平面 4 価グラフの実現問題の結び目射影図への拡張

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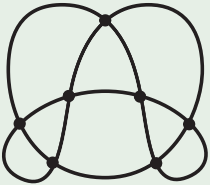
Introduction

G : connected loopless 4-regular planar graph on $\mathbb{S}^2$

- $f_n(G) := \sharp$ of n-sided faced of G

Example



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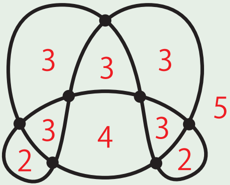

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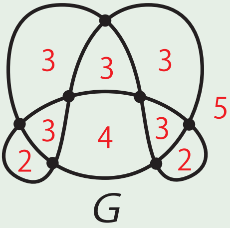
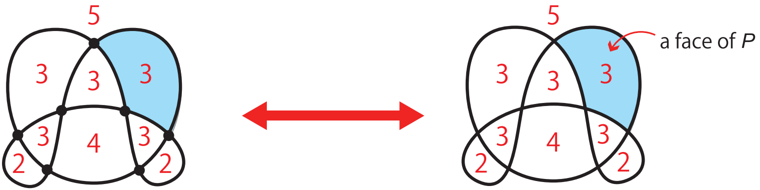


$$f_n(G) = \begin{cases} 2 & (n = 2) \\ 5 & (n = 3) \\ 1 & (n = 4) \\ 1 & (n = 5) \\ 0 & (otherwise) \end{cases}$$



G : 4-regular planar graph

P : knot/link projection

Assumption

Projections are non trivial and reduced.

P : knot projection (4-regular planar graph)
$f_i(P)$: the number of faces of i sides of P.
we have

$$
\begin{aligned}
e(P) &= \frac{2f_2(P) + 3f_3(P) + 4f_4(P) + \dots}{2} = \frac{4v(P)}{2}, \\
f(P) &= f_2(P) + f_3(P) + f_4(P) + \dots
\end{aligned}
$$

where v : the number of vertices, e : the number of edges,
f : the number of faces

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where v : the number of vertices, e : the number of edges,
f : the number of faces

$$(*)\ 2f_2(P) + f_3(P) = 8 + f_5(P) + 2f_6(P) + 3f_7(P) + \dots.$$

Theorem 1 [Jeong 1995]

$\{f_2, f_3, f_5, \ldots, f_m\}$: seq. of non-nega. int. satisfying $(*)$

$$f_2 = 0 \Longrightarrow \begin{array}{l} \exists f_4 \text{ : non-negative integer} \\ \exists P \text{ : knot proj. s.t } f_i(P) = f_i \ (2 \leq i \leq m) \end{array}$$

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Using Theorem 1, we showed the following before.

Theorem 2 [S-T]

$\forall K$: knot

$\{f_2, f_3, f_5, \ldots, f_m\}$: seq. of non-nega. int. satisfying $(*)$

$$f_2 = 0 \Longrightarrow \begin{array}{l} \exists f_4 \ : \text{non-negative integer} \\ \exists P \ : \text{proj of } K \text{ . s.t } f_i(P) = f_i \ (2 \leq i \leq m) \end{array}$$

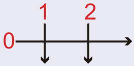
Conjecture

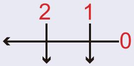
$\forall K$: knot
$\forall \{f_2, f_3, f_5, \cdots, f_m\}$: a seq. of non neg. int.s satisfying $(*)$
$\exists f_4$: non negative integer
$\exists P$: projection of K s.t. $f_i(P) = f_i \ (2 \leq i \leq m)$

Previous results.

Combining Theorem 1 and the following lemma, we obtain Theorem 2.

