}} + e + 1 \\
v_{\text{max}} - 1 & \text{if } v_{\text{max}} = h_{\text{max}} + e + 1 \text{ and } h^v_{\max} \text{ is injective} \\
v_{\text{max}} + 1 & \text{if } v_{\text{max}} = h_{\text{max}} + e + 1 \text{ and } h^v_{\max} \text{ is not injective},
\end{cases}
$$

$$
\delta^-_{\text{min}} = \begin{cases} 
\min\{v_{\text{min}} - 1, h_{\text{min}} + e\} & \text{if } v_{\text{min}} \neq h_{\text{min}} + e - 1 \\
v_{\text{min}} + 1 & \text{if } v_{\text{min}} = h_{\text{min}} + e - 1 \text{ and } h^v_{\min} \text{ is surjective} \\
v_{\text{min}} - 1 & \text{if } v_{\text{min}} = h_{\text{min}} + e - 1 \text{ and } h^v_{\min} \text{ is not surjective},
\end{cases}
$$

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v_{\text{max}} + 1 & \text{if } v_{\text{max}} = h_{\text{max}} + e - 1 \text{ and } h^v_{\max} \text{ is not injective}.\end{cases}
$$
Lemma (G.)

If \( D = N((\sigma_2 \sigma_3)^{3(m+1)} \sigma_2^{-1} \sigma_3^{-1} (\sigma_1 \sigma_2)^{3n} \cdot c) \), then \( Kh(D) \) is \([4m + l + 2, 6m + 2n + l + 4]\)-thick. Hence, \( w_{kh}(D) = m + n + 2 \).

Proof. (outline)
Lemma (G.)

If \( D = N((\sigma_2 \sigma_3)^3(m+1)\sigma_2^l \sigma_3^{-1}(\sigma_1 \sigma_2)^3 n \cdot c) \), then \( Kh(D) \) is \([4m + l + 2, 6m + 2n + l + 4]\)-thick. Hence, \( w_{Kh}(D) = m + n + 2 \).

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Lemma (G.)

Let $D$ be the closure of the 3-braid $(\sigma_2\sigma_3)^{3k}\sigma_2^r\sigma_3^{-1}(\sigma_1\sigma_2)^{3n}$ with $k, r, m \in \mathbb{N}$ and $k \geq 2$, then $Kh(D)$ is $[4(k + n) + r - 3, 6(k + n) + r - 3]$-thick.

Proof. Induction over $n$ by using the braid $\sigma_2^r\sigma_3^{-1}(\sigma_2\sigma_3)^{3k}(\sigma_1\sigma_2)^{3n}$. □
Lemma (G.)

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Proof. Induction over $n$ by using the braid $\sigma_2^r \sigma_3^{-1} (\sigma_2 \sigma_3)^{3k} (\sigma_1 \sigma_2)^{3n}$. □

Proposition (Lowrance, 2009)

Let $D$ be the closure of the braid $(\sigma_1 \sigma_2)^{3k} \sigma_1^a \sigma_2^{-1} \sigma_2$ where $a$ and $k$ are positive integers. Then $\text{Kh}(D)$ is $[4k + a - 2, 6k + a - 2]$-thick.
Kh($D^*_v$) is $[4(m + n) + l + 1, 6(m + n) + l + 3]$-thick

$\neg(D_v) = 4n + 1$ and $\neg(D^*_v) = 1$ $Kh^\delta(D_v) \cong Kh^{\delta+s}(D^*_v)$.

Therefore $Kh(D_v)$ is $[4m + l + 1, 6m + l + 3]$-thick.

Note that $D_{hv} = D^*_v$ and $Kh(D_{hh})$ is $[4m + l + 2, 6m + l + 4]$-thick.

$\neg(D_{hh}) - \neg(D_h) = 4n + 1 - 1$.

Then, $Kh(D_h)$ is $[4(m + n) + l + 2, 6(m + n) + l + 4]$-thick.

$e = \neg(D_h) - \neg(D_+) = -4n$,

since $4m + l + 1 \neq (4(m + n) + l + 2) + e + 1$ and

$6m + l + 3 \neq (6(m + n) + l + 4) + e + 1$.

It implies that $Kh(D_+)$ is $[4m + l + 2, 6m + 2n + l + 4]$-thick. Hence,

$w_{Kh}(N(D)) = m + n + 2$.

\[ \square \]
Theorem (G.)

