Rasmussen invariant and normalization of the canonical classes

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2018-12-26
Outline

1. Overview

2. Review: Khovanov homology theory

3. Generalized Lee’s class

4. Divisibility of Lee’s class and the invariant $s'_c$

5. Variance of $s'_c$ under cobordisms

6. Coincidence with the $s$-invariant

7. Appendix
Overview
History

- M. Khovanov, A categorification of the Jones polynomial. (2000)
  - Homology constructed from a planar link diagram.
  - Its graded Euler characteristic gives the Jones polynomial.

  - A variant Khovanov-type homology.
  - Introduced to prove the “Kight move conjecture” for the \(\mathbb{Q}\)-Khovanov homology of alternating knots.

- J. Rasmussen, Khovanov homology and the slice genus. (2004)
  - A knot invariant obtained from \(\mathbb{Q}\)-Lee homology.
  - Gives a lower bound of the slice genus, and also gives an alternative (combinatorial) proof for the Milnor conjecture.
Lee homology and the $s$-invariant

For a knot diagram $D$, there are 2 distinct classes $[\alpha], [\beta]$ in $H_{Lee}(D; \mathbb{Q})$, that form a generator of the $\mathbb{Q}$-Lee homology of $D$:

$$H_{Lee}(D; \mathbb{Q}) = \mathbb{Q} \langle [\alpha], [\beta] \rangle \cong \mathbb{Q}^2.$$

Rasmussen proved that they are are invariant (up to unit) under the Reidemeister moves. Thus are called the “canonical generators” of $H_{Lee}(K; \mathbb{Q})$ for the corresponding knot $K$.

The difference of the $q$-degrees of two classes $[\alpha + \beta]$ and $[\alpha - \beta]$ is exactly 2, and the $s$-invariant is defined as:

$$s(K) := \frac{q\deg([\alpha + \beta]) + q\deg([\alpha - \beta])}{2}.$$
Rasmussen’s theorems

Theorem ([1, Theorem 2])
s defines a homomorphism from the knot concordance group in $S^3$ to $2\mathbb{Z}$, 

$$s: \text{Conc}(S^3) \rightarrow 2\mathbb{Z}.$$ 

Theorem ([1, Theorem 1])
s gives a lower bound of the slice genus:

$$|s(K)| \leq 2g_*(K),$$

Corollary (The Milnor Conjecture)
The slice genus of the $(p, q)$ torus knot is $(p - 1)(q - 1)/2$. 
Our Questions

Now consider the Lee homology over $\mathbb{Z}$.

Question

Does $\{[\alpha(D, o)]\}$ generate $H_{\text{Lee}}(D; \mathbb{Z})/(\text{tors})$?

Answer

No.

Question

Is each $[\alpha(D, o)]$ invariant up to unit under the Reidemeister moves?

Answer

No.
Observations

Computational results showed that the components of $[\alpha], [\beta]$ with respect to a computed basis of $H_{Lee}(D; \mathbb{Z})_f \cong \mathbb{Z}^2$ were 2-powers.

<table>
<thead>
<tr>
<th>3_1</th>
<th>4_1</th>
<th>5_1</th>
<th>5_2</th>
</tr>
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<tbody>
<tr>
<td>$b \otimes a_{111}$</td>
<td>$2$, $-2$</td>
<td>$a \otimes b_{111}$</td>
<td>$-2$, $-2$</td>
</tr>
<tr>
<td>$a \otimes b_{111}$</td>
<td>$-2$, $-2$</td>
<td>$b \otimes a_{1011}$</td>
<td>$2$, $-2$</td>
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<td>$2$, $-2$</td>
<td>$a \otimes b_{1111}$</td>
<td>$-2$, $-2$</td>
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<tr>
<td>$b \otimes a \otimes a_{1111}$</td>
<td>$[-8$, $8]$</td>
<td>$a \otimes b \otimes a_{1111}$</td>
<td>$[-8$, $-8]$</td>
</tr>
</tbody>
</table>

The 2-divisibility of $[\alpha], [\beta]$ might give some important information. This observation can be extended to a more generalized setting.
Main theorems

Definition
Let $c \in R$. For any link $L$, define a link invariant as:

$$s'_c(L; R) := 2k_c(D; R) - r(D) + w(D) + 1.$$ 

Theorem (S.)
$s'_c$ defines a link concordance invariant in $S^3$.