Lemma 1 (S-T)

$\forall P_0$: knot projection with 

or 

$\forall K$: knot/link

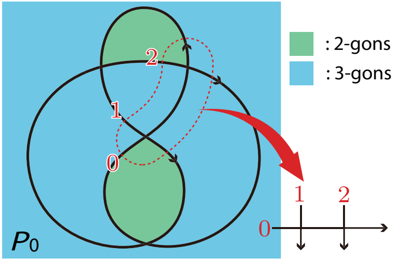
$\exists P$: projection of K s.t. $f_n(P) = f_n(P_0) \quad (n \neq 4)$

Recall (Theorem 1[Jeong 1995])

$\{f_2, f_3, f_5, \ldots, f_m\}$: seq. of non-nega. int. satisfying $(*)$

$$f_2 = 0 \Longrightarrow \begin{array}{l} \exists f_4 \text{ : non-negative integer} \\ \exists P \text{ : knot proj. s.t } f_i(P) = f_i \ (2 \leq i \leq m) \end{array}$$

Example of an application of Lemma 1



$$f_n(P_0) = \begin{cases} 2 & (n=2) \\ 4 & (n=3) \\ 0 & (\mathit{otherwise}) \end{cases}$$

A projection P s.t. $f_2(P) = 4$, $f_3(P) = 2$, $f_n(P) = 0$ $(n \neq 2, 3, 4)$ can be constructed for any knot/link by Lemma 1.

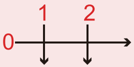
Recall(Theorem 1 [Jeong 1995])

$\{f_2, f_3, f_5, \ldots, f_m\}$: seq. of nonnega. int.s satisfying $(*)$

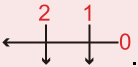
$$f_2 = 0 \Longrightarrow \begin{array}{l} \exists f_4 \ : \ \text{non negative integer} \\ \exists P \ : \ \text{knot projection} \\ \qquad\qquad \text{s.t. } f_i(P) = f_i \ (2 \leq i \leq m). \end{array}$$

Remark

The projection P in Jeong's Theorem has



or



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Main Result

Recall (Theorem 1 [Jeong 1995])

$\{f_2, f_3, f_5, \ldots, f_m\}$: seq. of nonnega. int. satisfying $(*)$

$$f_2 = 0 \Longrightarrow \begin{array}{l} \exists f_4 \ : \text{non-negative integer} \\ \exists P \ : \text{knot proj. s.t } f_i(P) = f_i \ (2 \leq i \leq m) \end{array}$$

Main Theorem

$\{f_2, f_3, f_5, \ldots, f_m\}$: seq. non-nega. int. satisfying $(*)$

$f_2 > 0, \ f_5 - 2f_2 + 2 \geq 0$

$$\Longrightarrow \begin{array}{l} \exists f_4 \ : \text{non-negative integer} \\ \exists P \ : \text{knot projection s.t. } f_i(P) = f_i \ (2 \leq i \leq m) \end{array}$$

We prepare the following two lemmas to prove our main theorem.

Lemma 2

$\{f_2, f_3, f_5, \ldots, f_m\}$: seq. of non-nega. int. satisfying $(*)$

$$f_2 = 1 \Longrightarrow \begin{array}{l} \exists f_4 \text{ : non-negative integer} \\ \exists P \text{ : knot proj. s.t. } f_i(P) = f_i \end{array}$$

Moreover, the 2-gon can be taken as

 or .

Lemma 3

$\forall n \in \mathbb{N}$

$\forall P$: knot proj. with  or .

$\exists P'$: knot projection

$$\text{s.t. } f_2(P') = f_2(P) + n,\ f_5(P') = f_5(P) + 2n,$$
$$f_i(P') = f_i(P)\ (i \neq 2, 4, 5).$$

A sketch proof of Lemma 3.

Recall (Lemma 3)

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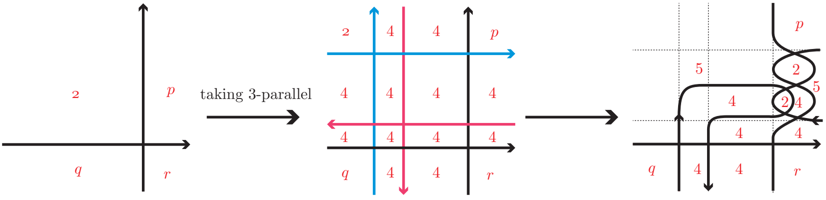
P : knot proj. with a 2-gon