For all pair $m$, $n$ of positive integers there exists a family of knots

$$\mathcal{F}^{m,n} = \{ N((\sigma_2\sigma_3)^{3(m+1)}\sigma_2\sigma_3^{-1}(\sigma_1\sigma_2)^{3n} \cdot c) \mid l \in \mathbb{N}, l \text{ is odd.} \}$$

such that, if $K \in \mathcal{F}^{m,n}$ then

$$dalt(K) = m + n \quad \text{and} \quad m - 1 \leq alt(K) \leq m + 1.$$ 

Proof.

Due to the previous lemma we have that

$$w_{Kh}(K) = m + n + 2.$$

Beside,

$$w_{Kh}(K) - 2 \leq g_T(K) \leq dalt(K).$$

It follows that

$$m + n \leq dalt(K).$$
\( alt(K) \leq m + n \)
\(\text{alt}(K) \leq m + n\)
\[ \text{alt}(K) \leq m + n \]

After \( m + n \) crossings changes we have an alternating diagram. Therefore, 
\[ \text{dalt}(K) = m + n. \]
After $m + n$ crossings changes we have an alternating diagram. Therefore, $d_{alt}(K) = m + n$. 
One crossing change.
One crossing change.
Alternation number

One crossing change.

\[(\sigma_1 \sigma_2)^3(m + 1)\sigma_l \sigma_2\]

which is conjugate to

\[(\sigma_1 \sigma_2)^3m\sigma_l + 4\sigma_2\]
Alternation number

One crossing change.
Alternation number

One crossing change.
Alternation number

One crossing change.
One crossing change.
Alternation number

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One crossing change.
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One crossing change.
Alternation number

One crossing change.

\[(\sigma_1 \sigma_2)^{3(m+1)} \sigma_1 \sigma_{-1}^2\]

which is conjugate to

\[(\sigma_1 \sigma_2)^{3m} \sigma_l^4 \sigma_2\]
Alternation number

One crossing change.

We obtain

\[(\sigma_1 \sigma_2)^{3(m+1)} \sigma_1 \sigma_2^{-1}\]
One crossing change.

We obtain

\[(\sigma_1 \sigma_2)^{3(m+1)} \sigma_1 \sigma_2^{-1}\]

which is conjugate to

\[(\sigma_1 \sigma_2)^{3m} \sigma_1^{l+4} \sigma_2\]
Theorem (Kanenobu, 2010)

For positive integers $m$, $r$ with $r$ odd and $r \geq 5$, we have that the closure of the 3-braid $(\sigma_1 \sigma_2)^{3m} \sigma_1' \sigma_2$, denoted by $K_{m,r}$, has alternation number equal to $m$.

It was used the following inequality.

$$\left|\sigma(K_{m,r}) - s(K_{m,r})\right| / 2 \leq \text{alt}(K_{m,r}).$$

(9)
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It was used the following inequality.

$$\frac{|\sigma(K_{m,r}) - s(K_{m,r})|}{2} \leq alt(K_{m,r}).$$  \hspace{2cm} (9)

$$\frac{|\sigma(K_{m,r}) - s(K_{m,r})|}{2} = alt(K_{m,r}).$$  \hspace{2cm} (10)

Then, $alt(K) \leq m + 1$. 

Skein relation

\[ 0 \leq \sigma(D_+) - \sigma(D_-) \leq 2. \quad (11) \]
\[ 0 \leq s(D_+) - s(D_-) \leq 2. \quad (12) \]

\( D_+ \) is a diagram of \( K \), \( D_- = K_{m,r} \)

\[ m - 1 \leq |\sigma(K) - s(K)|/2 \leq m + 1. \quad (13) \]

Then, \( alt(K) \geq m - 1 \).

Therefore,

\[ m - 1 \leq alt(K) \leq m + 1. \]
Theorem (G.)

For all pair \( m, n \) of positive integers there exists a family of knots

\[
\mathcal{F}^{m,n} = \{ N((\sigma_2\sigma_3)^{3(m+1)}\sigma_2^{-1}\sigma_1\sigma_2^{3n} \cdot c) \mid l \in \mathbb{N}, l \text{ is odd.} \}
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Thank you for your attention