Proposition (S.)
$s'_c$ gives a lower bound of the slice genus:

$$|s'_c(K)| \leq 2g_*(K),$$

Corollary (The Milnor Conjecture)
The slice genus of the $(p, q)$ torus knot is $(p - 1)(q - 1)/2$. 
Main theorems

Theorem (S.)
Consider the polynomial ring $R = \mathbb{Q}[h]$. The knot invariant $s'_h$ coincides with the Rasmussen’s $s$-invariant:

$$s(K) = s'_h(K; \mathbb{Q}[h]).$$

More generally, for any field $F$ of char $F \neq 2$ we have:

$$s(K; F) = s'_h(K; F[h]).$$

Remark
We do not know (at the time of writing) whether there exists a pair $(R, c)$ such that $s'_c$ is distinct from any of $s(−; F)$.
Review: Khovanov homology theory
Construction of the chain complex $C_A(D)$

Let $D$ be an (oriented) link diagram with $n$ crossings. The $2^n$ possible resolutions of the crossings form a commutative cube of cobordisms.

By applying a TQFT determined by a Frobenius algebra $A$, we obtain a chain complex $C_A(D)$, and the homology $H_A(D)$. 
Khovanov homology and its variants

Khovanov homology and some variants are given by the following Frobenius algebras:

- $A = R[X]/(X^2) \rightarrow$ Khovanov’s theory
- $A = R[X]/(X^2 - 1) \rightarrow$ Lee’s theory
- $A = R[X]/(X^2 - hX) \rightarrow$ Bar-Natan’s theory

Khovanov unified these theories by considering the following Frobenius algebra determined by two elements $h, t \in R$:

$$A_{h,t} = R[X]/(X^2 - hX - t).$$

Denote the corresponding chain complex by $C_{h,t}(D; R)$ and its homology by $H_{h,t}(D; R)$. The isomorphism class of $H_{h,t}(D; R)$ is invariant under Reidemeister moves, thus gives a link invariant.
Lee homology

Consider $H_{\text{Lee}}(-; \mathbb{Q}) = H_{0,1}(-; \mathbb{Q})$. For an $\ell$-component link diagram $D$, there are $2^\ell$ distinct classes $\{[\alpha(D, o)]\}_o$, one determined for each alternative orientation $o$ of $D$:

These classes form a generator of the $\mathbb{Q}$-Lee homology of $D$:

$$H_{\text{Lee}}(D; \mathbb{Q}) = \mathbb{Q} \langle [\alpha(D, o)] \rangle_o \cong \mathbb{Q}^{2^\ell}$$

Question

*Does this construction generalize to $H_{h,t}(D; R)$?*
Generalized Lee’s class
Generalized Lee’s classes (1/2)

We assume \((R, h, t)\) satisfies:

**Condition**
*There exists* \(c \in R\) *such that* \(h^2 + 4t = c^2\) *and* \((h \pm c)/2 \in R\).*

With \(c = \sqrt{h^2 + 4t}\) (fix one such square root), let

\[
    u = (h - c)/2, \quad v = (h + c)/2 \in R.
\]

Then \(X^2 - hX - t\) factors as \((X - u)(X - v)\) in \(R[X]\).

The special case \(c = 2, \ (u, v) = (-1, 1)\) gives Lee’s theory.
Generalized Lee’s classes (2/2)

Let
\[ a = X - u, \quad b = X - v \in A. \]

We define the \( \alpha \)-classes by the exact same procedure.