Q : 3-comp. link proj. obtained by taking a 3-parallel of P



P Q : projection of 3 component link P' : knot projection

Using the operation, we obtain a knot proj. P' satisfying

$$f_2(P') = f_2(P) + 1,\ f_5(P) = f_5(P') + 2,\ f_k(P') = f_k(P) \quad (k \neq 2, 4, 5).$$

Repeating the operation, we obtain the desired projection. $\square$

Proof of Main Theorem

Recall (Main Theorem)

$\{f_2, f_3, f_5, \dots, f_m\}$: seq. nonnega. int. satisfying $(*)$

$f_2 > 0,\ f_5 - 2f_2 + 2 \geq 0$

$$\Longrightarrow \begin{array}{l} \exists f_4 : \text{non-negative integer} \\ \exists P : \ \text{knot projection s.t. } f_i(P) = f_i \end{array}$$

$\{f_2', f_3', f_5', \dots, f_m'\}$: a seq. non-nega. int.
s.t. $f_2' = 1,\ f_5' = f_5 - 2f_2 + 2,\ f_i' = f_i\ (i \neq 2, 4, 5)$.

By Lemma 2,

$\exists Q$: knot projection $f_i(Q) = f_i'\ (2 \leq i \leq m)$

Moreover, the 2-gon can be taken as  or .

By Lemma 3 with $n = f_2 - 1$,

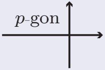
$$\exists P : \text{knot proj. s.t} \begin{cases} f_2(P) &= f_2(Q) + (f_2 - 1) = 1 + (f_2 - 1) \\ &= f_2, \\ f_5(P) &= f_5(Q) + 2(f_2 - 1) \\ &= (f_5 - 2f_2 + 2) + 2(f_2 - 1) = f_5, \\ f_i(P) &= f_i(Q) = f_i \ (i \neq 2, 4, 5). \end{cases}$$

The projection P is desired one. $\square$

Remark

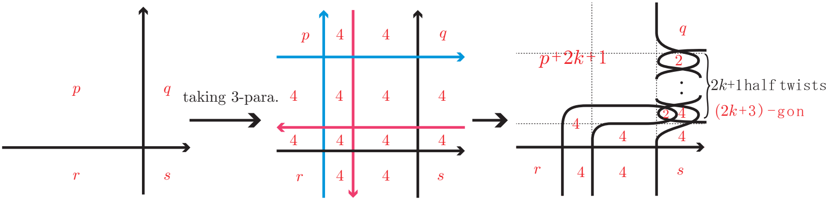
Lemma 3 can be extended as follows.

Lemma 3'

$\forall n \in \mathbb{N}$, $\forall P$: knot proj. with 

$\exists P'$: knot proj. s.t.

$$\begin{aligned} f_2(P') &= f_2(P) + 2k - 1, \\ f_{p+2k+1}(P') &= f_{p+2k+1}(P) + 1, \\ f_{2k+3}(P') &= f_{2k+3}(P) + 1, \\ f_i(P') &= f_i(P) \ (i \neq 2, 4, p+2k+1). \end{aligned}$$



P $\quad$ Q : proj. of 3 comp. link $\quad$ P' : knot projection

We obtain the following as a corollary of Lemma 3'.

Corollary

$\forall n \in \mathbb{N}$, $\forall l, \forall k$: non-nega. integer

$\forall P$: knot projection with a 2-gon.

$\exists p_4$: non-nega. integer

$\exists P'$: knot projection s.t. $f_2(P') = f_2(P) + (2n-1)l$

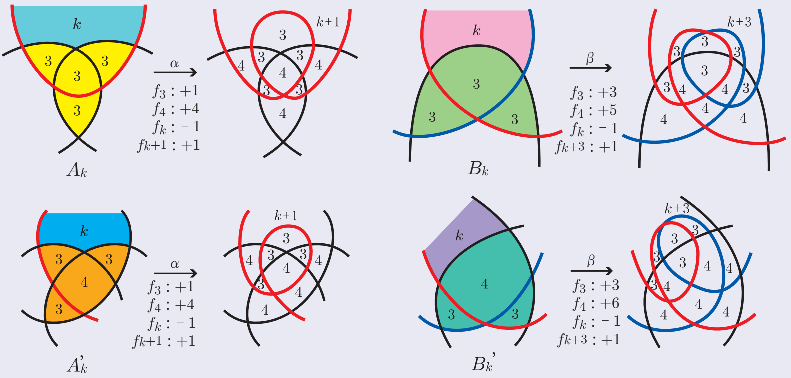
$$f_{2n+3}(P') = f_{2n+3}(P) + 2l$$

$$f_i(P') = f_i(P) \ (i \neq 2, 4, 2n+3)$$

Outline of a proof of Lemma 2

To prove Lemma 2, we use the following operations introduced by Jeong.

