Rasmussen invariant and normalization of the canonical classes

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Outline

Overview

History

- M. Khovanov, A categorification of the Jones polynomial. (2000)
  - Homology constructed from a planar link diagram.
  - Its graded Euler characteristic gives the Jones polynomial.

- E. S. Lee, An endomorphism of the Khovanov invariant. (2002)
  - A variant Khovanov-type homology.
  - Introduced to prove the "Kight move conjecture" for the $\mathbb{Q}$-Khovanov homology of alternating knots.

- J. Rasmussen, Khovanov homology and the slice genus. (2004)
  - A knot invariant obtained from $\mathbb{Q}$-Lee homology.
  - Gives a lower bound of the slice genus, and also gives an alternative (combinatorial) proof for the Milnor conjecture.

Lee homology and the s-invariant

For a knot diagram D, there are 2 distinct classes $[\alpha], [\beta]$ in $H_{Lee}(D; \mathbb{Q})$, that form a generator of the $\mathbb{Q}$-Lee homology of D:

$$H_{Lee}(D; \mathbb{Q}) = \mathbb{Q} \langle [\alpha], [\beta] \rangle \cong \mathbb{Q}^2.$$

Rasmussen proved that they are are invariant (up to unit) under the Reidemeister moves. Thus are called the "canonical generators" of $H_{Lee}(K; \mathbb{Q})$ for the corresponding knot K.

The difference of the q-degrees of two classes $[\alpha + \beta]$ and $[\alpha - \beta]$ is exactly 2, and the s-invariant is defined as:

$$s(K) := \frac{\mathrm{qdeg}([\alpha + \beta]) + \mathrm{qdeg}([\alpha - \beta])}{2}.$$

Rasmussen's theorems

Theorem ([1, Theorem 2])

s defines a homomorphism from the knot concordance group in S^3 to $2\mathbb{Z}$,

$$s\colon Conc(S^3) \to 2\mathbb{Z}.$$

Theorem ([1, Theorem 1])

s gives a lower bound of the slice genus:

$$|s(K)| \leq 2g_*(K),$$

Corollary (The Milnor Conjecture)

The slice genus of the (p, q) torus knot is $(p-1)(q-1)/2$.

Our Questions

Now consider the Lee homology over $\mathbb{Z}$.

Question

Does $\{[\alpha(D, o)]\}_o$ *generate* $H_{Lee}(D; \mathbb{Z})/(tors)$ *?*

Answer

No.

Question

Is each $[\alpha(D, o)]$ *invariant up to unit under the Reidemeister moves?*

Answer

No.