Proposition

If \( c = \sqrt{h^2 + 4t} \) is invertible, then \( H_{h,t}(D; R) \) is freely generated by \( \{ [\alpha(D, o)] \} \circ R \).

Our main concern is when \( c \) is not invertible.
Correspondence under Reidemeister moves (1/2)

The following is a generalization of the invariance of $[\alpha]$ over $\mathbb{Q}$ (which implies that $[\alpha]$ is not invariant when $c$ is non-invertible)

Proposition (S.)

Suppose $D, D'$ are two diagrams related by a single Reidemeister move. Under the isomorphism corresponding to the move:

$$\rho : H_{h,t}(D; R) \rightarrow H_{h,t}(D'; R)$$

there exists some $j \in \{0, \pm 1\}$ such that $[\alpha(D)]$ in $H_{h,t}(D; R)$ and $[\alpha(D')]$ in $H_{h,t}(D'; R)$ are related as:

$$[\alpha(D')] = \pm c^j \rho[\alpha(D)].$$

(Here $c$ is not necessarily invertible, so when $j < 0$ the equation $z = c^j w$ is to be understood as $c^{-j} z = w$.)
Proposition (continued)

Moreover \( j \) is determined as in the following table:

<table>
<thead>
<tr>
<th>Type</th>
<th>( \Delta r )</th>
<th>( j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( RM1_L )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( RM1_R )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( RM2 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( RM3 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>-2</td>
<td>-1</td>
</tr>
</tbody>
</table>

where \( \Delta r \) is the difference of the numbers of Seifert circles. Alternatively, \( j \) can be written as:

\[
j = \frac{\Delta r - \Delta w}{2},
\]

where \( \Delta w \) is the difference of the writhes.
Divisibility of Lee’s class and the invariant $s'_c$
Let $R$ be an integral domain, and $c \in R$ be a non-zero non-invertible element. Denote $H_{h,t}(D; R)_f = H_{h,t}(D; R)/(\text{tor})$.

**Definition**
For any link diagram $D$, define:

$$k_c(D) = \max_{k \geq 0} \{ [\alpha(D)] \in c^k H_{h,t}(D; R)_f \}.$$
Proposition

Let $D, D'$ be two diagrams a same link $L$. Then

$$\Delta k_c = \frac{\Delta r - \Delta w}{2}.$$  

Proof.

Follows from the previous proposition, by taking any sequence of Reidemeister moves that transforms $D$ to $D'$. 

$\square$
Definition of $s'_c$

Thus the following definition is justified:

**Definition**
For any link $L$, define

$$s'_c(L; R) := 2k_c(D; R) - r(D) + w(D) + 1.$$  

where $D$ is any diagram of $L$, and

- $k_c(D)$ – the $c$-divisibility of Lee’s class $[\alpha] \in H_c(D; R)_f$,
- $r(D)$ – the number of Seifert circles of $D$, and
- $w(D)$ – the writhe of $D$. 
Variance of $s'_c$ under cobordisms
Proposition (S.)

If $S$ is a oriented cobordism between links $L, L'$ such that every component of $S$ has a boundary in $L$, then

$$s'_c(L') - s'_c(L) \geq \chi(S).$$

If also every component of $S$ has a boundary in both $L$ and $L'$, then

$$|s'_c(L') - s'_c(L)| \leq -\chi(S).$$
Behaviour under cobordisms (2/2)

Proof sketch.

Figure 1: The cobordism map

Decompose $S$ into elementary cobordisms such that each corresponds to a Reidemeister move or a Morse move. Inspect the successive images of the $\alpha$-class at each level.
Consequences

The previous proposition implies many properties of $s'_c$ that are common to the $s$-invariant:

**Theorem**
$s'_c$ is a link concordance invariant in $S^3$.

**Proposition**
For any knot $K$,

$$|s'_c(K)| \leq 2g_*(K),$$

**Corollary (The Milnor Conjecture)**
The slice genus of the $(p, q)$ torus knot is $(p - 1)(q - 1)/2.$
Coincidence with the $s$-invariant
Normalizing Lee’s classes

Now we focus on knots, and

\[(R, c) = (F[h], h)\]

with \(F\) a field of char \(F \neq 2\) and \(\deg h = -2\).