Jeong's operations



Recall (Lemma 2)

$\{f_2, f_3, f_5, \dots, f_m\}$: seq. of nonnega. int. satisfying $(*)$

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Moreover, the 2-gon can be taken as

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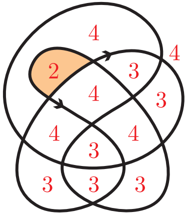
We start with an appropriate knot projecion P_0.



P_0

$$\begin{cases} f_2(P_0) = 1, \\ f_3(P_0) = 6, \\ f_4(P_0) = 5, \\ f_k(P_0) = 0 \ (k \neq 2, 3, 4). \end{cases}$$

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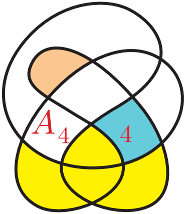

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The knot projection P_0 has A_4 and B_3 as subprojections.

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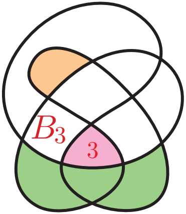

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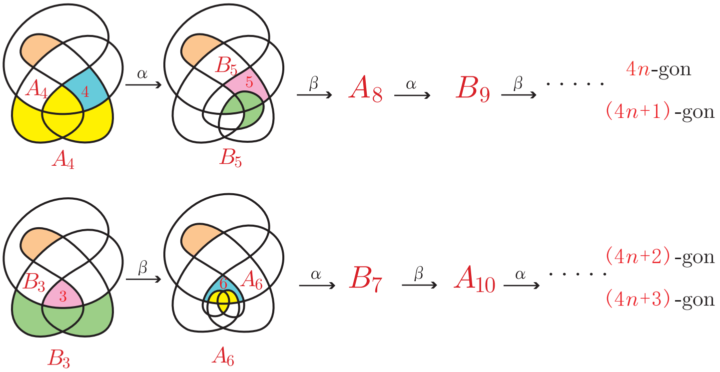
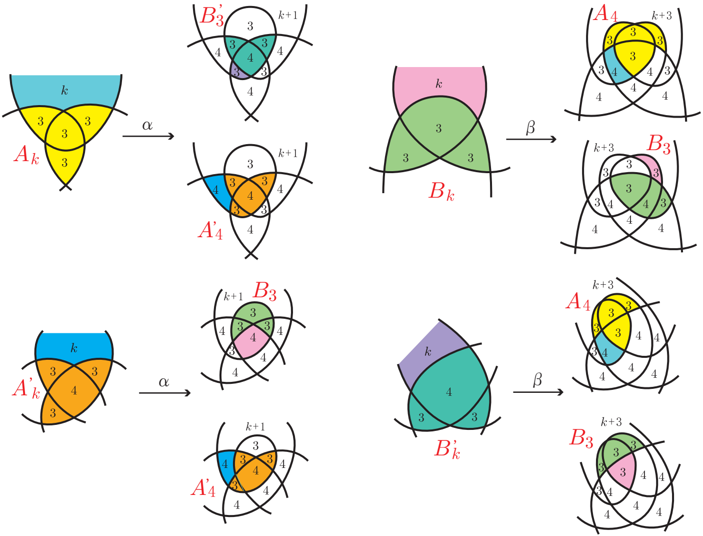
The knot projection P_0 has A_4 and B_3 as subprojections.

Using Jeong's operation the number of k-gons ($\forall n \geq 5$) can be increased by one without changing the number of 3 and 4-gons.



In the resulting projection, A_4 or A_4', and B_3 or B_3' can be found.



We can repeat these operations until the number of k-gons become f_k ($\forall k \geq 5$).

Thank you

for your attention!