Observations

Computational results showed that the components of $[\alpha], [\beta]$ with respect to a computed basis of $H_{Lee}(D; \mathbb{Z})_f \cong \mathbb{Z}^2$ were 2-powers.

```
3_1
b⊗a₍₁₁₁₎ = [2, -2]
a⊗b₍₁₁₁₎ = [-2, -2]

4_1
a⊗b⊗a₍₀₀₁₁₎ = [-2, -2]
b⊗a⊗b₍₀₀₁₁₎ = [2, -2]

5_1
b⊗a₍₁₁₁₁₁₎ = [2, -2]
a⊗b₍₁₁₁₁₁₎ = [-2, -2]

5_2
b⊗a⊗b⊗a₍₁₁₁₁₁₎ = [-8, 8]
a⊗b⊗a⊗b₍₁₁₁₁₁₎ = [-8, -8]
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The 2-divisibility of $[\alpha], [\beta]$ might give some important information. This observation can be extended to a more generalized setting.

Main theorems

Definition

Let $c \in R$. For any link L, define a link invariant as:

$$s'_c(L; R) := 2k_c(D; R) - r(D) + w(D) + 1.$$

Theorem (S.)

s'_c defines a link concordance invariant in S^3.

Proposition (S.)

s'_c gives a lower bound of the slice genus:

$$|s'_c(K)| \leq 2g_*(K),$$

Corollary (The Milnor Conjecture)

The slice genus of the (p, q) torus knot is $(p-1)(q-1)/2$.

Main theorems

Theorem (S.)

Consider the polynomial ring $R = \mathbb{Q}[h]$. The knot invariant s'_h coincides with the Rasmussen's s-invariant :

$$s(K) = s'_h(K; \mathbb{Q}[h]).$$

More generally, for any field F of char $F \neq 2$ *we have:*

$$s(K; F) = s'_h(K; F[h]).$$

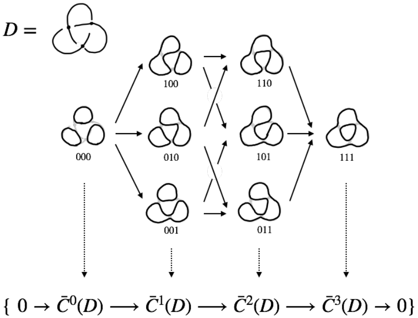
Remark

We do not know (at the time of writing) whether there exists a pair (R, c) such that s'_c is distinct from any of $s(-; F)$.

Review: Khovanov homology theory

Construction of the chain complex $C_A(D)$

Let D be an (oriented) link diagram with n crossings. The 2^n possible resolutions of the crossings form a commutative cube of cobordisms.



By applying a TQFT determined by a Frobenius algebra A, we obtain a chain complex $C_A(D)$, and the homology $H_A(D)$.

Khovanov homology and its variants

Khovanov homology and some variants are given by the following Frobenius algebras:

- $A = R[X]/(X^2)$ $\rightarrow$ Khovanov's theory
- $A = R[X]/(X^2 - 1)$ $\rightarrow$ Lee's theory
- $A = R[X]/(X^2 - hX)$ $\rightarrow$ Bar-Natan's theory

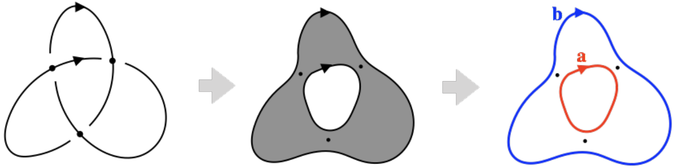
Khovanov unified these theories by considering the following Frobenius algebra determined by two elements $h, t \in R$:

$$A_{h,t} = R[X]/(X^2 - hX - t).$$

Denote the corresponding chain complex by $C_{h,t}(D; R)$ and its homology by $H_{h,t}(D; R)$. The isomorphism class of $H_{h,t}(D; R)$ is invariant under Reidemeister moves, thus gives a link invariant.

Lee homology

Consider $H_{Lee}(-;\mathbb{Q}) = H_{0,1}(-;\mathbb{Q})$. For an ℓ-component link diagram D, there are 2^ℓ distinct classes $\{[\alpha(D,o)]\}_o$, one determined for each alternative orientation o of D:



These classes form a generator of the $\mathbb{Q}$-Lee homology of D:

$$H_{Lee}(D;\mathbb{Q}) = \mathbb{Q}\left\langle [\alpha(D,o)] \right\rangle_o \cong \mathbb{Q}^{2^\ell}$$

Question

Does this construction generalize to $H_{h,t}(D;R)$?

Generalized Lee's class

Generalized Lee's classes (1/2)

We assume (R, h, t) satisfies:

Condition

There exists $c \in R$ such that $h^2 + 4t = c^2$ and $(h \pm c)/2 \in R$.