We normalize Lee’s classes and obtain a basis \{ \([\zeta], [X\zeta]\) \} of \(H_{h,0}(D; F[h])_f\) such that

\[
[\alpha] = h^k( [X\zeta] + (h/2)[\zeta] ) \\
[\beta] = (-h)^k( [X\zeta] - (h/2)[\zeta] )
\]

where \(k = k_h(D; F[h])\).

Proposition

\{ \([\zeta], [X\zeta]\) \} are invariant under the Reidemeister moves. Moreover they are invariant under concordance.
The homomorphism property of $s'_h$

Using the normalized generators, we obtain the following:

**Theorem (S.)**

$s'_h$ defines a homomorphism from the concordance group of knots in $S^3$ to $2\mathbb{Z}$,

$$s'_h: Conc(S^3) \to 2\mathbb{Z}.$$
Coincidence with the Rasmussen’s invariant (1/2)

Theorem (S.)
For any knot $K$,

$$s(K; F) = s'_h(K; F[h]).$$

Proof.
It suffices to prove:

$$s(K; F) \geq s'_h(K; F[h]).$$

The ring homomorphism $\pi : F[h] \to F$, $h \mapsto 2$ gives

$$\text{qdeg}([\alpha]) = \text{qdeg}(\pi_* [\alpha_h])$$

$$= \text{qdeg}(\pi_* [\alpha'_h])$$

$$\geq \text{qdeg}([\alpha'_h])$$

$$= 2k_h(D) + w(D) - r(D).$$
Corollary

\[ s(K; F) = \text{qdeg}[\zeta] - 1. \]
The normalization of Lee’s class also works for \((R, c) = (\mathbb{Z}, 2)\), the integral Lee theory.

Computational results show that \(s'_2(K; \mathbb{Z})\) coincide with \(s(K; \mathbb{Q})\) for knots of crossing number up to 11.

**Question**

Is \(s'_2(K; \mathbb{Z})\) distinct from any of \(s(K; F)\)?

arXiv preprint coming soon...
Appendix
Let $R$ be a commutative ring with unity. A Frobenius algebra over $R$ is a quintuple $(A, m, \iota, \Delta, \varepsilon)$ satisfying:

1. $(A, m, \iota)$ is an associative $R$-algebra with multiplication $m : A \otimes A \to A$ and unit $\iota : R \to A$,

2. $(A, \Delta, \varepsilon)$ is a coassociative $R$-coalgebra with comultiplication $\Delta : A \to A \otimes A$ and counit $\varepsilon : A \to R$, and

3. the Frobenius relation holds:

$$\Delta \circ m = (id \otimes m) \circ (\Delta \otimes id) = (m \otimes id) \circ (id \otimes \Delta).$$
A (co)commutative Frobenius algebra $A$ determines a $1+1$ TQFT

$$\mathcal{F}_A : \text{Cob}_2 \longrightarrow \text{Mod}_R,$$

by mapping:

- **Objects:**
  $$\bigcirc \sqcup \cdots \sqcup \bigcirc \longrightarrow A \otimes \cdots \otimes A$$

- **Morphisms:**
  $$\begin{array}{ccc}
  \cdots & \longrightarrow & \begin{array}{c}
  A \\
  \scriptstyle{\iota} \\
  \scriptstyle{\epsilon} \\
  \scriptstyle{\Delta}
  \end{array} \\
  \begin{array}{c}
  R \\
  A \\
  A \\
  A \otimes A
  \end{array} & \longrightarrow & \begin{array}{c}
  R \\
  A
  \end{array}
  \end{array}$$
Jacob Rasmussen.
Khovanov homology and the slice genus.