With $c = \sqrt{h^2 + 4t}$ (fix one such square root), let

$$u = (h - c)/2, \quad v = (h + c)/2 \ \in \ R.$$

Then $X^2 - hX - t$ factors as $(X - u)(X - v)$ in $R[X]$.

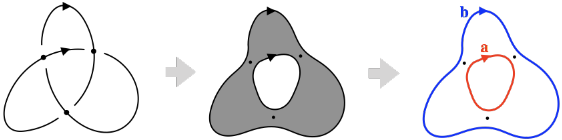
The special case $c = 2$, $(u, v) = (-1, 1)$ gives Lee's theory.

Generalized Lee's classes (2/2)

Let

$$\mathbf{a} = X - u, \quad \mathbf{b} = X - v \;\in\; A.$$

We define the α-*classes* by the exact same procedure.



Proposition

If $c = \sqrt{h^2 + 4t}$ *is invertible, then* $H_{h,t}(D; R)$ *is freely generated by* $\{[\alpha(D, o)]\}_o$ *R.*

Our main concern is when c is not invertible.

Correspondence under Reidemeister moves (1/2)

The following is a generalization of the invariance of $[\alpha]$ over $\mathbb{Q}$ (which implies that $[\alpha]$ is *not* invariant when c is non-invertible)

Proposition (S.)

Suppose D, D' are two diagrams related by a single Reidemeister move. Under the isomorphism corresponding to the move:

$$\rho : H_{h,t}(D; R) \to H_{h,t}(D'; R)$$

there exists some $j \in \{0, \pm 1\}$ such that $[\alpha(D)]$ in $H_{h,t}(D; R)$ and $[\alpha(D')]$ in $H_{h,t}(D'; R)$ are related as:

$$[\alpha(D')] = \pm c^j \rho[\alpha(D)].$$

(Here c is not necessarily invertible, so when $j < 0$ the equation $z = c^j w$ is to be understood as $c^{-j} z = w$.)

Correspondence under Reidemeister moves (2/2)

Proposition (continued)

Moreover j is determined as in the following table:

Type	Δr	j
$RM1_L$	1	0
$RM1_R$	1	1
$RM2$	0	0
	2	1
$RM3$	0	0
	2	1
	-2	-1

where Δr is the difference of the numbers of Seifert circles.
Alternatively, j can be written as:

$$j = \frac{\Delta r - \Delta w}{2},$$

where Δw is the difference of the writhes.

Divisibility of Lee's class and the invariant s_c'

c-divisibility of the α-class

Let R be an integral domain, and $c \in R$ be a non-zero non-invertible element. Denote $H_{h,t}(D; R)_f = H_{h,t}(D; R)/(\mathrm{tor})$.

Definition

For any link diagram D, define:

$$k_c(D) = \max_{k \geq 0} \{ \ [\alpha(D)] \in c^k H_{h,t}(D; R)_f \ \}.$$

Variance of k_c under Reidemeister moves

Proposition

Let D, D' be two diagrams a same link L. Then

$$\Delta k_c = \frac{\Delta r - \Delta w}{2}.$$

Proof.

Follows from the previous proposition, by taking any sequence of Reidemeister moves that transforms D to D'. □

Definition of s'_c

Thus the following definition is justified:

Definition

For any link L, define

$$s'_c(L; R) := 2k_c(D; R) - r(D) + w(D) + 1.$$

where D is any diagram of L, and

- $k_c(D)$ – the c-divisibility of Lee's class $[\alpha] \in H_c(D; R)_f$,
- $r(D)$ – the number of Seifert circles of D, and
- $w(D)$ – the writhe of D.

Variance of s_c' under cobordisms

Behaviour under cobordisms (1/2)

Proposition (S.)

If S is a oriented cobordism between links L, L' such that every component of S has a boundary in L, then

$$s'_c(L') - s'_c(L) \geq \chi(S).$$

If also every component of S has a boundary in both L and L', then

$$|s'_c(L') - s'_c(L)| \leq -\chi(S).$$

Behaviour under cobordisms (2/2)

Proof sketch.

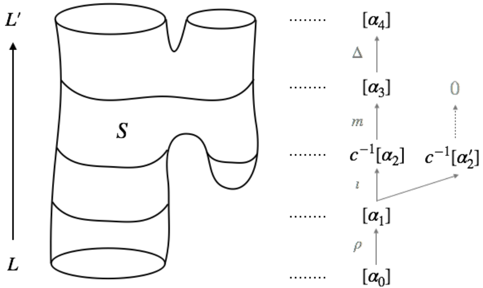


Figure 1: The cobordism map

Decompose S into elementary cobordisms such that each corresponds to a Reidemeister move or a Morse move. Inspect the successive images of the α-class at each level. $\square$

Consequences

The previous proposition implies many properties of s'_c that are common to the s-invariant:

Theorem

s'_c is a link concordance invariant in S^3.

Proposition

For any knot K,

$$|s'_c(K)| \leq 2g_*(K),$$

Corollary (The Milnor Conjecture)

The slice genus of the (p, q) torus knot is $(p-1)(q-1)/2$.

Coincidence with the s-invariant

Normalizing Lee's classes

Now we focus on knots, and

$$(R, c) = (F[h], h)$$

with F a field of char $F \neq 2$ and $\deg h = -2$.

We normalize Lee's classes and obtain a basis $\{\ [\zeta], [X\zeta]\ \}$ of $H_{h,0}(D; F[h])_f$ such that

$$\begin{aligned} [\alpha] &= \quad h^k(\ [X\zeta] + (h/2)[\zeta]\ ) \\ [\beta] &= (-h)^k(\ [X\zeta] - (h/2)[\zeta]\ ) \end{aligned}$$

where $k = k_h(D; F[h])$.

Proposition

$\{\ [\zeta], [X\zeta]\ \}$ *are invariant under the Reidemeister moves. Moreover they are invariant under concordance.*

The homomorphism property of s_h'

Using the normalized generators, we obtain the following:

Theorem (S.)

s_h' defines a homomorphism from the concordance group of knots in S^3 to $2\mathbb{Z}$,

$$s_h' \colon Conc(S^3) \to 2\mathbb{Z}.$$

Coincidence with the Rasmussen's invariant (1/2)

Theorem (S.)

For any knot K,

$$s(K;F) = s'_h(K;F[h]).$$

Proof.

It suffices to prove:

$$s(K;F) \geq s'_h(K;F[h]).$$

The ring homomorphism $\pi : F[h] \to F,\ h \mapsto 2$ gives

$$\begin{aligned}
\mathrm{qdeg}([\alpha]) &= \mathrm{qdeg}(\pi_*[\alpha_h]) \\
&= \mathrm{qdeg}(\pi_*[\alpha'_h]) \\
&\geq \mathrm{qdeg}([\alpha'_h]) \\
&= 2k_h(D) + w(D) - r(D).
\end{aligned}$$

□

Coincidence with the Rasmussen's invariant (2/2)

Corollary

$$s(K; F) = \mathrm{qdeg}[\zeta] - 1.$$

Final remark

The normalization of Lee's class also works for $(R, c) = (\mathbb{Z}, 2)$, the integral Lee theory.

Computational results show that $s'_2(K; \mathbb{Z})$ coincide with $s(K; \mathbb{Q})$ for knots of crossing number up to 11.

Question

Is $s'_2(K; \mathbb{Z})$ distinct from any of $s(K; F)$?

arXiv preprint coming soon...

Appendix

Frobenius algebra

Let R be a commutative ring with unity. A *Frobenius algebra* over R is a quintuple $(A, m, \iota, \Delta, \varepsilon)$ satisfying:

1. (A, m, ι) is an associative R-algebra with multiplication $m : A \otimes A \to A$ and unit $\iota : R \to A$,
2. (A, Δ, ε) is a coassociative R-coalgebra with comultiplication $\Delta : A \to A \otimes A$ and counit $\varepsilon : A \to R$, and
3. the Frobenius relation holds:

$$\Delta \circ m = (id \otimes m) \circ (\Delta \otimes id) = (m \otimes id) \circ (id \otimes \Delta).$$

1+1 TQFT

A (co)commutative Frobenius algebra A determines a 1+1 TQFT

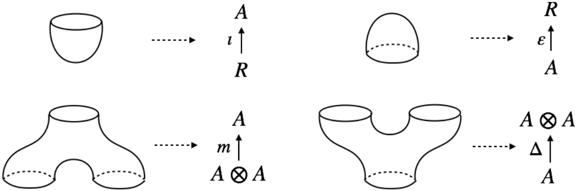
$$\mathcal{F}_A : Cob_2 \longrightarrow Mod_R,$$

by mapping:

- Objects:

$$\underbrace{\bigcirc \sqcup \cdots \sqcup \bigcirc}_{r} \longrightarrow \underbrace{A \otimes \cdots \otimes A}_{r}$$

- Morphisms:



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